## Lecture – 5

## Date: 19.01.2015

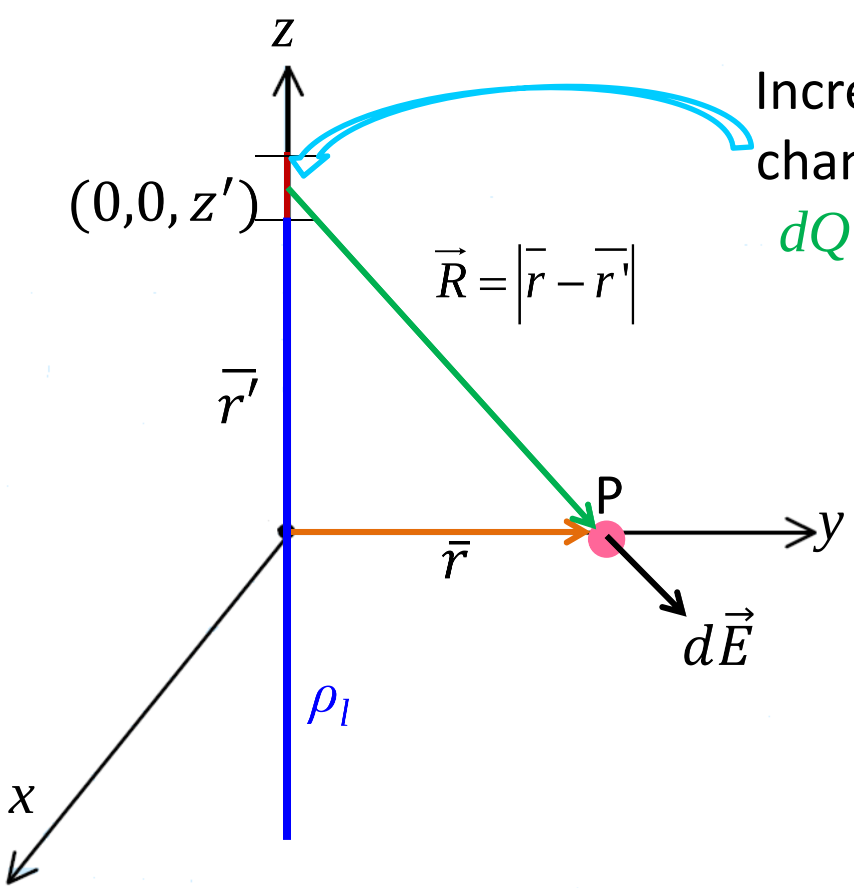
- Electric Field due to a Line Charge (contd.)
- Electric Field due to a Surface Charge
- Electric Field Lines
- Electric Flux
- Gauss Law
- Applications of Gauss Law

# Electric Field due to a Line Charge

- Let us assume an infinite straight-line charge, with charge density $\rho_l$ C/m, lying along the z-axis.
- What is electric field $\vec{E}$ at $P(0, y, 0)$?

# Electric Field due to a Line Charge (contd.)

- For the calculation of electric field $\vec{E}$ at $P(0, y, 0)$, the first step is to determine the incremental field at P due to the incremental charge $dQ = \rho_l dz'$




**We have:**

$$\overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} = \frac{\rho_l dz'}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$$

$$\bar{r} = y\hat{a}_y = \rho\hat{a}_\rho$$

$$\bar{r}' = z'\hat{a}_z$$

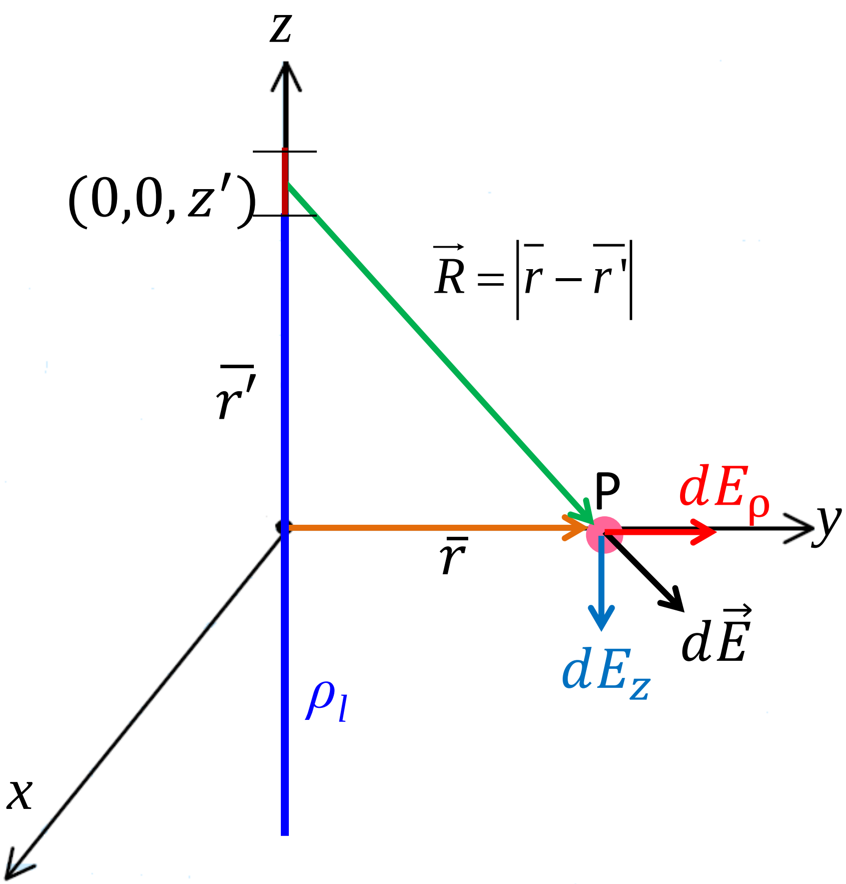
$$\therefore \vec{R} = \bar{r} - \bar{r}' = \rho\hat{a}_\rho - z'\hat{a}_z$$

# Electric Field due to a Line Charge (contd.)

$$\Rightarrow \overrightarrow{dE} = \frac{\rho_l dz'}{4\pi\varepsilon_0} \frac{\left(\rho\hat{a}_\rho - z'\hat{a}_z\right)}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\therefore \overrightarrow{dE} = \frac{\rho_l \rho dz'}{4\pi\varepsilon_0\left(\rho^2 + z'^2\right)^{3/2}}\hat{a}_\rho - \frac{\rho_l z' dz'}{4\pi\varepsilon_0\left(\rho^2 + z'^2\right)^{3/2}}\hat{a}_z$$

$d\vec{E}_\rho$ $\quad$ $d\vec{E}_z$




$$\therefore \overrightarrow{dE} = \hat{a}_\rho dE_\rho - \hat{a}_z dE_z$$

# Electric Field due to a Line Charge (contd.)

**Now:**

$$dE_\rho = \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l \rho}{4\pi\varepsilon_0 \left(\rho^2 + z'^2\right)^{3/2}} dz' = \frac{\rho_l \rho}{4\pi\varepsilon_0} \int_{z'=-\infty}^{z'=\infty} \frac{dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\Rightarrow E_\rho = \frac{\rho_l \rho}{4\pi\varepsilon_0} \left[ \frac{1}{\rho^2} \frac{z'}{\left(\rho^2 + z'^2\right)^{1/2}} \right]_{z'=-\infty}^{z'=\infty}$$

$$\therefore E_\rho = \frac{\rho_l}{2\pi\varepsilon_0 \rho}$$

**AND:**

$$dE_z = \frac{\rho_l}{4\pi\varepsilon_0} \int_{z'=-\infty}^{z'=\infty} \frac{z' dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\therefore E_z = \frac{\rho_l}{4\pi\varepsilon_0} \times (0) = 0$$

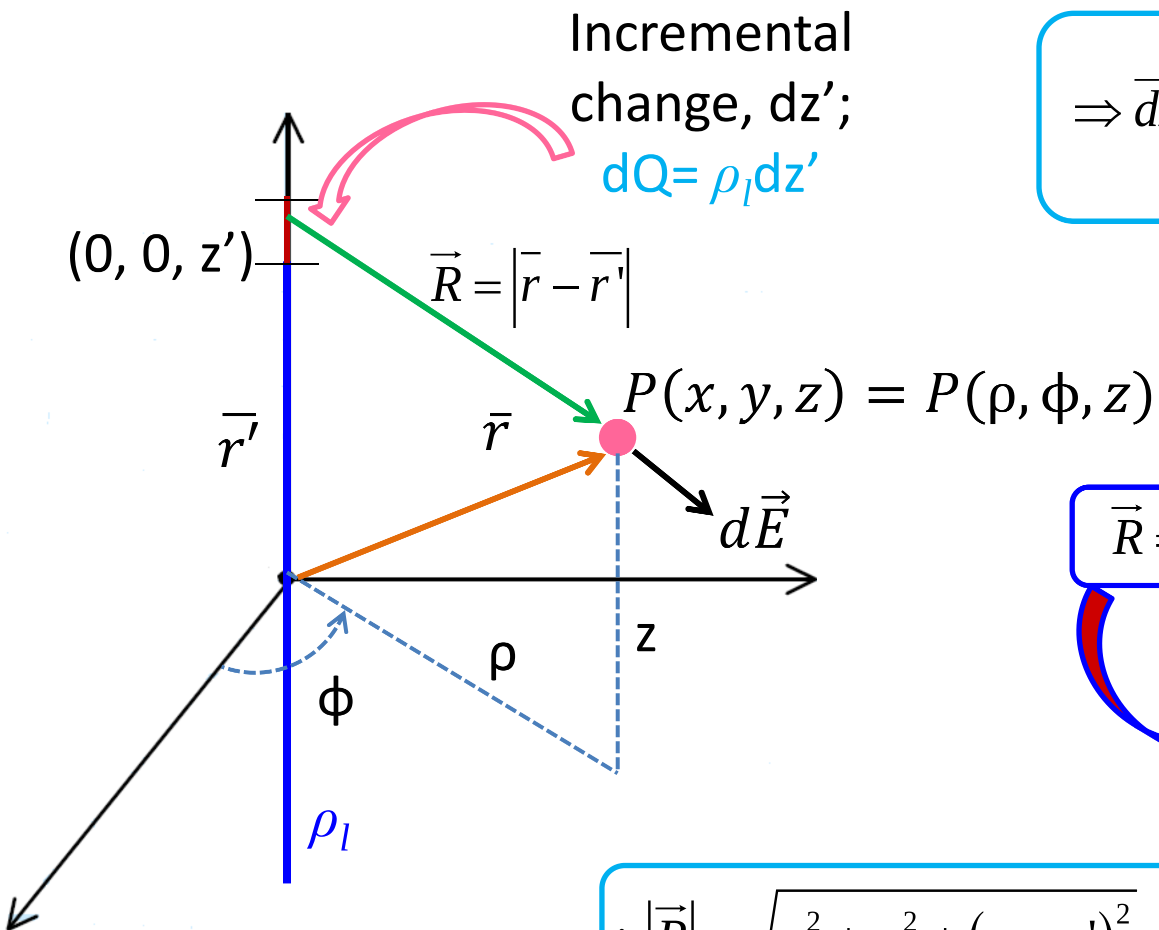
**Therefore:**

$$\vec{E}\left(\bar{r}\right) = E_\rho \hat{a}_\rho - E_z \hat{a}_z = \frac{\rho_l}{2\pi\varepsilon_0 \rho} \hat{a}_\rho$$

BTW, there are multiple ways of solving this problem.
You can master this art through practice!

# Example – 1

- Determine electric field $\vec{E}$ at P(x, y, z)




$$\Rightarrow \overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0}\frac{\left(\bar{r}-\bar{r}'\right)}{\left|\bar{r}-\bar{r}'\right|^3} = \frac{\rho_l dz'}{4\pi\varepsilon_0}\frac{\left(\bar{r}-\bar{r}'\right)}{\left|\bar{r}-\bar{r}'\right|^3}$$

$$\vec{R} = \bar{r}-\bar{r}' = x\hat{a}_x + y\hat{a}_y + \left(z-z'\right)\hat{a}_z$$

$$\vec{R} = \rho\hat{a}_\rho + \left(z-z'\right)\hat{a}_z$$

$$\therefore \left|\vec{R}\right| = \sqrt{x^2+y^2+\left(z-z'\right)^2} = \sqrt{\rho^2+\left(z-z'\right)^2}$$

# Example – 1 (contd.)

Now:

$$\vec{E}(\bar{r}) = \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l dz'}{4\pi\varepsilon_0} \frac{\rho \hat{a}_\rho + (z-z')\hat{a}_z}{\left[\rho^2 + (z-z')^2\right]^{3/2}}$$

$$\vec{E}(\bar{r}) = \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l \rho dz'}{4\pi\varepsilon_0 \left[\rho^2 + (z-z')^2\right]^{3/2}} \hat{a}_\rho + \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l (z-z') dz'}{4\pi\varepsilon_0 \left[\rho^2 + (z-z')^2\right]^{3/2}} \hat{a}_z$$

$$\Rightarrow \vec{E}(\bar{r}) = \frac{\rho_l}{4\pi\varepsilon_0} \left\{ \left[ \hat{a}_\rho \frac{\rho}{\rho^2} \frac{-(z-z')}{\left[\rho^2 + (z-z')^2\right]^{1/2}} \right]_{z'=-\infty}^{z'=\infty} + \left[ \hat{a}_z \frac{-(z-z')}{\left[\rho^2 + (z-z')^2\right]^{1/2}} \right]_{z'=-\infty}^{z'=\infty} \right\}$$

$$= \frac{\rho_l}{4\pi\varepsilon_0} \left[ \hat{a}_\rho \frac{2}{\rho} + \hat{a}_z \times (0) \right]$$

$$\therefore \vec{E}(\bar{r}) = \frac{\rho_l}{2\pi\varepsilon_0 \rho} \hat{a}_\rho$$

**same result**

# Electric Field due to a Line Charge (contd.)

$$\vec{E}(\bar{r}) = \frac{\rho_l}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

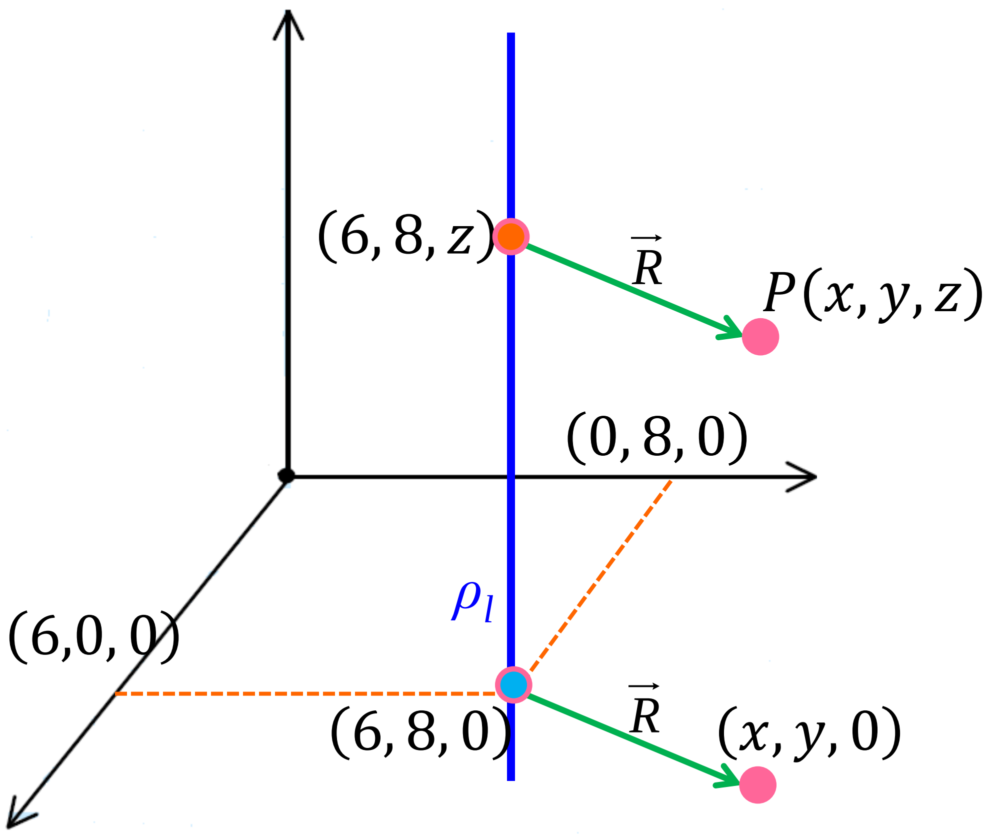
Recall unit vector $\hat{a}_\rho$ is the direction that **points away from** the z-axis.

Thus, the electric field produced by the uniform line charge points away from the line charge, just like the electric field produced by a point charge points away from the charge.

- Note the **magnitude** of the electric field is **proportional** to $1/\rho$, therefore the electric field **diminishes** as we get further from the line charge.
- Note however, the electric field **does not diminish** as **quickly** as that generated by a point charge. Recall in the case of point charge, the magnitude of the electric field diminishes as $1/r^2$.

# Example – 2

- **Oh yes!** It is important to note that **not all the line charges** will be located along the $z$-axis.
- For example, let us consider an infinite line charge parallel to the $z$-axis at $x = 6, y = 8$. We wish to find $\vec{E}$ at the general field point $P(x, y, z)$.




$$\vec{R}$$

$$\therefore |\vec{R}| = \rho = \sqrt{(x-6)^2 + (y-8)^2}$$

$$\Rightarrow \hat{a}_\rho = \frac{(x-6)\hat{a}_x + (y-8)\hat{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

$$\therefore \vec{E} = \frac{\rho_l}{2\pi\varepsilon_0} \frac{(x-6)\hat{a}_x + (y-8)\hat{a}_y}{(x-6)^2 + (y-8)^2}$$

again, not a function of **z**

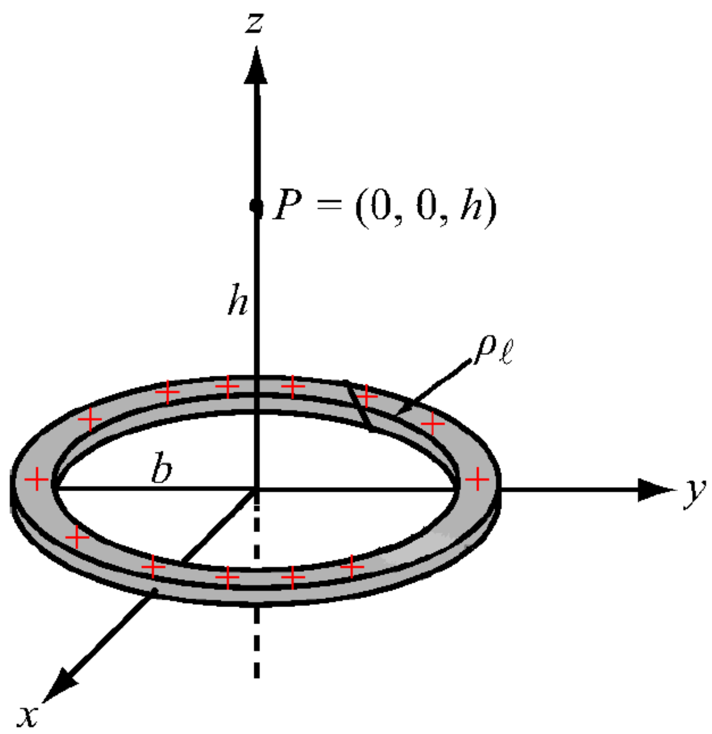
# Example – 3

- A ring of charge of radius $b$ is characterized by a uniform line charge density of positive polarity $\rho_l$. The ring, with its center at (0,0,0), resides in free medium and is positioned in the $xy$-plane.

i. Determine $\vec{E}$ at point $P = (0, 0, h)$ along the axis of the ring at a distance $h$ from the center.

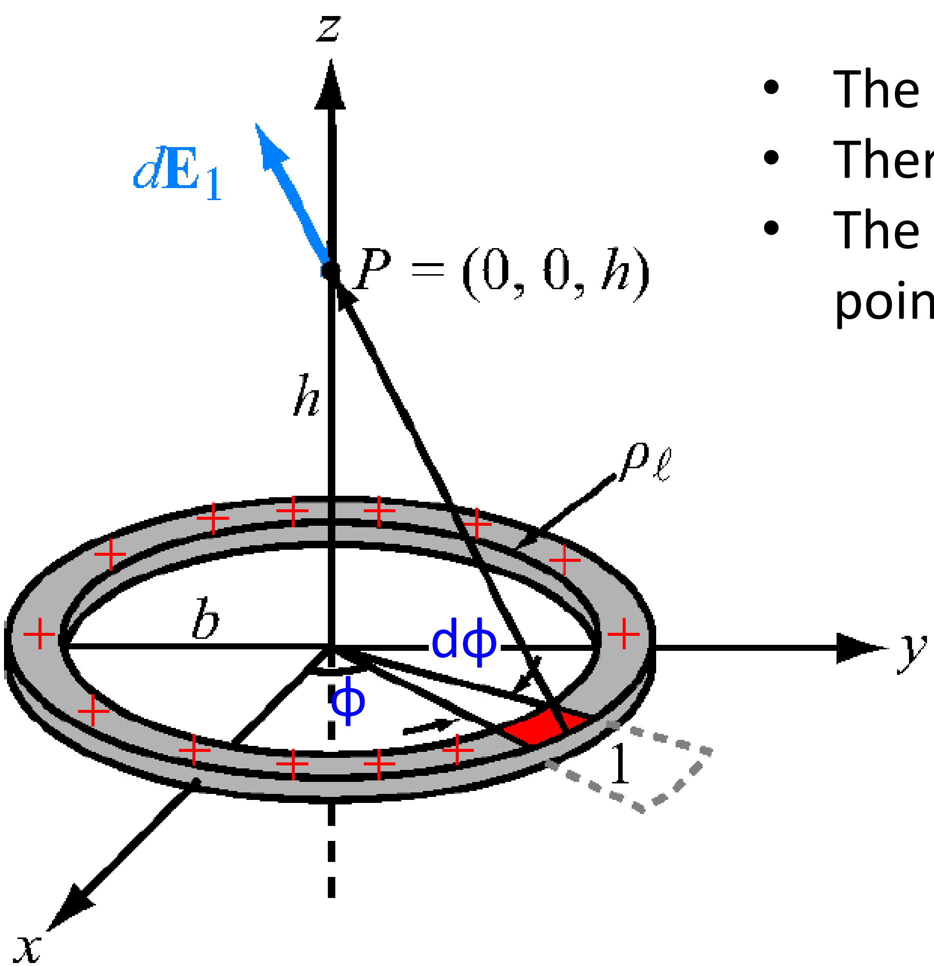
ii. What values of h gives the maximum value of $\vec{E}$.

iii. If the total charge on the ring is Q, find $\vec{E}$ as $b \to 0$.



# Example – 3 (contd.)

**(i)** Let us start by considering the electric field generated by a differential ring segment -1 with cylindrical coordinates (b, ϕ, 0) as shown.




- The line segment has length $dl = bd\phi$
- Therefore contains charge $dQ = \rho_l bd\phi$
- The directed distance from segment-1 to point P is:

$$\vec{r}_1 = -b\hat{a}_\rho + h\hat{a}_z$$

This gives:

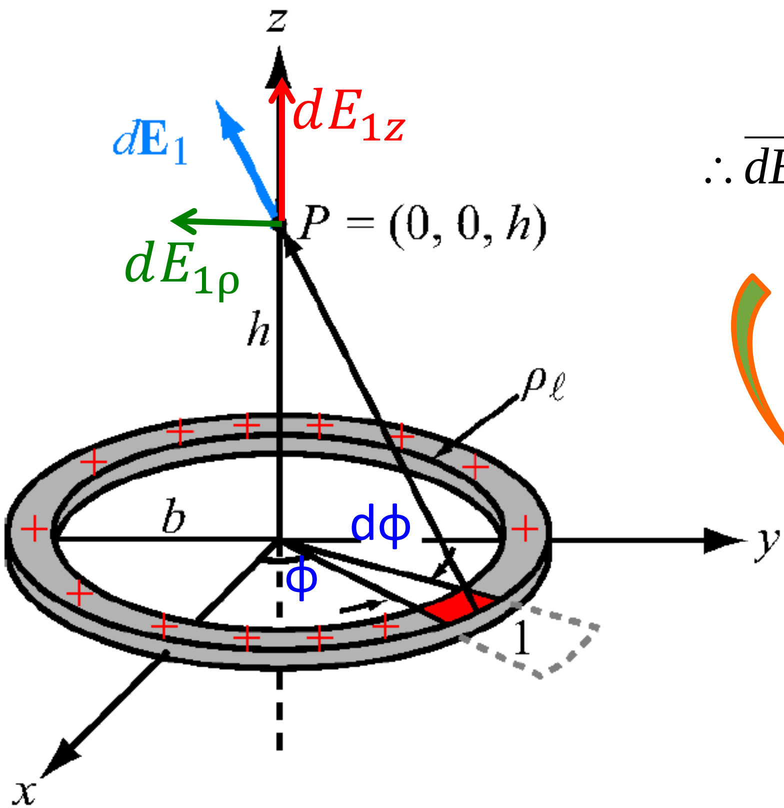
$$r_1 = \left|\vec{r}_1\right| = \sqrt{b^2 + h^2}$$

$$\hat{a}_{r_1} = \frac{\vec{r}_1}{\left|\vec{r}_1\right|} = \frac{-b\hat{a}_\rho + h\hat{a}_z}{\sqrt{b^2 + h^2}}$$

# Example – 3 (contd.)

**Therefore:** Electric Field at P = (0, 0, h) due to charge in segment-1 is:

$$\overrightarrow{dE_1} = \frac{dQ}{4\pi\varepsilon_0}\frac{\hat{a}_{r_1}}{r_1^2} \Rightarrow \overrightarrow{dE_1} = \frac{\rho_l b}{4\pi\varepsilon_0}\frac{-b\hat{a}_\rho + h\hat{a}_z}{\left(b^2+h^2\right)^{3/2}}$$

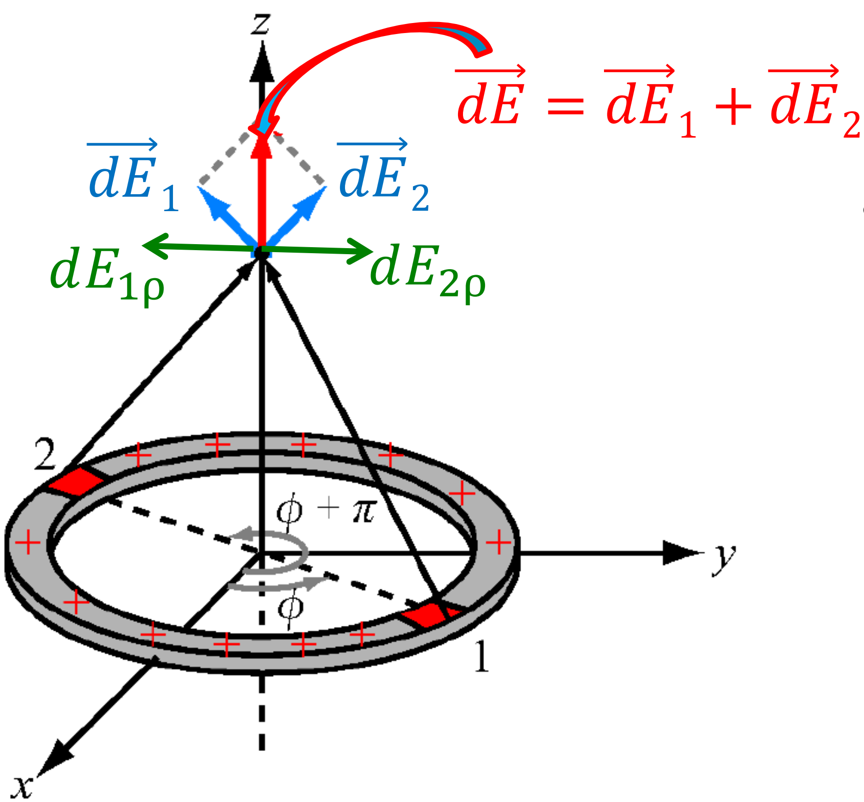



$$\therefore \overrightarrow{dE_1} = -\underbrace{\frac{\rho_l b^2}{4\pi\varepsilon_0\left(b^2+h^2\right)^{3/2}}d\phi}_{dE_{1\rho}}\hat{a}_\rho + \underbrace{\frac{\rho_l bh}{4\pi\varepsilon_0\left(b^2+h^2\right)^{3/2}}d\phi}_{dE_{1z}}\hat{a}_z$$

$$\therefore \overrightarrow{dE_1} = -\hat{a}_\rho dE_{1_\rho} + \hat{a}_z dE_{1_z}$$

# Example – 3 (contd.)

- The field $\overrightarrow{dE}_1$ has component $\overrightarrow{dE}_{1\rho}$ along $-\hat{a}_\rho$ and $\overrightarrow{dE}_{1z}$ along $\hat{a}_z$.
- From symmetry, the field $\overrightarrow{dE}_2$ generated by segment-2, which is located diametrically opposite to segment-1, is identical to $\overrightarrow{dE}_1$ except that the $\hat{a}_\rho$ components **in the sum** cancel and $\hat{a}_z$ components add.




- The sum of two contributions are:

$$\overrightarrow{dE} = \overrightarrow{dE}_1 + \overrightarrow{dE}_2 = \frac{\rho_l bh}{2\pi\varepsilon_0\left(b^2+h^2\right)^{3/2}} d\phi \hat{a}_z$$

# Example – 3 (contd.)

- For every segment-1 in the first half, there is a segment-2 in the other half.
- Therefore, we can set the integration limit as: $0 < \phi < \pi$.
- Thus the total electric field $\vec{E}$ is:

$$\vec{E} = \frac{\rho_l bh}{2\pi\varepsilon_0\left(b^2+h^2\right)^{3/2}}\hat{a}_z\int_0^{\pi} d\phi \quad \Rightarrow \quad \therefore \vec{E} = \frac{\rho_l bh}{2\varepsilon_0\left(b^2+h^2\right)^{3/2}}\hat{a}_z$$

**(ii)** For maximum $\vec{E}$: $\quad \frac{d\left|\vec{E}\right|}{dh} = 0$

$$\frac{d\left|\vec{E}\right|}{dh} = \frac{\rho_l b}{2\varepsilon_0}\left\{\frac{\left(b^2+h^2\right)^{3/2}(1) - \frac{3}{2}(h)2h\left(b^2+h^2\right)^{1/2}}{\left(b^2+h^2\right)^{3/2}}\right\}$$

# Example – 3 (contd.)

$$\Rightarrow \frac{\rho_l b}{2\varepsilon_0}\left\{\frac{\left(b^2+h^2\right)^{3/2}(1)-\frac{3}{2}(h)2h\left(b^2+h^2\right)^{1/2}}{\left(b^2+h^2\right)^{3/2}}\right\}=0$$

$$\left(b^2+h^2\right)^{1/2}\left[h^2+b^2-3h^2\right]=0 \Rightarrow b^2-2h^2=0 \Rightarrow h=\pm\frac{b}{\sqrt{2}}$$

**(iii)** Since the charge is uniformly distributed, the line charge density is:

$$\rho_l=\frac{Q}{2\pi b}$$

**Therefore:**

$$\vec{E}=\frac{\rho_l bh}{2\varepsilon_0\left(b^2+h^2\right)^{3/2}}\hat{a}_z=\frac{Qh}{4\pi\varepsilon_0\left(b^2+h^2\right)^{3/2}}\hat{a}_z$$

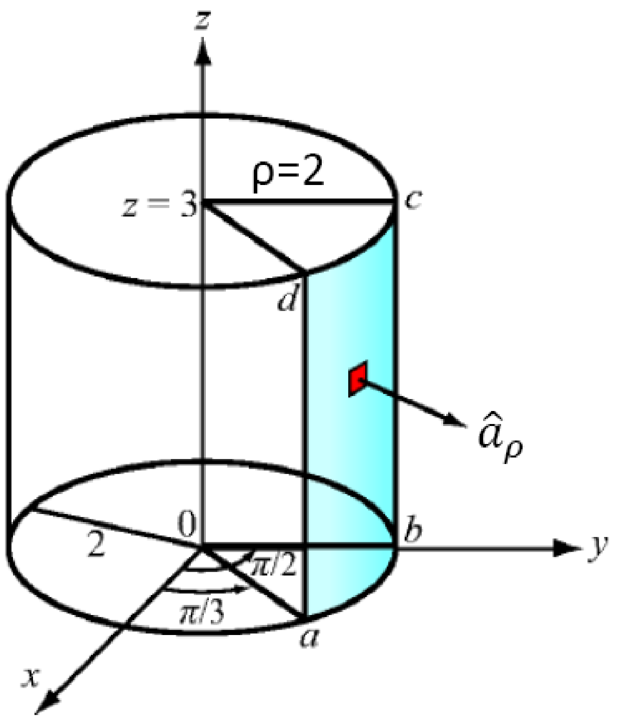
**For, $b \to 0$:**

$$\vec{E}=\frac{Q}{4\pi\varepsilon_0 h^2}\hat{a}_z$$

Same as that of a point charge !

# Example – 4

A vector field $\vec{B} = (cos\phi/\rho)\hat{a}_\rho$ , verify Stoke's Theorem for a segment of a cylindrical surface defined by $\rho = 2, \frac{\pi}{3} \le \phi \le \frac{\pi}{2}, and\ 0 \le z \le 3$ (as shown).




## Stoke's Theorem:

$$\iint_S \left(\nabla \times \vec{B}\right)\bullet\overline{ds} = \oint_C \vec{B}\bullet\overline{dl}$$

# Example – 4 (contd.)

## Left Hand Side

$$\nabla\times\vec{B}=\left(\frac{1}{\rho}\frac{\partial B_z}{\partial\phi}-\frac{\partial B_\phi}{\partial z}\right)\hat{a}_\rho+\left(\frac{\partial B_\rho}{\partial z}-\frac{\partial B_z}{\partial\rho}\right)\hat{a}_\phi+\frac{1}{\rho}\left(\frac{\partial\left(\rho B_\phi\right)}{\partial\rho}-\frac{\partial B_z}{\partial\phi}\right)\hat{a}_z$$

$$\therefore\nabla\times\vec{B}=-\frac{\sin\phi}{\rho^2}\hat{a}_\rho+\frac{\cos\phi}{\rho^2}\hat{a}_\phi$$

- The integration of $\nabla\times\vec{B}$ over the specified surface $S$ is:

$$\oiint_S\left(\nabla\times\vec{B}\right)\bullet\overline{ds}=\int_{z=0}^{3}\int_{\phi=\frac{\pi}{3}}^{\frac{\pi}{2}}\left(-\frac{\sin\phi}{\rho^2}\hat{a}_\rho+\frac{\cos\phi}{\rho^2}\hat{a}_\phi\right)\bullet\rho\,d\phi\,dz\,\hat{a}_\rho$$

$$=\int_{z=0}^{3}\int_{\phi=\frac{\pi}{3}}^{\frac{\pi}{2}}-\frac{\sin\phi}{\rho}d\phi\,dz\quad=-\frac{3}{2\rho}$$

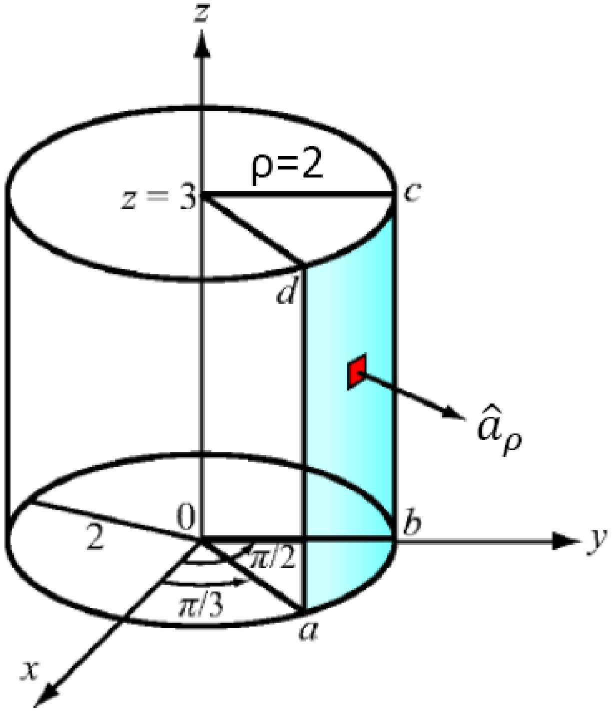
- Given, $\rho=2$: $$\therefore\oiint_S\left(\nabla\times\vec{B}\right)\bullet\overline{ds}=-\frac{3}{4}$$

# Example – 4 (contd.)

## Right Hand Side

- The direction of $C$ is chosen such that it is compatible with the surface normal $\hat{a}_\rho$ by the right hand rule.

$$\oint_C \vec{B}.\overline{dl} = \int_a^b \vec{B}_{\boldsymbol{ab}}.\overline{dl} + \int_b^c \vec{B}_{\boldsymbol{bc}}.\overline{dl} + \int_c^d \vec{B}_{\boldsymbol{cd}}.\overline{dl} + \int_d^a \vec{B}_{\boldsymbol{da}}.\overline{dl}$$




- Over segments $ab$ and $cd$ the integral is zero considering that $\vec{B}.\overline{dl} = 0$ over these segments.
- Over segment $bc$, $\varphi = \frac{\pi}{2}$. Therefore:

$$\vec{B}_{bc} = 0$$

# Example – 4 (contd.)

- Therefore: $\oint_C \vec{B}.\overline{dl} = \int_d^a \vec{B}_{\boldsymbol{da}}.\overline{dl}$
- Over segment $da$, $\varphi = \frac{\pi}{3}$. Therefore: $\vec{B}_{da} = \frac{\cos\frac{\pi}{3}}{2}\hat{a}_z = \frac{1}{4}\hat{a}_z$

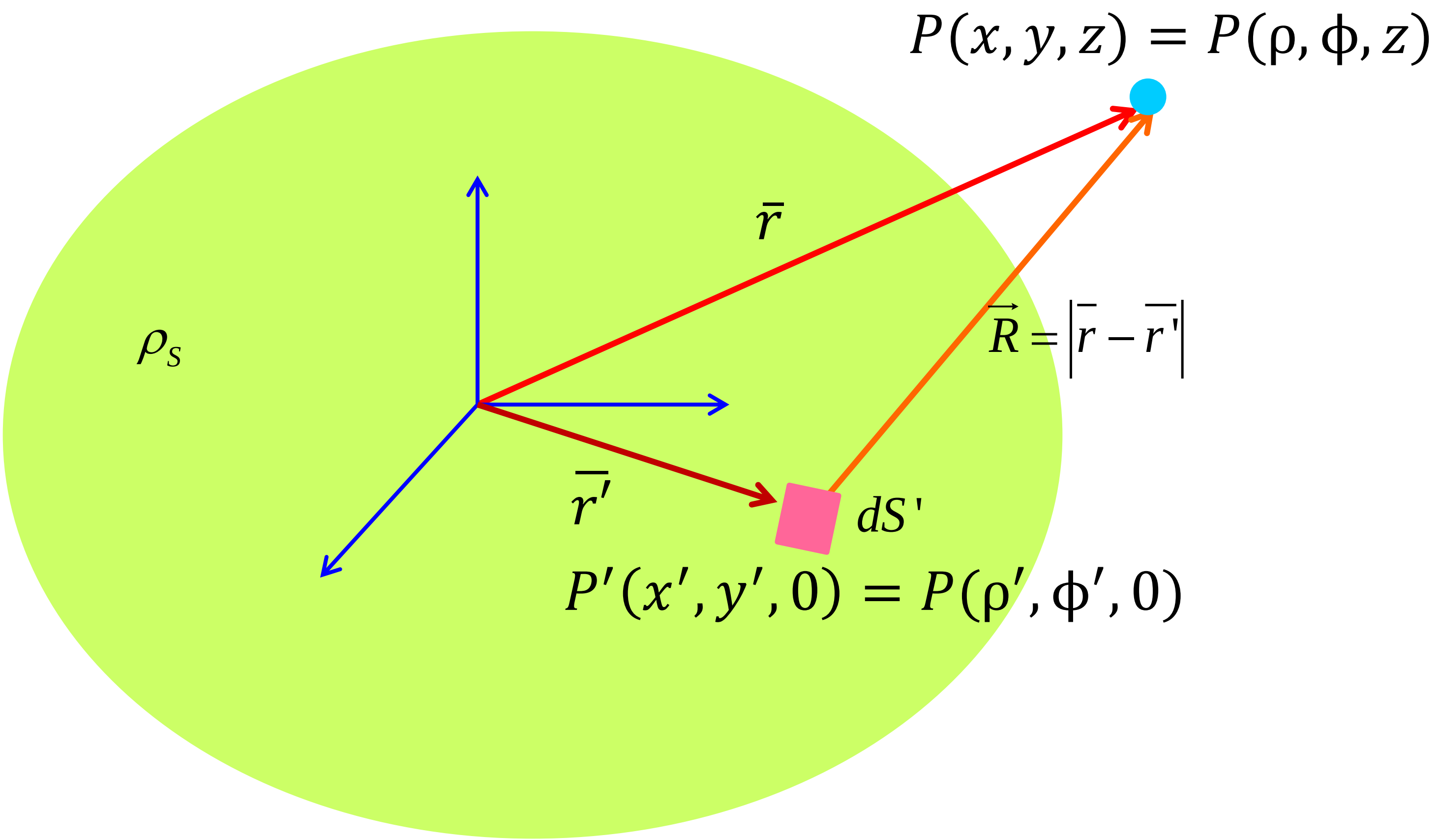
$$\overline{dl} = -dz\hat{a}_z$$

- So: $\oint_C \vec{B}.\overline{dl} = \int_0^3 \frac{1}{4}\hat{a}_z.(-dz\hat{a}_z)$ ⇒ $\oint_C \vec{B}.\overline{dl} = -\frac{3}{4}$

Hence, Stoke's Theorem Verified

# Electric Field due to a Surface Charge

- Consider a **disk of radius a**, centered at the origin, and lying entirely on the xy-plane (i.e., z =0 plane). Let us also assume that this disk carries a uniform charge density of $\rho_S$ C/m$^2$.




**Challenge:** determine electric field at point P

# Electric Field due to a Surface Charge (contd.)

- From Coulomb's Law:

$$\vec{E}(\bar{r}) = \iint_S \frac{\rho_S(r')}{4\pi\varepsilon_0} \frac{\bar{r}-\bar{r}'}{\left|\bar{r}-\bar{r}'\right|^3} dS'$$

$$dS' = \rho' d\rho' d\phi' \qquad 0 < \rho' < a \qquad 0 < \phi' < 2\pi$$

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \qquad \bar{r}' = x'\hat{a}_x + y'\hat{a}_y$$

$$\therefore \vec{R} = \bar{r} - \bar{r}' = (x - x')\hat{a}_x + (y - y')\hat{a}_y + z\hat{a}_z$$

Convert to cylindrical

$$\therefore \vec{R} = \bar{r} - \bar{r}' = (x - \rho'\cos\phi')\hat{a}_x + (y - \rho'\sin\phi')\hat{a}_y + z\hat{a}_z$$

$$\left|\vec{R}\right|^3 = \left|\bar{r} - \bar{r}'\right|^3 = \left[(x - \rho'\cos\phi')^2 + (y - \rho'\sin\phi')^2 + z^2\right]^{3/2}$$

# Electric Field due to a Surface Charge (contd.)

$$\vec{E}(\bar{r}) = \iint_S \frac{\rho_S(\bar{r}')}{4\pi\varepsilon_0} \frac{\bar{r}-\bar{r}'}{\left|\bar{r}-\bar{r}'\right|^3} dS'$$

$$= \frac{\rho_S}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{(x-\rho'\cos\phi')\hat{a}_x + (y-\rho'\sin\phi')\hat{a}_y + z\hat{a}_z}{\left[(x-\rho'\cos\phi')^2 + (y-\rho'\sin\phi')^2 + z^2\right]^{3/2}} \rho' d\rho' d\phi'$$

Yikes! What a **mess**!

- To **simplify** our integration let's determine the electric field $\vec{E}(\bar{r})$ along the **z-axis** only. In other words, set x = 0 and y = 0.

$$\Rightarrow \vec{E}(x=0, y=0, z) = \frac{\rho_S}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{(0-\rho'\cos\phi')\hat{a}_x + (0-\rho'\sin\phi')\hat{a}_y + z\hat{a}_z}{\left[(0-\rho'\cos\phi')^2 + (0-\rho'\sin\phi')^2 + z^2\right]^{3/2}} \rho' d\rho' d\phi'$$

# Electric Field due to a Surface Charge (contd.)

$$\Rightarrow \vec{E}(\bar{r}) = \frac{\rho_S}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(\rho'\cos\phi')\hat{a}_x + (\rho'\sin\phi')\hat{a}_y + z\hat{a}_z}{\left[(\rho'\cos\phi')^2 + (\rho'\sin\phi')^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi'$$

$$= \frac{\rho_S}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(\rho'\cos\phi')\hat{a}_x + (\rho'\sin\phi')\hat{a}_y + z\hat{a}_z}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi'$$

$$= \frac{\rho_S}{4\pi\varepsilon_0}\left[\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(\rho'\cos\phi')\hat{a}_x}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi' + \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(\rho'\sin\phi')\hat{a}_y}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi' + \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{z\hat{a}_z}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi'\right]$$

=0

**We know:**

$$\int_0^{2\pi}\sin\phi d\phi = 0 = \int_0^{2\pi}\cos\phi d\phi$$

# Electric Field due to a Surface Charge (contd.)

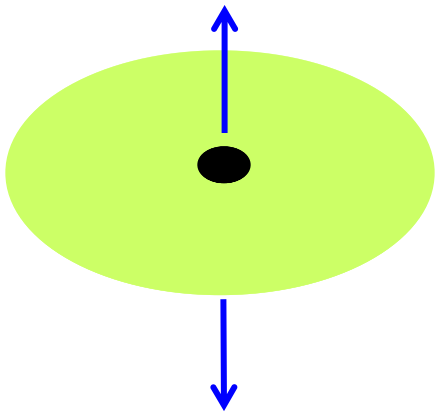
$$\vec{E}(x=0,y=0,z)=\frac{\rho_S}{4\pi\varepsilon_0}\left[\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{z\hat{a}_z}{\left[\rho'^2+z^2\right]^{3/2}}\rho' d\rho' d\phi'\right]$$

$$\vec{E}(x=0,y=0,z)=\begin{cases}\dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z>0\\[2ex] \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[-1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z<0\end{cases}$$

From this expression, we can conclude **two** things. The first is that **above** the disk (z > 0), the electric field points in the direction $\hat{a}_z$, and below the disk (z < 0), it points in the direction $-\hat{a}_z$.

# Electric Field due to a Surface Charge (contd.)

- What a surprise (not)! The electric field **points away** from the charge. It appears to be **diverging** from the charged disk



- Likewise, it is evident that as we move further and **further from** the disk, the electric field will **diminish**. In fact, as distance z goes to **infinity**, the magnitude of the electric field approaches **zero**. This of course is similar to the **point** or **line** charge; as we move an infinite distance away, the electric field diminishes to **nothing**.

# Electric Field due to a Surface Charge (contd.)

- Say that we have a **very large** charge disk. So large, in fact, that its radius $a$ approaches **infinity** !

**Q:** What electric field is created by this infinite plane?

**A:** We **already** know! Just evaluate the charge disk solution for the case where the disk **radius** $a$ is **infinity**.

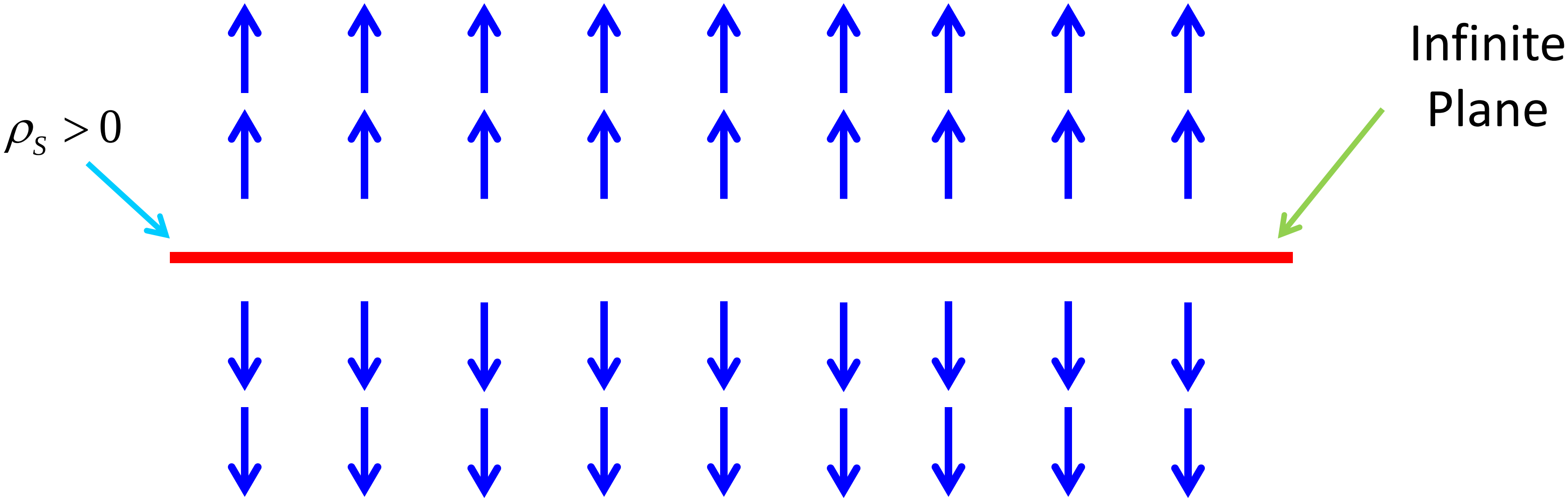
$$\lim_{a\to\infty} \vec{E}(x=0, y=0, z) = \begin{cases} \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z>0 \\[2ex] \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[-1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z<0 \end{cases}$$

$$= \begin{cases} \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z & \text{If } z>0 \\[2ex] \dfrac{-\rho_S}{2\varepsilon_0}\hat{a}_z & \text{If } z<0 \end{cases}$$

**Think** about what **this** says!

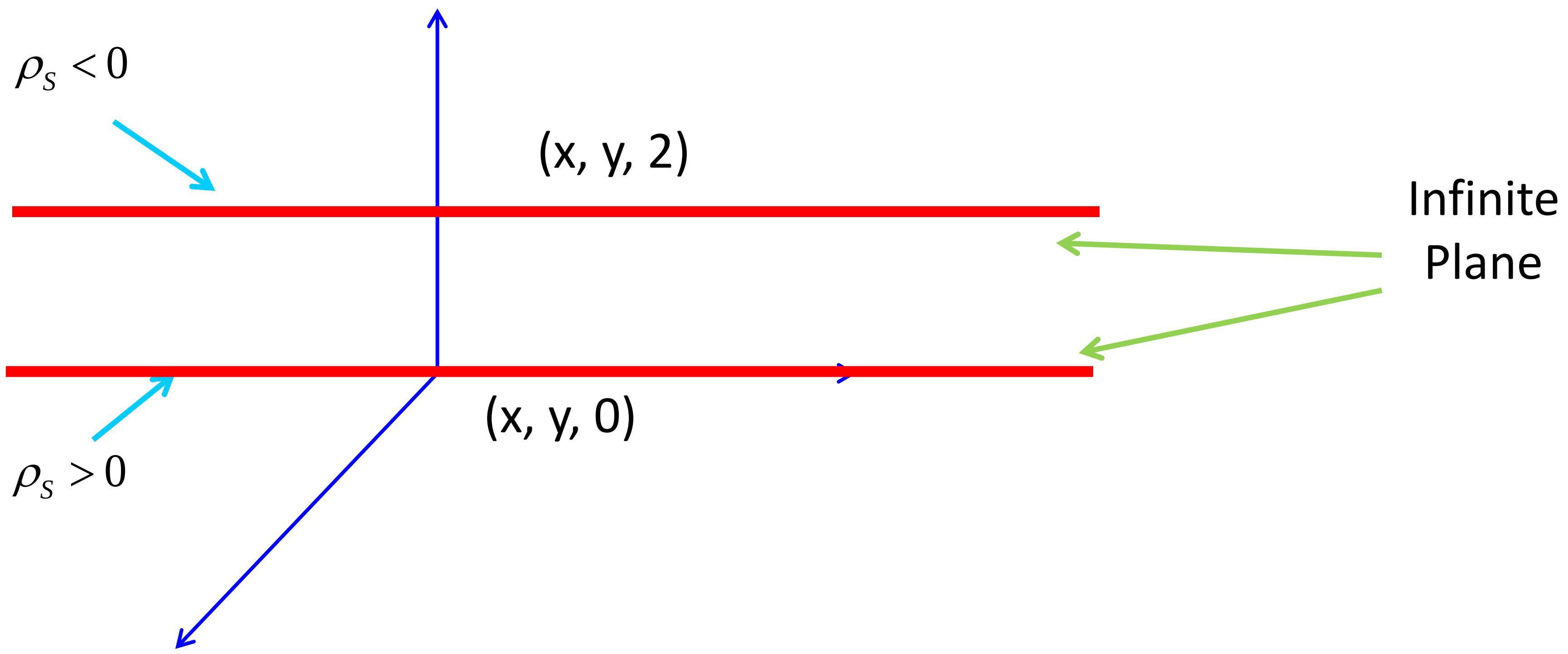
# Electric Field due to a Surface Charge (contd.)

- First, we note that the electric field **points away** from the plane if $\boldsymbol{\rho_S}$ is positive, and toward the plane if $\boldsymbol{\rho_S}$ is negative.
- Second, we notice that the magnitude of the electric field is a **constant**—the magnitude is **independent** of the distance from the infinite plane!



# Example – 5

- An infinite sheet with uniform surface charge density **$\rho_s$ is located at z=0 (x-y plane)**, and another infinite sheet with **$-\rho_s$ is located at z=2m**, both in free space. Determine $\vec{E}$ everywhere.



# Example – 5 (contd.)

**For the sheet at z =0**

$$\vec{E}_1 = \begin{cases} \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for} \quad z > 0 \\ \dfrac{-\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for} \quad z < 0 \end{cases}$$

**For the sheet at z =2m**

$$\vec{E}_2 = \begin{cases} \dfrac{-\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for} \quad z > 2\text{m} \\ \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for} \quad z < 2\text{m} \end{cases}$$

**Therefore:**

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \begin{cases} 0 & \text{for} \quad z < 0 \\ \dfrac{\rho_S}{\varepsilon_0}\hat{a}_z & \text{for} \quad 0 < z < 2\text{m} \\ 0 & \text{for} \quad z > 2\text{m} \end{cases}$$

**Any thought on this outcome !**

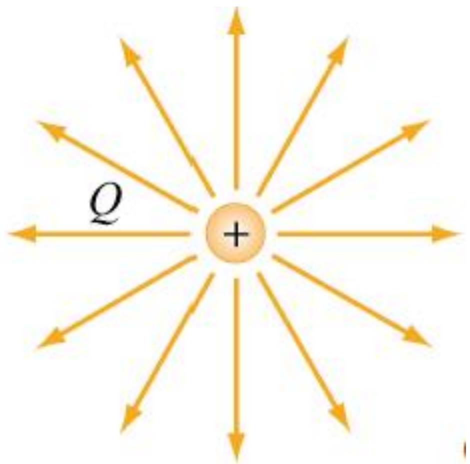
# Electric Field of a Charged Sphere

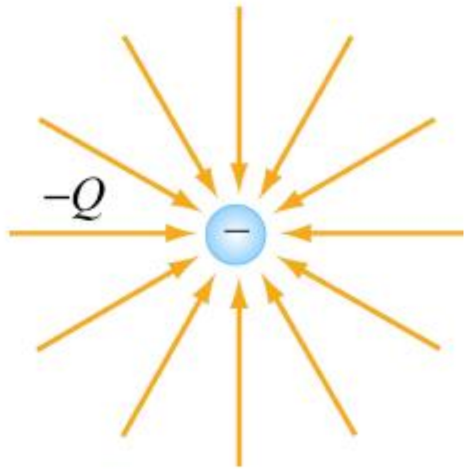
Involves tricky triple integration. Lets first learn Gauss Theorem. It will simplify this problem.

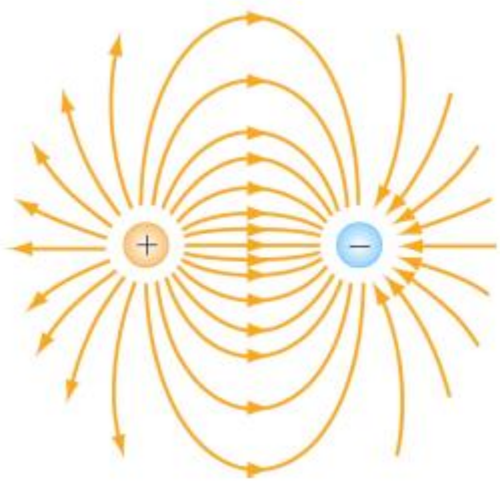
# Electric Field Lines

- Electric *field lines* are a pictorial representation of the electric field. These consist of directed lines indicating the direction of electric field at various points in space.
- There is no rule as to how many lines are to be shown. However, it is customary to draw number of lines proportional to the charge. Thus if N number of lines are drawn from or into a charge Q, 2N number of lines would be drawn for charge 2Q.
- Lines are dense close to a source of the electric field and become sparse as one moves away.
- Lines originate from a positive charge and end either on a negative charge or move to infinity.
- Lines of force due to a solitary negative charge is assumed to start at infinity and end at the negative charge.
- Field lines do not cross each other. ( if they did, the field at the point of crossing will not be uniquely defined.)

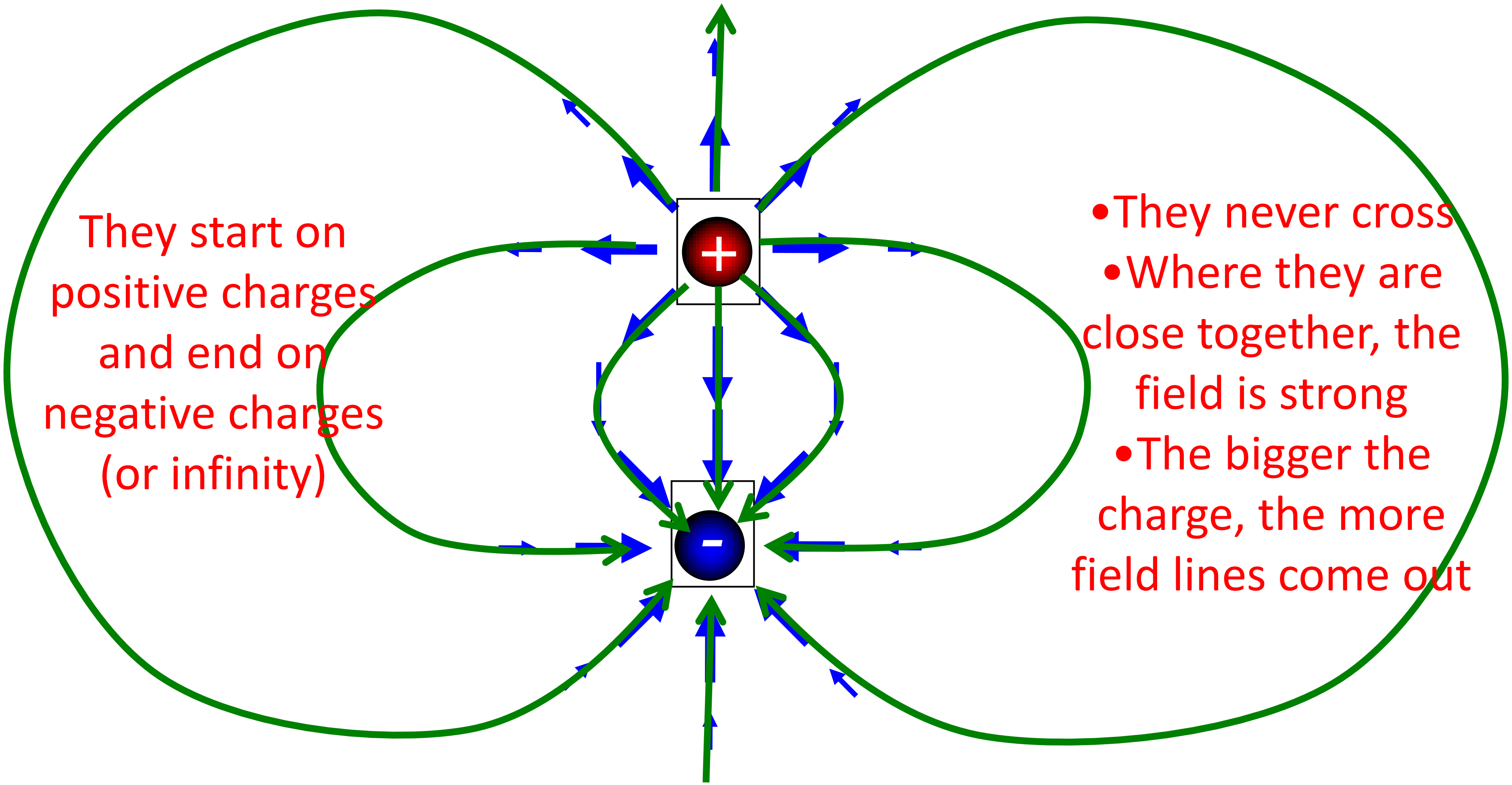
# Electric Field Lines (contd.)



# Electric Field Lines (contd.)



They start on positive charges and end on negative charges (or infinity)

- They never cross
- Where they are close together, the field is strong
- The bigger the charge, the more field lines come out

# Electric Field Lines (contd.)

**Note:**

- **Near field:** very close to a charge, the field due to that charge predominates. Therefore, the lines are radial and spherically symmetric.

- **Far field:** far from the system of charges, the pattern should look like that of a single point charge of value $Q = \sum_i Q_i$. Thus, the lines should be radially inward or outward, unless $Q = 0$.

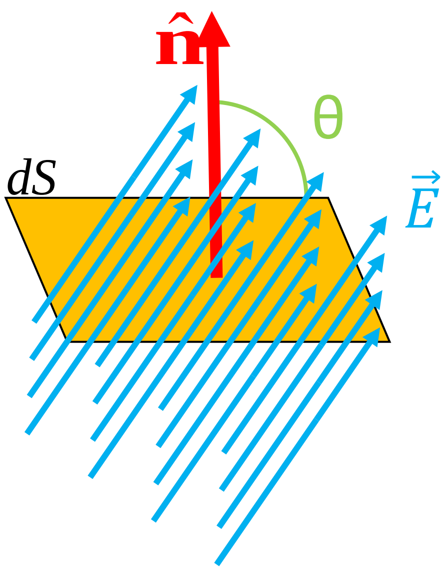
- The direction of the electric field vector $\overrightarrow{E}$ at a point is always tangent to the field lines.

# Electric Flux

- The concept of ***flux*** is borrowed from flow of water through a surface.
  - The amount of water flowing through a surface **depends** on the velocity of water, the area of the surface and the orientation of the surface with respect to the direction of velocity of water.
  - **Similarly**, the electric flux through a surface **depends** on the electric field, the area of the surface and the orientation of the surface to the direction of electric field lines.
- Though an area is generally considered as a scalar, an element of area may be considered to be a vector because:
  - It has magnitude (measured in $m^2$).
  - **If the area is infinitesimally small**, it can be considered to be in a plane. We can then associate a direction with it. **For instance**, if the area element lies in the x-y plane, it can be considered to be directed along the z–direction. **(Conventionally, the direction of the area is taken to be along the outward normal.)**

# Electric Flux (contd.)

- Simply speaking, electric flux is the amount of electric field going through a surface. It is defined in terms of a direction, unit vector, perpendicular to the surface.




$$d\psi = \vec{E}.\hat{n}dS$$

$$\therefore d\psi = |\vec{E}| dS \cos\theta$$

- For an arbitrary surface S, the flux is obtained by integrating over all the surface elements.

$$\psi = \int_S d\psi = \int_S \vec{E}.\hat{n}dS$$

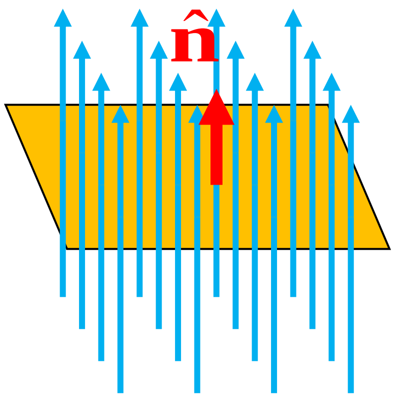
- If the electric field is uniform, the angle θ is constant and we have:

$$\psi = E(S\cos\theta)$$

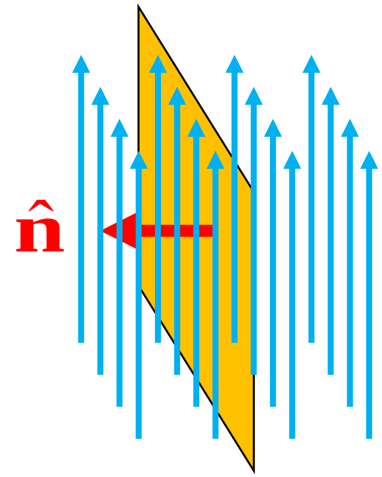
Thus the flux is equal to the product of magnitude of the electric field and the projection of area perpendicular to the field.

# Electric Flux (contd.)

- When the surface is flat, and the fields are constant, you can just use multiplication to get the flux.




$$\psi = S\left|\vec{E}\right|$$



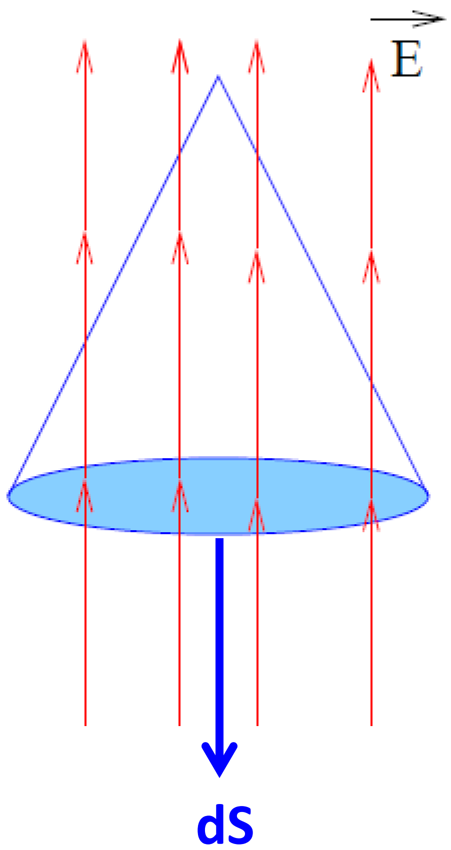

$$\psi = 0$$

- When the surface is curved, or the fields are not constant, you have to perform an integration:

$$\psi = \int \vec{E} \cdot \hat{\mathbf{n}}\, dS$$

# Example – 6

Calculate the flux through the base of the cone of radius R shown below.




$$S = \pi R^2$$

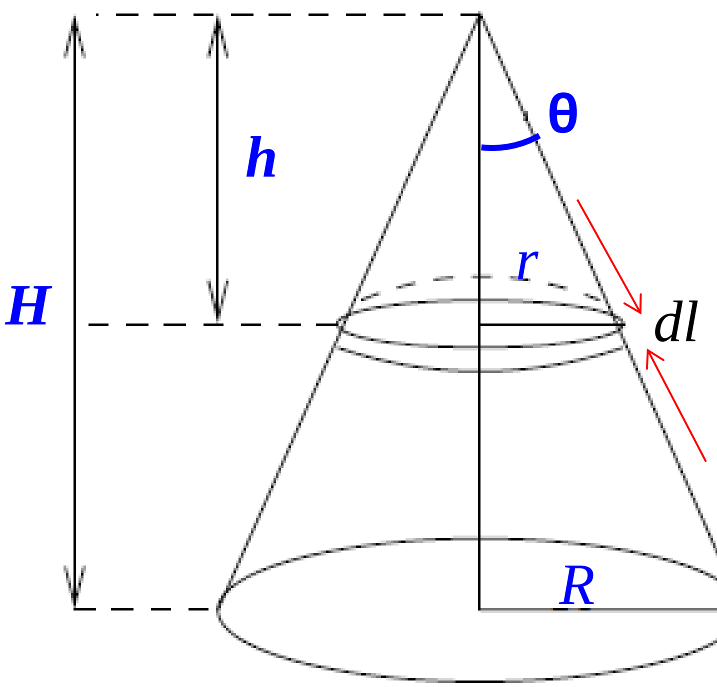
$$\theta = 180^o$$

$$\psi = E\left(S\cos\theta\right) = E\left(\pi R^2 \cos 180^o\right)$$

$$\therefore \psi = E\left(\pi R^2 \times (-1)\right) = -\pi R^2 E$$

# Example – 7

- Calculate the flux coming out through the curved surface of the cone in the example – 6.




- Let us consider a circular strip of radius $r$ at a depth $h$ from the apex of the cone. The angle between the electric field through the strip and the vector $\vec{dS}$ is (π/2)-θ, where θ is the semi-angle of the cone.
- If $dl$ is the length element along the slope, the area of the strip is $2\pi r dl$.
- **Then:**

$$\vec{E}.\vec{dS} = 2\pi r dl \left|\vec{E}\right| \cos\left(\frac{\pi}{2} - \theta\right) = 2\pi r dl \left|\vec{E}\right| \sin\theta$$

- We have, $dl = dh/\cos\theta$. Further, $r = h\tan\theta$. Substituting, we get:

$$\vec{E}.\vec{dS} = 2\pi h \tan^2\theta \left|\vec{E}\right| dh \qquad \therefore \psi = \int_{h=0}^{h=H} 2\pi h \tan^2\theta \left|\vec{E}\right| dh = E\pi H^2 \tan^2\theta = \pi R^2 E$$

# Gauss Law

- In practice, electric field intensity is dependent on the medium in which the charge is placed (free space in our discussion).
- Let us define a new vector $\vec{D}$ that relates the medium and the electric field as:

$$\vec{D} = \varepsilon_0 \vec{E}$$

Electric Flux Density

- According to Gauss **[full name: Carl Friedrich Gauss]**, the total electric flux ψ **through any closed surface** is equal to the total charge enclosed by that surface.

$$\psi = Q_{enc}$$

$$\psi = \oint_S d\psi = \oint_S \vec{D}.\vec{dS}$$

$$Q_{enc} = \int_v \rho_v dv$$

$$\Rightarrow \oint_S \vec{D}.\vec{dS} = \int_v \rho_v dv$$

- From Divergence theorem:

$$\Rightarrow \oint_S \vec{D}.\vec{dS} = \int_v \nabla.\vec{D} dv$$

# Gauss Law (contd.)

- Therefore: $\rho_v = \nabla . \vec{D}$ ← First of the four Maxwell's equations

**Gauss's Law can be used to solve three types of problems:**

1. Finding the total charge in a region when you know the electric field outside that region
2. Finding the total flux out of a region when the charge is known
   - It can also be used to find the flux out of one side in symmetrical problems ↔ In such cases, you must first argue from symmetry that the flux is identical through each side
3. Finding the electrical field in highly symmetrical situations
   - One must first use reason to find the direction of the electric field everywhere
   - Then draw a Gaussian surface over which the electric field is constant
   - Use this surface to find the electric field using Gauss's Law
   - Works generally only for spherical, cylindrical, or planar-type problems

# Gauss Law (contd.)

1. A continuous charge distribution has rectangular symmetry if it depends only on $x\ (or\ y\ or\ z)$, cylindrical symmetry if it depends only on $\rho$, and spherical symmetry if it depends only on $r\ (independent\ of\ \theta\ and\ \varphi)$.
2. Gauss's Law is also valid for asymmetric charge distribution. However, you can't apply Gauss's Law to determine $\vec{E}$ or $\vec{D}$. In such situations, apply Coulomb's Law.
3. Gaussian surface is chosen **such that $\vec{D}$ is normal or tangential to the surface.** **When $\vec{D}$ is normal to the surface then $\vec{D}.\overline{ds} = \vec{D}$ and for tangential $\vec{D}$ we get $\vec{D}.\overline{ds} = 0$.**

# Applications of Gauss's Law

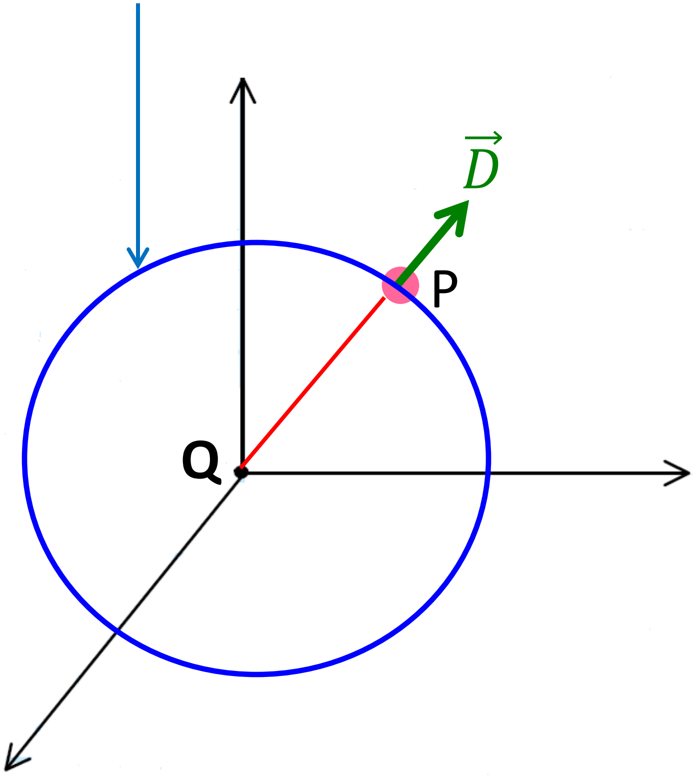
## Point Charge

- Suppose a point charge is located at origin.
- Determine $\vec{D}$ at a point P.

- Choose a spherical surface containing P.
- $\vec{D}$ is everywhere normal to the Gaussian surface.

$$Q = \oint_S \vec{D}.\vec{dS} = \oint_S DdS\cos 0^o = \oint_S DdS = D\oint_S dS = D \times 4\pi r^2$$

$$\therefore D = \frac{Q}{4\pi r^2}$$



# Applications of Gauss's Law (contd.)

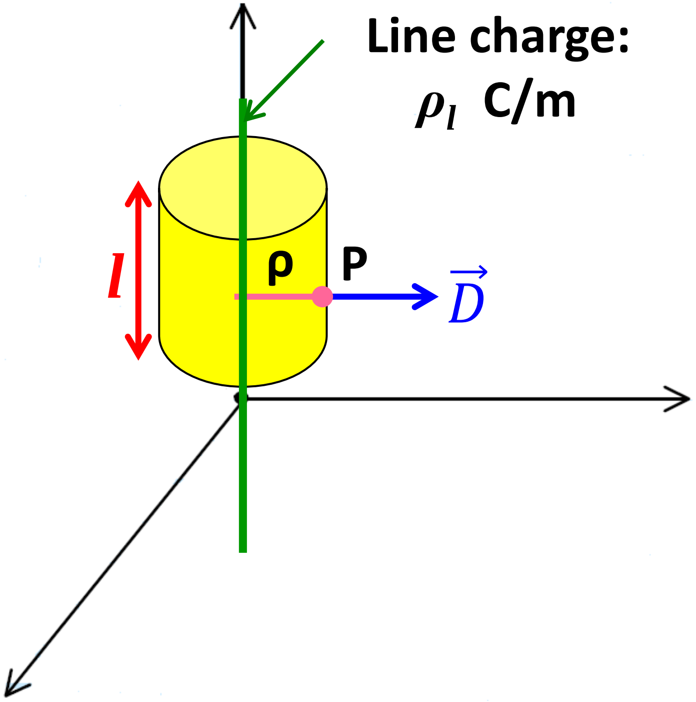
## Infinite Line Charge

- To determine $\vec{D}$ at a point P, lets choose a cylindrical surface of arbitrary length ***l***.
- $\vec{D}$ is normal to side surface, doesn't exist on the top and bottom surface (because there is no z-component of $\vec{D}$).
- **Therefore:**

$$Q = \rho_l l = \oint_S \vec{D}.\vec{dS} = D_\rho \oint_S dS = D_\rho \times 2\pi\rho l$$

$$D_\rho = \frac{\rho_l}{2\pi\rho}$$

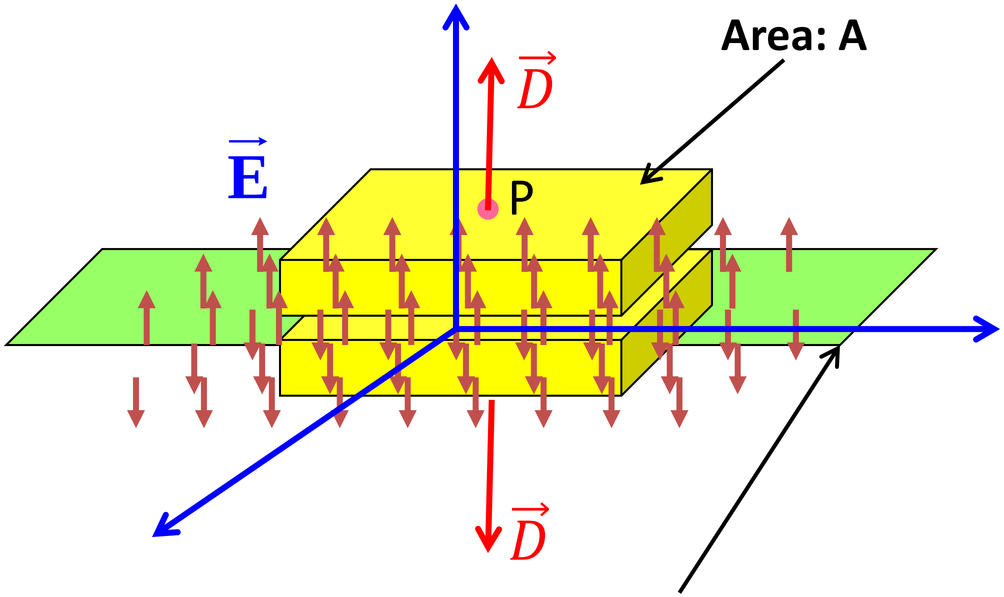
$$\therefore D = \frac{\rho_l}{2\pi\rho}\hat{a}_\rho$$



# Applications of Gauss's Law (contd.)

## Infinite Sheet Charge

- To determine $\vec{D}$ at a point P, lets choose a rectangular box with top and bottom area A
- $\vec{D}$ is normal to the top and bottom, doesn't exist on the side surface




- **Therefore:**

$$Q = \rho_S \int dS = \oint_S \vec{D}.\vec{dS} = D_z \left[ \int_{top} dS + \int_{bottom} dS \right]$$

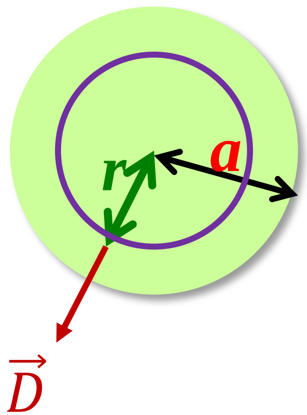
$$\Rightarrow \rho_S A = D_z [A + A] \quad \Rightarrow D_z = \frac{\rho_S}{2}$$

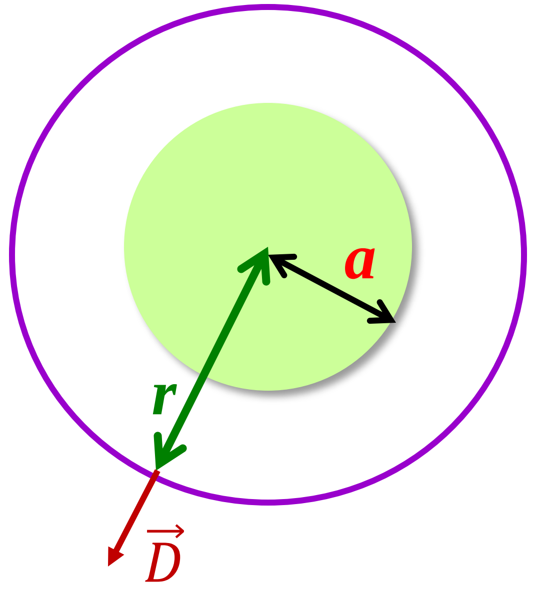
$$\therefore \vec{D} = \frac{\rho_S}{2} \hat{a}_z \Rightarrow \therefore \vec{E} = \frac{\rho_S}{2\varepsilon_0} \hat{a}_z$$

# Applications of Gauss's Law (contd.)

## Uniformly Charged Sphere

- A sphere of radius $a$ has uniform charge density $\rho_v$ **throughout**. What is the direction and magnitude of the electric field everywhere?
  - To determine $\vec{D}$ **everywhere**, **let us construct Gaussian surfaces for cases $\mathbf{r} \leq \boldsymbol{a}$** and $\boldsymbol{r} \geq \boldsymbol{a}$ separately.



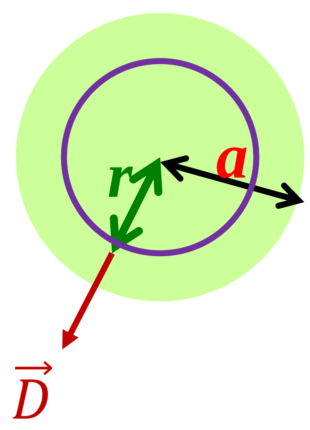




- Clearly, all directions are created equal in this problem
- Certainly the electric field will point away from the sphere at all points
- The electric field must depend *only* on the distance

# Applications of Gauss's Law (contd.)

## Uniformly Charged Sphere

**Case-I: $r \le a$**




- When computing the flux for a Gaussian surface, only include the electric charges *inside* the surface. Here, the enclosed charge is:

$$Q_{enc} = \int_v \rho_v dv = \rho_v \int_v dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta dr d\theta d\phi$$

$$\therefore Q_{enc} = \rho_v \frac{4}{3}\pi r^3$$

- **The total flux:**

$$\psi = \oint_S \vec{D}.\vec{dS} = D_r \oint_S dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\phi$$

$$\therefore \psi = D_r 4\pi r^2$$

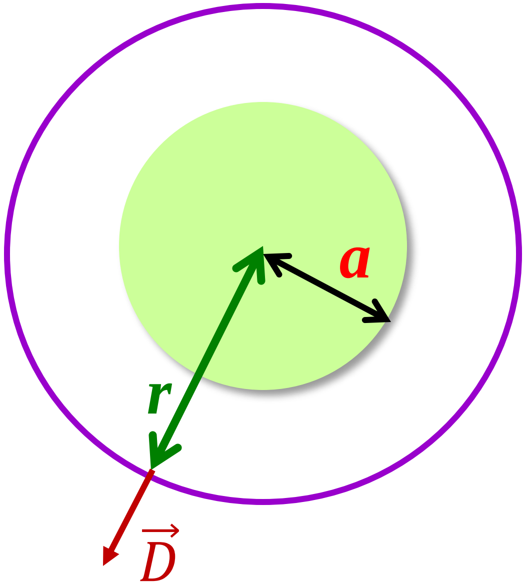
- **From Gauss's Law:** $\psi = Q_{enc}$ $\Rightarrow D_r 4\pi r^2 = \rho_v \frac{4}{3}\pi r^3$ $\Rightarrow D_r = \frac{\rho_v}{3} r$

$$\therefore \vec{D} = \frac{r}{3}\rho_v \hat{a}_r$$

# Applications of Gauss's Law (contd.)

## Uniformly Charged Sphere

### Case-II: $r \geq a$




- The charge enclosed in this case is the entire charge:

$$Q_{enc} = \int_v \rho_v dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta dr d\theta d\phi$$

$$\therefore Q_{enc} = \rho_v \frac{4}{3}\pi a^3$$

- **While:**

$$\psi = \oint_S \vec{D}.\vec{dS} = D_r \oint_S dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\phi$$

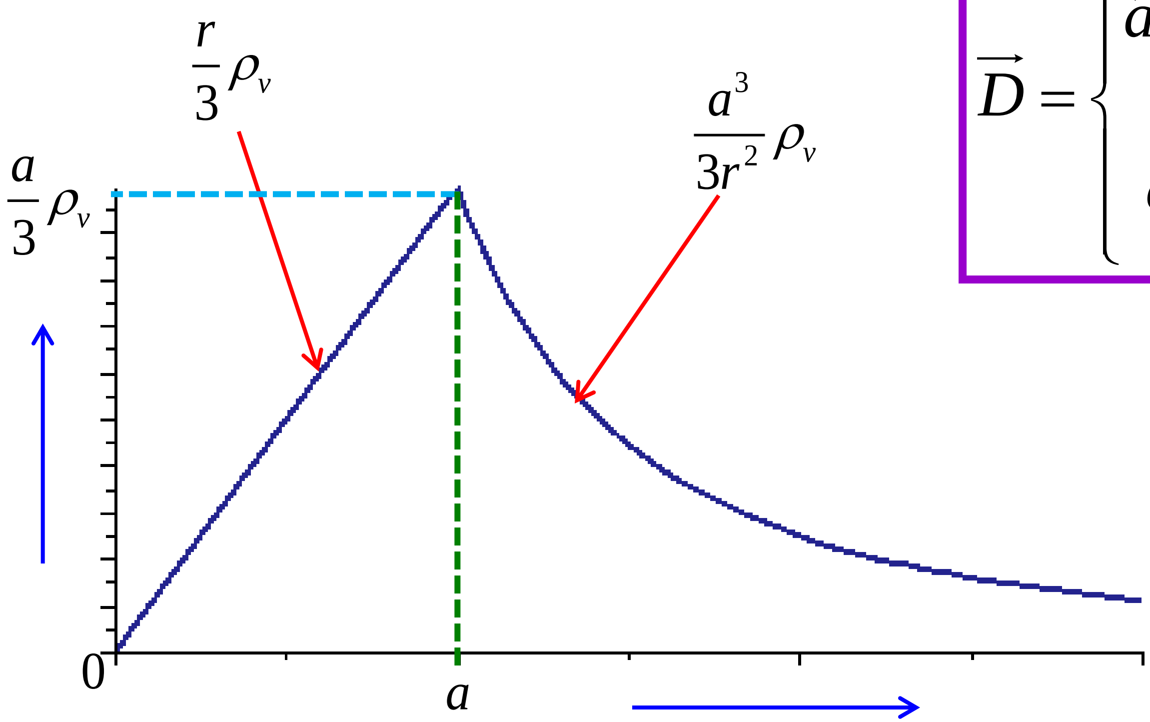
$$\therefore \psi = D_r 4\pi r^2$$

- **From Gauss's Law:** $\psi = Q_{enc}$ $\Rightarrow D_r 4\pi r^2 = \rho_v \frac{4}{3}\pi a^3$ $\Rightarrow D_r = \frac{a^3}{3r^2}\rho_v$

$$\therefore \vec{D} = \frac{a^3}{3r^2}\rho_v \hat{a}_r$$

# Applications of Gauss's Law (contd.)

## Uniformly Charged Sphere




$$\vec{D} = \begin{cases} \hat{a}_r \rho_v \dfrac{a^3}{3r^2} & \text{for } r > a, \\ \hat{a}_r \dfrac{r}{3} \rho_v & \text{for } r < a. \end{cases}$$