Lecture – 2

Date: 08.01.2015

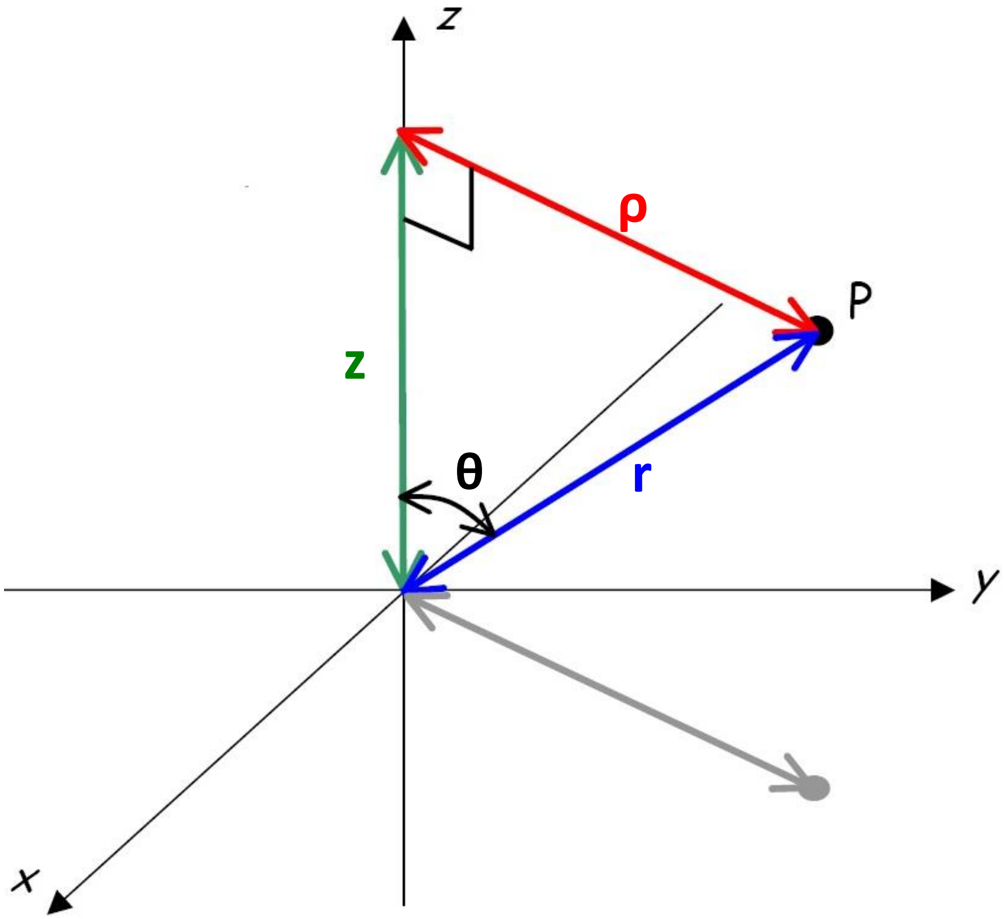
- Coordinate Transformations
- Base Vectors
- Position Vector
- Contours (Cartesian, Cylindrical, and Spherical)
- Surfaces (Cartesian, Cylindrical, and Spherical)
- Volume

Coordinate Transformations

- Say we **know** the location of a point, or the description of some scalar field in terms of **Cartesian** coordinates (e.g., T (x, y, z)).
- What if we decide to express this point or this scalar field in terms of **cylindrical** or **spherical** coordinates **instead**?
- We see that the coordinate values ***z, ρ, r,*** and θ are all variables of a **right triangle**! We can use our knowledge of trigonometry to relate them to each other.
- In fact, we can **completely derive** the relationship between **all six** independent coordinate values by considering just **two very important right triangles**!
  - Hint: **Memorize these 2 triangles!!!**

Coordinate Transformations (contd.)

Right Triangle #1



$$z = r \times \cos\theta = \rho \times \cot\theta = \sqrt{r^2 - \rho^2}$$

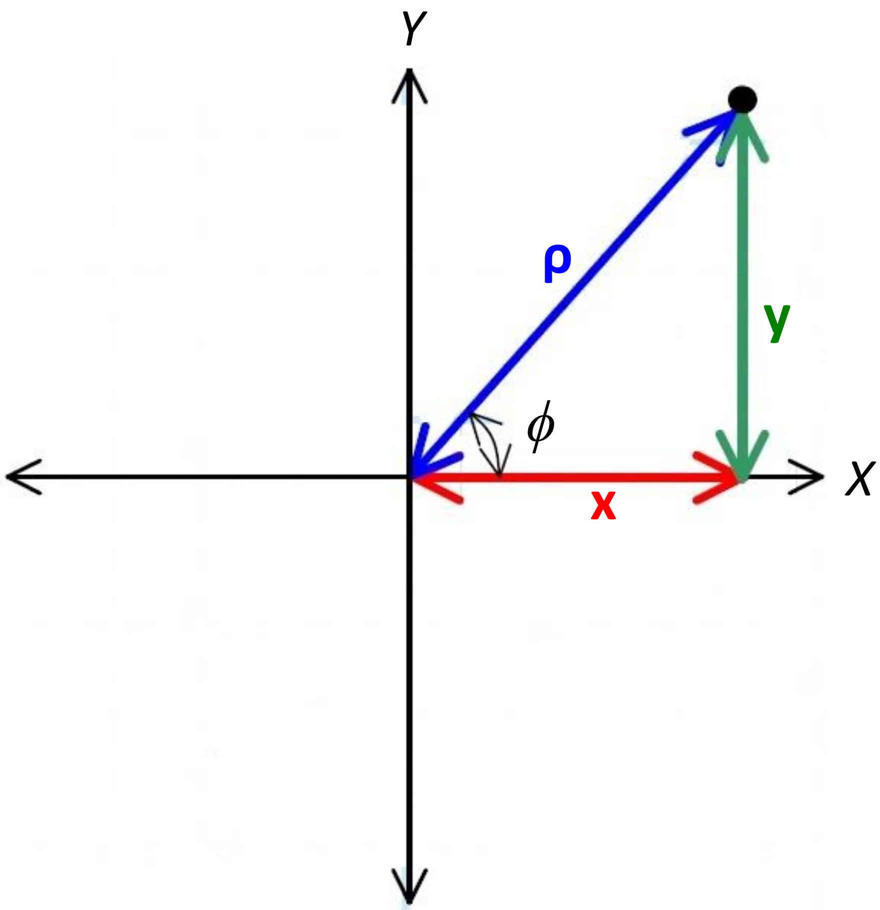
$$\rho = r \times \sin\theta = z \times \tan\theta = \sqrt{r^2 - z^2}$$

$$r = \sqrt{\rho^2 + z^2} = \rho \times \operatorname{cosec}\theta = z \times \sec\theta$$

$$\theta = \tan^{-1}\left[\frac{\rho}{z}\right] = \sin^{-1}\left[\frac{\rho}{r}\right] = \cos^{-1}\left[\frac{z}{r}\right]$$

Coordinate Transformations (contd.)

Right Triangle #2



$$x = \rho \times \cos\phi = y \times \cot\phi = \sqrt{\rho^2 - y^2}$$

$$y = \rho \times \sin\phi = x \times \tan\phi = \sqrt{\rho^2 - x^2}$$

$$\rho = \sqrt{x^2 + y^2} = x \times \sec\phi = y \times \operatorname{cosec}\phi$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right] = \sin^{-1}\left[\frac{y}{\rho}\right] = \cos^{-1}\left[\frac{x}{\rho}\right]$$

Coordinate Transformations (contd.)

Combining the results of the two triangles allows us to write each coordinate set in terms of each other

- **Cartesian and Cylindrical**

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right]$$

$$z = z$$

⇕

$$x = \rho \times \cos\phi$$

$$y = \rho \times \sin\phi$$

$$z = z$$

- **Cartesian and Spherical**

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right]$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right]$$

⇕

$$x = r \times \sin\theta \times \cos\phi$$

$$y = r \times \sin\theta \times \sin\phi$$

$$z = r \times \cos\theta$$

Coordinate Transformations

- **Cylindrical and Spherical**

$$\rho = r \times \sin\theta$$
$$\phi = \phi$$
$$z = r \times \cos\theta$$

⟺

$$r = \sqrt{\rho^2 + z^2}$$
$$\theta = \tan^{-1}\left[\frac{\rho}{z}\right]$$
$$\phi = \phi$$

Base Vectors

Q: We know that **vector** quantities (either discrete or field) have **both magnitude** and **direction**. But how do we **specify** direction in 3-D space? Do we use **coordinate** values (e.g., x, y, z)??

A: It is very important that you understand that **coordinates only** allow us to specify **position** in 3-D space. They **cannot** be used to specify **direction**!

The most convenient way for us to specify the direction of a vector quantity is by using a well-defined **orthonormal set** of vectors known as **base vectors**.

Base Vectors (contd.)

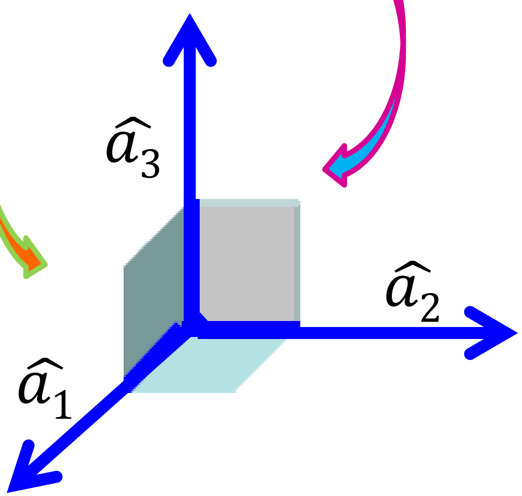
- Recall that an orthonormal set of unity vectors, say $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ have the following properties:

- Each vector is a **unit** vector: $\hat{a}_1.\hat{a}_1 = \hat{a}_2.\hat{a}_2 = \hat{a}_3.\hat{a}_3 = 1$

- Each vector is mutually orthogonal: $\hat{a}_1.\hat{a}_2 = \hat{a}_2.\hat{a}_3 = \hat{a}_3.\hat{a}_1 = 0$

- **Additionally,** a set of base vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ must be arranged such that:

$$\hat{a}_1 \times \hat{a}_2 = \hat{a}_3 \qquad \hat{a}_2 \times \hat{a}_3 = \hat{a}_1 \qquad \hat{a}_3 \times \hat{a}_1 = \hat{a}_2$$

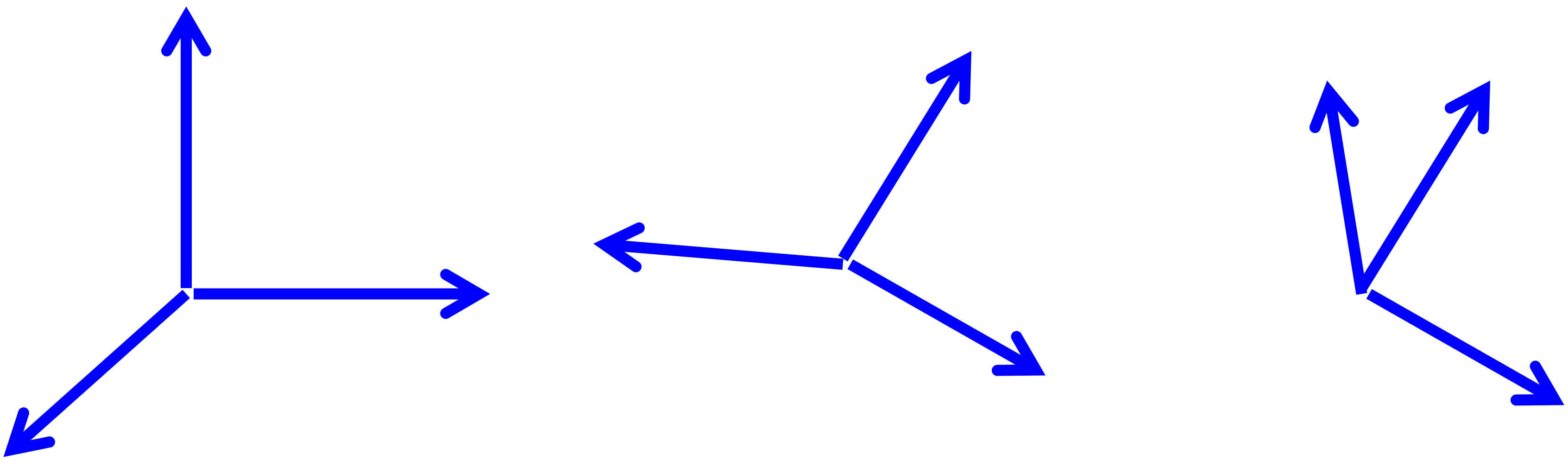


An orthonormal set with this property is known as a **right handed** system.

All base vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ must form a **right-handed, orthonormal** set.

Base Vectors (contd.)

Recall that we use **unit vectors** to define **direction**. Thus, a set of base vectors define three distinct directions in our 3-D space!



Q: But, **what three** directions do we use?? I remember, there are an **infinite** number of possible **orientations** of an orthonormal set!!

A: We will define several systematic, mathematically **precise methods** for defining the orientation of base vectors. Generally speaking, we will find that the orientation of these base vectors will **not be fixed**, but will in fact vary with **position** in space (i.e., as a function of coordinate values)!

Base Vectors (contd.)

- Essentially, we will define at **each** and every point in space a **different** set of base vectors, which can be used to uniquely define the direction of any vector quantity **at that point**!

Q: Good golly! Defining a **different** set of base vectors for **every** point in space just seems **confusing.** Why can't we just **fix** a set of base vectors such that their orientation is the **same** at **all** points in space?

A: We will in fact study **one** method for defining base vectors that **does** in fact result in an orthonormal set whose orientation is **fixed**—the same at **all** points in space **(Cartesian base vectors).**

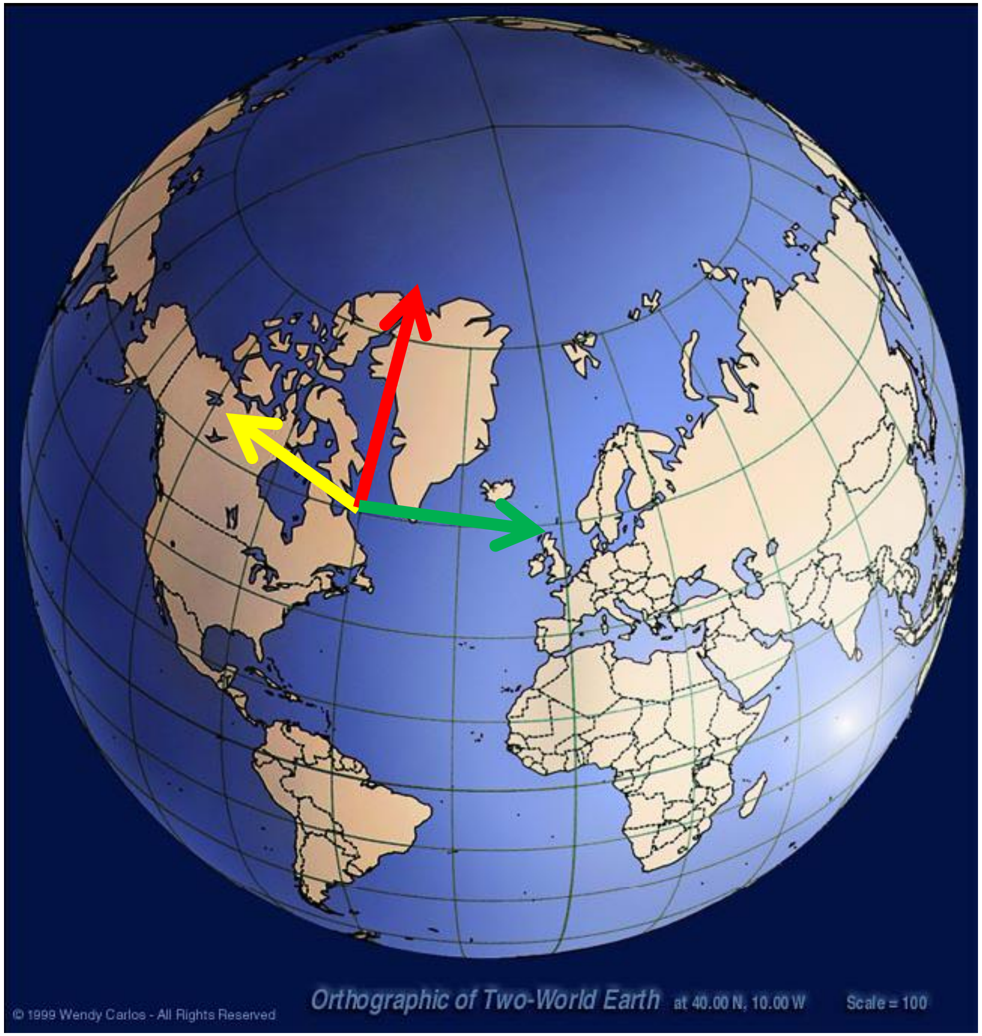
However, we will study **two other** methods where the orientation of base vectors is **different** at all points in space (spherical and cylindrical base vectors). We use these two methods to define base vectors because for **many** physical problems, it is actually **easier** and **wiser** to do so!

Base Vectors (contd.)

For example, consider how we define direction on **Earth**: North/South, East/West, Up/Down.

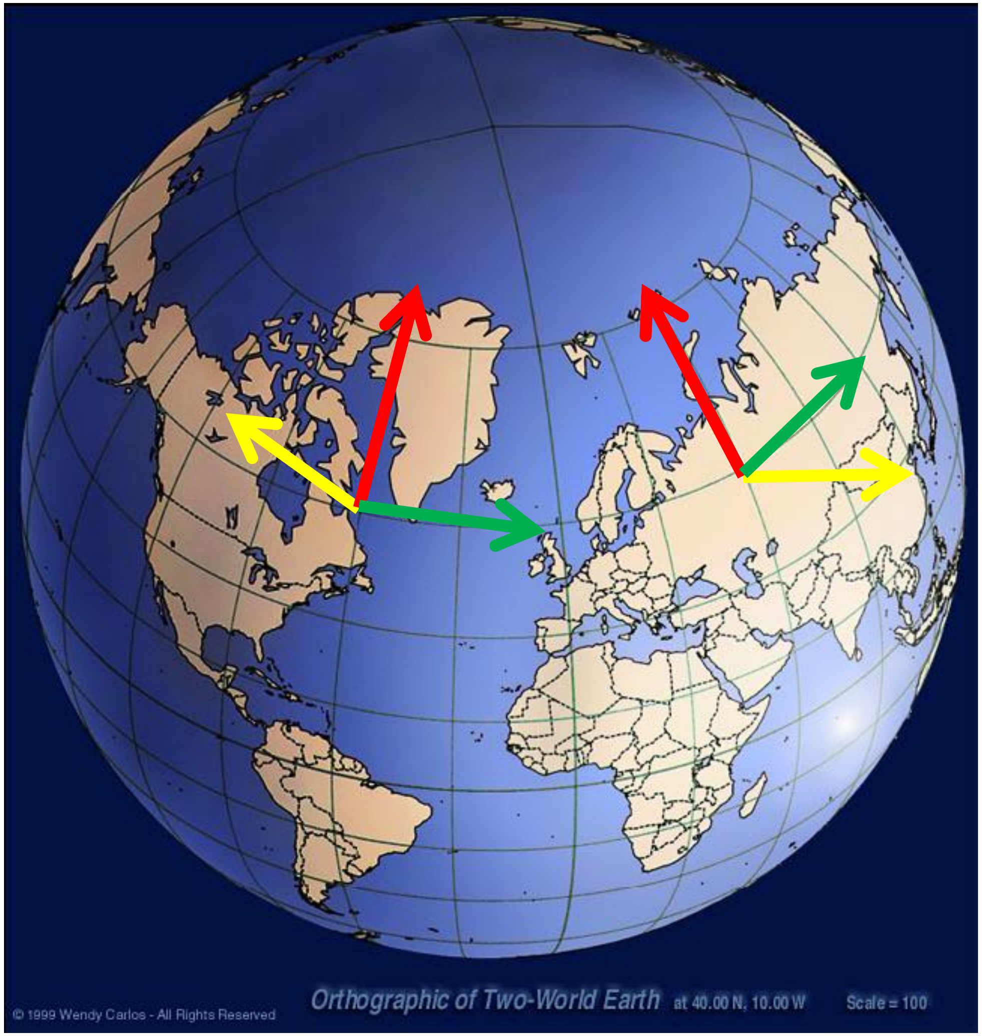
Each of these directions can be represented by a **unit vector**, and the three unit vectors together form a set of **base vectors**.

Think about, however, how these base vectors are oriented! Since we live on the surface of a **sphere** (i.e., the Earth), it makes sense for us to orient the base vectors with **respect to the spherical surface**.



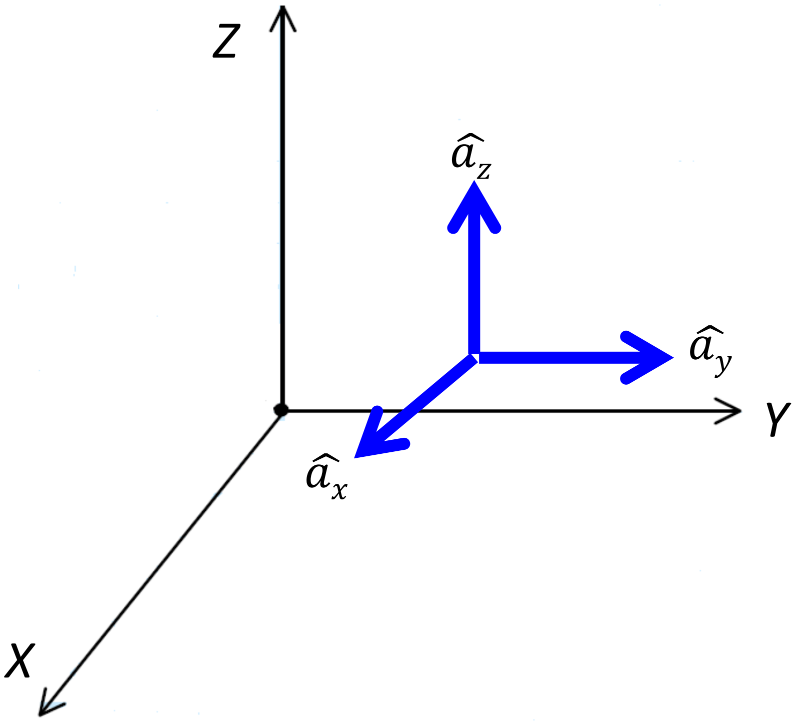
Base Vectors (contd.)

What this means, of course, is that **each location** on the Earth will orient its “base vectors” differently. This orientation is thus **different** for every point on Earth—a method that makes **perfect sense**!



Cartesian Base Vectors

- As the name implies, the Cartesian base **vectors** are related to the Cartesian **coordinates**.
- Specifically, the unit vector $\hat{a}_x$ points in the **direction of increasing x**. In other words, it points away from the y-z (x=0) plane.
- Similarly, $\hat{a}_y$ and $\hat{a}_z$ point in the direction of **increasing** y and z, respectively.



It was said that the directions of base vectors **generally** vary with location in space—Cartesian base vectors are the **exception**! Their directions are the same **regardless** of where you are in space.

Vector Expansion using Base Vectors

- Having defined an orthonormal set of base vectors, we can express **any** vector in terms of these unit vectors as:

$$\vec{A} = \mathrm{A_x}\hat{a}_x + \mathrm{A_y}\hat{a}_y + \mathrm{A_z}\hat{a}_z$$

- Note therefore that any vector can be written as a sum of three vectors!
- Each of these three vectors point in one of the **three orthogonal directions** $\hat{a}_x$, $\hat{a}_y$, and $\hat{a}_z$.
- The **magnitude** of each of these three vectors are determined by the scalar values A_x, A_y, and A_z.
- The values A_x, A_y, and A_z are called the **scalar components** of vector $\vec{\mathrm{A}}$.
- The vectors $A_x\hat{a}_x$, $A_y\hat{a}_y$ and $A_z\hat{a}_z$ are called the **vector components** of $\vec{\mathrm{A}}$**.**

Q: What the heck are scalar components A_x, A_y, and A_z and how do we determine them ??

A: Use the **dot product** to evaluate the expression above!

Vector Expansion using Base Vectors (contd.)

- Begin by taking the **dot product** of the above expression with unit vector $\hat{a}_x$

$$\vec{A}.\hat{a}_x = (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z).\,\hat{a}_x \quad \Rightarrow \quad A_x = \vec{A}.\hat{a}_x$$

- In other words, the scalar component A_x is just the value of the **dot product** of vector $\vec{A}$ and base vector $\hat{a}_x$. Similarly, we find that:

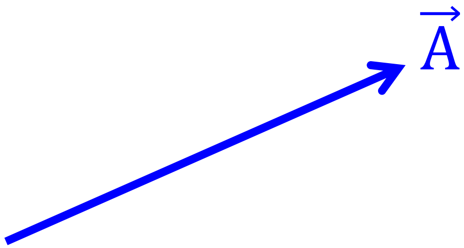
$$A_y = \vec{A}.\hat{a}_y \qquad\qquad A_z = \vec{A}.\hat{a}_z$$

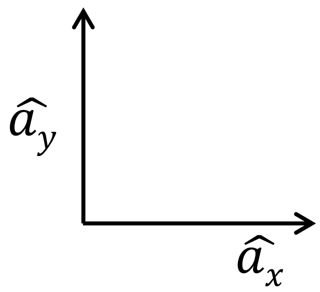
- Thus, **any vector** can be expressed specifically as:

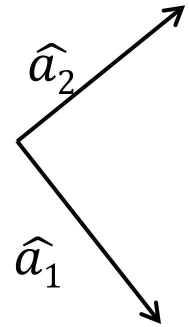
$$\boxed{\vec{A} = (\vec{A}.\hat{a}_x)\hat{a}_x + (\vec{A}.\hat{a}_y)\hat{a}_y + (\vec{A}.\hat{a}_z)\hat{a}_z}$$

Vector Expansion using Base Vectors (contd.)

- **For example**, consider a vector $\vec{A}$, along with **two different** sets of orthonormal base vectors:







- The **scalar components** of vector $\vec{A}$, in the direction of each base vector are:

$A_x = \vec{A}.\hat{a}_x = 2.0$ $\quad$ $A_1 = \vec{A}.\hat{a}_1 = 0.0$

$A_y = \vec{A}.\hat{a}_y = 1.5$ $\quad$ $A_2 = \vec{A}.\hat{a}_2 = 2.5$

$A_z = \vec{A}.\hat{a}_z = 0.0$ $\quad$ $A_3 = \vec{A}.\hat{a}_3 = 0.0$

- Using the **first** set of base vectors, we can write the vector $\vec{A}$ as:

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z = 2.0\hat{a}_x + 1.5\hat{a}_y$$

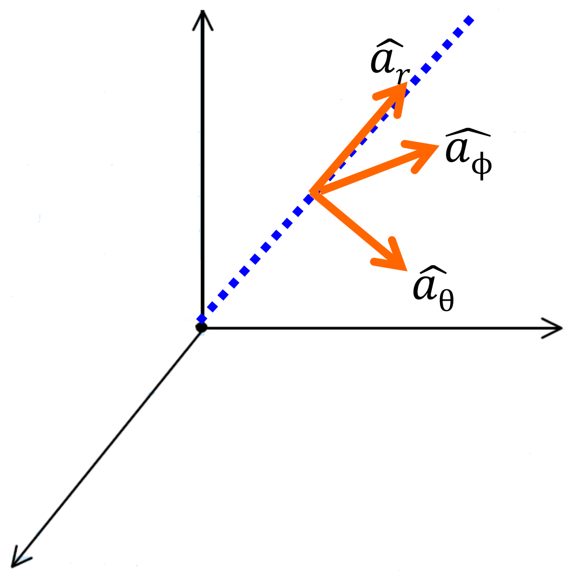
Vector Expansion using Base Vectors (contd.)

- using the **second** set, we find that: $\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3 = 2.5\hat{a}_2$
- It is **very** important to realize that: $\vec{A} = 2.0\hat{a}_x + 1.5\hat{a}_y = 2.5\hat{a}_2$

In other words, both expressions represent **exactly** the same vector! The difference in the representations is a result of using **different base vectors**, not because vector $\vec{A}$ is somehow "different" for each representation.

Spherical Base Vectors (contd.)

- Spherical base vectors are the “natural” base vectors of a **sphere**.
  - $\hat{a}_r$ points in the direction of **increasing** r. In other words $\hat{a}_r$ points **away from the origin**. This is analogous to the direction we call **up**.
  - $\hat{a}_\theta$ points in the direction of **increasing** θ. This is analogous to the direction we call **south.**
  - $\hat{a}_\phi$ a points in the direction of **increasing** ϕ. This is analogous to the direction we call **east.**



Spherical Base Vectors (contd.)

IMPORTANT NOTE: The directions of spherical base vectors are **dependent on position**. First you must determine **where** you are in space (using coordinate values), **then** you can define the directions of $\hat{a}_r$, $\hat{a}_\phi$, and $\hat{a}_\theta$.

Reminder: **Cartesian** base vectors are **special**, in that their directions are **independent** of location—they have the same directions throughout all space.

- Thus, it is prudent to define spherical base vectors **in terms of** Cartesian base vectors. It can be shown that:

$$\hat{a}_r . \hat{a}_x = \sin\theta \ \cos\phi \qquad \hat{a}_\theta . \hat{a}_x = \cos\theta \ \cos\phi \qquad \hat{a}_\phi . \hat{a}_x = -\sin\phi$$

$$\hat{a}_r . \hat{a}_y = \sin\theta \ \sin\phi \qquad \hat{a}_\theta . \hat{a}_y = \cos\theta \ \sin\phi \qquad \hat{a}_\phi . \hat{a}_y = \cos\phi$$

$$\hat{a}_r . \hat{a}_z = \cos\theta \qquad \hat{a}_\theta . \hat{a}_z = -\sin\theta \qquad \hat{a}_\phi . \hat{a}_z = 0$$

Spherical Base Vectors (contd.)

- **any** vector $\vec{A}$ can be written as: $\vec{A} = (\vec{A}.\hat{a}_x)\hat{a}_x + (\vec{A}.\hat{a}_y)\hat{a}_y + (\vec{A}.\hat{a}_z)\hat{a}_z$

- Therefore, we can write unit vector $\hat{a}_r$ as:

$$\hat{a}_r = (\hat{a}_r.\hat{a}_x)\hat{a}_x + (\hat{a}_r.\hat{a}_y)\hat{a}_y + (\hat{a}_r.\hat{a}_z)\hat{a}_z$$

$$\hat{a}_r = \sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z$$

This result explicitly shows that $\hat{a}_r$ is a function of θ and ϕ.

- For **example**, at the point in space r = 7.239, θ = 90° and ϕ =0°, we find that $\hat{a}_r$ = $\hat{a}_x$. In other words, at this point in space, the direction $\hat{a}_r$ points in the **x**-direction.
- **Or**, at the point in space r = 2.735, θ = 90° and ϕ =90°, we find that $\hat{a}_r$ = $\hat{a}_y$. In other words, at this point in space, $\hat{a}_r$ points in the **y**-direction.

Spherical Base Vectors (contd.)

- Additionally, we can write $\widehat{a_\phi}$, and $\widehat{a_\theta}$ as:

$$\widehat{a_\theta} = (\widehat{a_\theta}.\widehat{a_x})\widehat{a_x} + (\widehat{a_\theta}.\hat{a}_y)\hat{a}_y + (\widehat{a_\theta}.\widehat{a_z})\widehat{a_z}$$

$$\widehat{a_\phi} = (\widehat{a_\phi}.\widehat{a_x})\widehat{a_x} + (\widehat{a_\phi}.\hat{a}_y)\hat{a}_y + (\widehat{a_\phi}.\widehat{a_z})\widehat{a_z}$$

- Alternatively, we can write **Cartesian** base vectors in terms of spherical base vectors, i.e.,

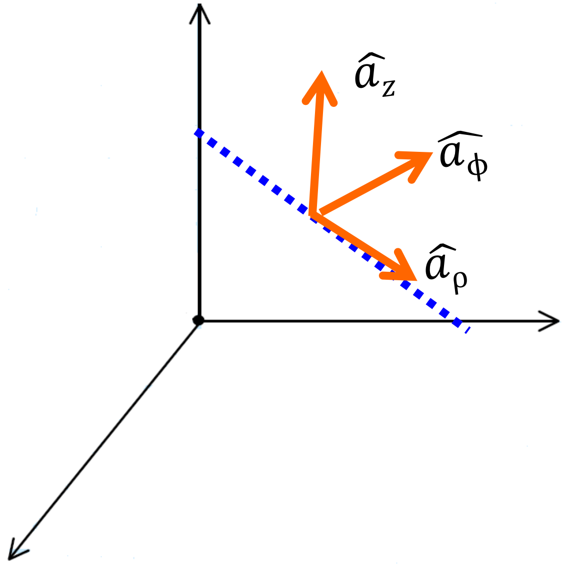
$$\hat{a}_x = (\hat{a}_x.\widehat{a_r})\widehat{a_r} + (\hat{a}_x.\widehat{a_\theta})\widehat{a_\theta} + (\hat{a}_x.\hat{a}_\phi)\hat{a}_\phi$$

$$\hat{a}_y = (\hat{a}_y.\widehat{a_r})\widehat{a_r} + (\hat{a}_y.\widehat{a_\theta})\widehat{a_\theta} + (\hat{a}_y.\hat{a}_\phi)\hat{a}_\phi$$

$$\hat{a}_z = (\hat{a}_z.\widehat{a_r})\widehat{a_r} + (\hat{a}_z.\widehat{a_\theta})\widehat{a_\theta} + (\hat{a}_z.\hat{a}_\phi)\hat{a}_\phi$$

Cylindrical Base Vectors

- Cylindrical base vectors are the **natural** base vectors of a **cylinder**.
  - $\hat{a}_\rho$ points in the direction of **increasing** ρ. In other words, $\hat{a}_\rho$ points **away from the z-axis**.
  - $\hat{a}_\phi$ points in the direction of **increasing** ϕ. This is precisely the **same** base vector we described for **spherical** base vectors.
  - $\hat{a}_z$ points in the direction of **increasing z**. This is precisely the **same** base vector we described for **Cartesian** base vectors.



It is evident, that like spherical base vectors, the cylindrical base vectors are **dependent on position**. A vector that points **away** from the z-axis (e.g., $\hat{a}_\rho$), will point in a direction that is **dependent** on where we are in space!

Cylindrical Base Vectors (contd.)

- We can express cylindrical base vectors in terms of **Cartesian** base vectors. First, we find that:

$$\hat{a}_\rho \,.\, \hat{a}_x = \cos\phi \qquad \widehat{a_\phi} \,.\, \hat{a}_x = -\sin\phi \qquad \hat{a}_z \,.\, \hat{a}_x = 0$$

$$\hat{a}_\rho \,.\, \hat{a}_y = \sin\phi \qquad \widehat{a_\phi} \,.\, \hat{a}_y = \cos\phi \qquad \hat{a}_z \,.\, \hat{a}_y = 0$$

$$\hat{a}_\rho \,.\, \hat{a}_z = 0 \qquad \widehat{a_\phi} \,.\, \hat{a}_z = 0 \qquad \hat{a}_z \,.\, \hat{a}_z = 1$$

- We can use these results to write **cylindrical** base vectors in terms of **Cartesian** base vectors, or vice versa!

$$\hat{a}_\rho = (\hat{a}_\rho . \hat{a}_x)\hat{a}_x + (\hat{a}_\rho . \hat{a}_y)\hat{a}_y + (\hat{a}_\rho . \hat{a}_z)\hat{a}_z$$

$$\hat{a}_\rho = \cos\phi\, \hat{a}_x + \sin\phi\, \hat{a}_y$$

- **or**

$$\hat{a}_x = (\hat{a}_x . \hat{a}_\rho)\hat{a}_\rho + (\hat{a}_y . \hat{a}_\rho)\hat{a}_\rho + (\hat{a}_z . \hat{a}_\rho)\hat{a}_\rho$$

$$\hat{a}_x = \cos\phi\, \hat{a}_\rho - \sin\phi\, \widehat{a_\phi}$$

Cylindrical Base Vectors (contd.)

- Finally, we can write **cylindrical** base vectors in terms of **spherical** base vectors, or vice versa, using the following relationships:

$$\hat{a}_\rho \cdot \hat{a}_r = \sin\theta \qquad \hat{a}_\phi \cdot \hat{a}_r = 0 \qquad \hat{a}_z \cdot \hat{a}_r = \cos\theta$$

$$\hat{a}_\rho \cdot \hat{a}_\theta = \cos\theta \qquad \hat{a}_\phi \cdot \hat{a}_\theta = 0 \qquad \hat{a}_z \cdot \hat{a}_\theta = -\sin\theta$$

$$\hat{a}_\rho \cdot \hat{a}_\phi = 0 \qquad \hat{a}_\phi \cdot \hat{a}_\phi = 1 \qquad \hat{a}_z \cdot \hat{a}_\phi = 0$$

- **For example:**

$$\hat{a}_\rho = (\hat{a}_\rho . \hat{a}_r)\hat{a}_r + (\hat{a}_\rho . \hat{a}_\theta)\hat{a}_\theta + (\hat{a}_\rho . \hat{a}_\phi)\hat{a}_\phi$$

$$\hat{a}_\rho = \sin\theta\, \hat{a}_r + \cos\theta\, \hat{a}_\theta$$

- **or**

$$\hat{a}_\theta = (\hat{a}_\theta . \hat{a}_\rho)\hat{a}_\rho + (\hat{a}_\theta . \hat{a}_\phi)\hat{a}_\phi + (\hat{a}_\theta . \hat{a}_z)\hat{a}_z$$

$$\hat{a}_\theta = \cos\theta\, \hat{a}_\rho - \sin\theta\, \hat{a}_z$$

Vector Algebra Using Orthonormal Base Vectors

Q: Just why do we express a vector in terms of 3 orthonormal base vectors? Doesn't this just make things even more complicated ??

A: Actually, it makes things **much** simpler. The **evaluation** of vector operations such as addition, subtraction, multiplication, dot product, and cross product all become straightforward **if** all vectors are expressed using the **same** set of base vectors.

Vector Algebra Using Orthonormal Base Vectors (contd.)

Dot Product

Say we take the **dot product** of $\vec{A}$ and $\vec{B}$:

$$\vec{A}.\vec{B} = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right).\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$

$$= A_x\hat{a}_x.\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$
$$+ A_y\hat{a}_y.\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$
$$+ A_z\hat{a}_z.\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$

$$= A_xB_x\left(\hat{a}_x.\hat{a}_x\right) + A_xB_y\left(\hat{a}_x.\hat{a}_y\right) + A_xB_z\left(\hat{a}_x.\hat{a}_z\right)$$
$$+ A_yB_x\left(\hat{a}_y.\hat{a}_x\right) + A_yB_y\left(\hat{a}_y.\hat{a}_y\right) + A_yB_z\left(\hat{a}_y.\hat{a}_z\right)$$
$$+ A_zB_x\left(\hat{a}_z.\hat{a}_x\right) + A_zB_y\left(\hat{a}_z.\hat{a}_y\right) + A_zB_z\left(\hat{a}_z.\hat{a}_z\right)$$

Q: I thought this was suppose to make things **easier** !?!

Vector Algebra Using Orthonormal Base Vectors (contd.)

A: Be patient! Recall that these are **orthonormal** base vectors, therefore:

$$\hat{a}_x.\hat{a}_x = \hat{a}_y.\hat{a}_y = \hat{a}_z.\hat{a}_z = 1 \qquad \hat{a}_x.\hat{a}_y = \hat{a}_y.\hat{a}_z = \hat{a}_z.\hat{a}_x = 0$$

- As a result, our **dot product** expression reduces to this simple expression:

$$\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

We can apply this to the expression for determining the **magnitude** of a vector:

$$\left|\vec{A}\right|^2 = \vec{A}.\vec{A} = A_x^2 + A_x^2 + A_z^2 \qquad \left|\vec{A}\right| = \sqrt{\vec{A}.\vec{A}} = \sqrt{A_x^2 + A_x^2 + A_z^2}$$

Vector Algebra Using Orthonormal Base Vectors (contd.)

- Let us revisit previous example, where we expressed a vector using two different sets of basis vectors:

$$\vec{A} = 2.0\hat{a}_x + 1.5\hat{a}_y \qquad \vec{A} = 2.5\hat{a}_y$$

- Therefore, the magnitude of $\vec{A}$ is determined to be:

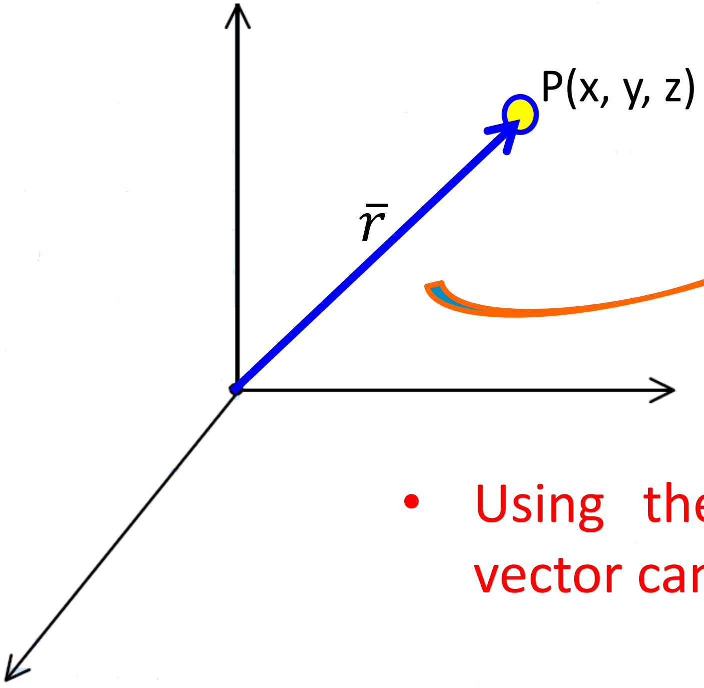
$$\left|\vec{A}\right| = \sqrt{2^2 + 1.5^2} = 2.5 \qquad \left|\vec{A}\right| = \sqrt{2.5^2} = 2.5$$

Q: Hey! We get the **same** answer from both expressions; is this a **coincidence**?

A: No! Remember, both expressions represent the **same** vector, only using different sets of base vectors. The magnitude of vector $\vec{A}$ is 2.5, **regardless** of how we choose to express $\vec{A}$.

The Position Vector

- Consider a point whose location in space is specified with Cartesian coordinates (e.g., P(x, y, z)). Now consider the directed distance (a vector quantity!) extending from the origin to this point.



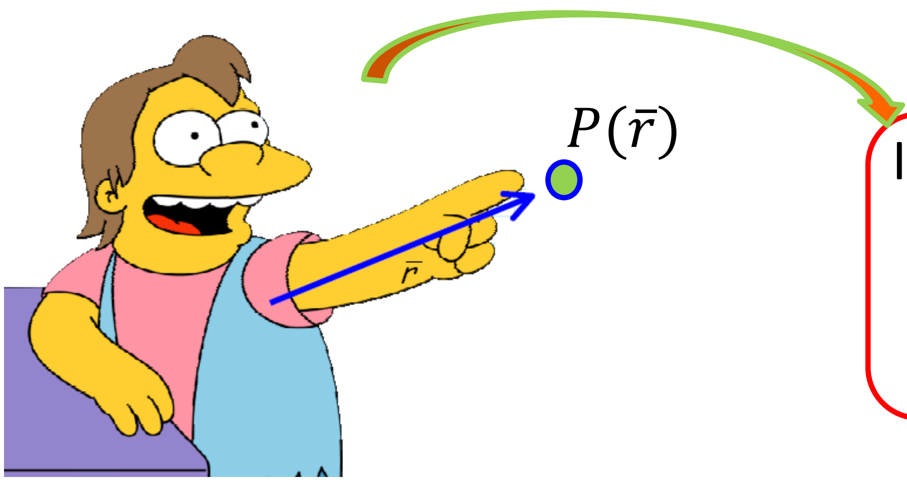
This **particular** directed distance—a vector beginning at the **origin** and extending outward to a point—is a **very important** and fundamental directed distance known as the **position vector** $\bar{r}$

- Using the **Cartesian** coordinate system, the position vector can be explicitly written as:

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

The Position Vector (contd.)

- Note that given the **coordinates** of some point (e.g., x =1, y =2, z =-3), we can easily determine the **corresponding position vector** (e.g., $\bar{r} = \hat{a}_x + 2\hat{a}_y - 3\hat{a}_z$).
- Moreover, given some **specific position vector** (e.g., $\bar{r} = 4\hat{a}_y - 2\hat{a}_z$), we can easily determine the **corresponding coordinates of that point** (e.g., x =0, y =4, z =-2).
- In other words, a position vector $\bar{r}$ is an alternative way to denote the location of a point in space! We can use **three coordinate values** to specify a point's location, **or** we can use a **single position vector** $\bar{r}$.



I see! The position vector is essentially a **pointer.** Look at the end of the vector, and you will find the **point specified**!

The magnitude of $\bar{r}$

- Note the **magnitude** of any and all position vectors is:

$$\left|\bar{r}\right| = \sqrt{\bar{r}.\bar{r}} = \sqrt{x^2 + y^2 + z^2} = r$$

Q: Hey, this makes **perfect sense**! Doesn't the coordinate value r have a **physical** interpretation as the **distance** between the **point** and the **origin**?

A: That's right! The **magnitude** of a **directed distance** vector is equal to the **distance** between the two points—in this case the distance between the **specified point** and the **origin**!

Alternative forms of the position vector

- Be **careful**! Although the position vector **is correctly** expressed as:

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

- It is **NOT CORRECT** to express the position vector as:

$$\bar{r} \neq \rho\hat{a}_\rho + \phi\hat{a}_\phi + z\hat{a}_z$$

$$\bar{r} \neq r\hat{a}_r + \theta\hat{a}_\theta + \phi\hat{a}_\phi$$

NEVER, EVER express the position vector in either of these two ways!

It should be **readily apparent** that the two expression above **cannot** represent a position vector—because **neither** is even a directed distance!

Alternative forms of the position vector (contd.)

Q: Why sure—it is **of course** readily apparent to me—but why don't you go ahead and explain it to those with **less** insight!

A: Recall that the **magnitude** of the position vector $\bar{r}$ has units of **distance**. Thus, the **scalar components** of the position vector must **also** have units of distance (e.g., meters). The coordinates x, y, z, ρ and r **do** have units of distance, but coordinates θ and ϕ do **not**.

Thus, the vectors $\theta\hat{a}_\theta$ and $\phi\hat{a}_\phi$ **cannot** be vector components of a position vector—or for that matter, any other **directed distance**!

Alternative forms of the position vector (contd.)

- Instead, we can use **coordinate transforms** to show that:

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

$$= \rho\cos\phi\hat{a}_x + \rho\sin\phi\hat{a}_y + z\hat{a}_z$$

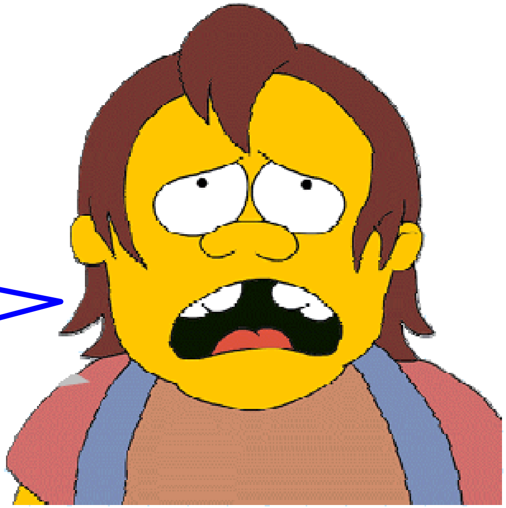
$$= r\sin\theta\cos\phi\hat{a}_x + r\sin\theta\sin\phi\hat{a}_y + r\cos\theta\hat{a}_z$$

ALWAYS use one of these three expressions of a position vector!!

Note that in **each** of the three expressions above, we use **Cartesian base vectors**. The **scalar components** can be expressed using Cartesian, cylindrical, or spherical **coordinates**, but we must always use **Cartesian base vectors**.

Alternative forms of the position vector (contd.)

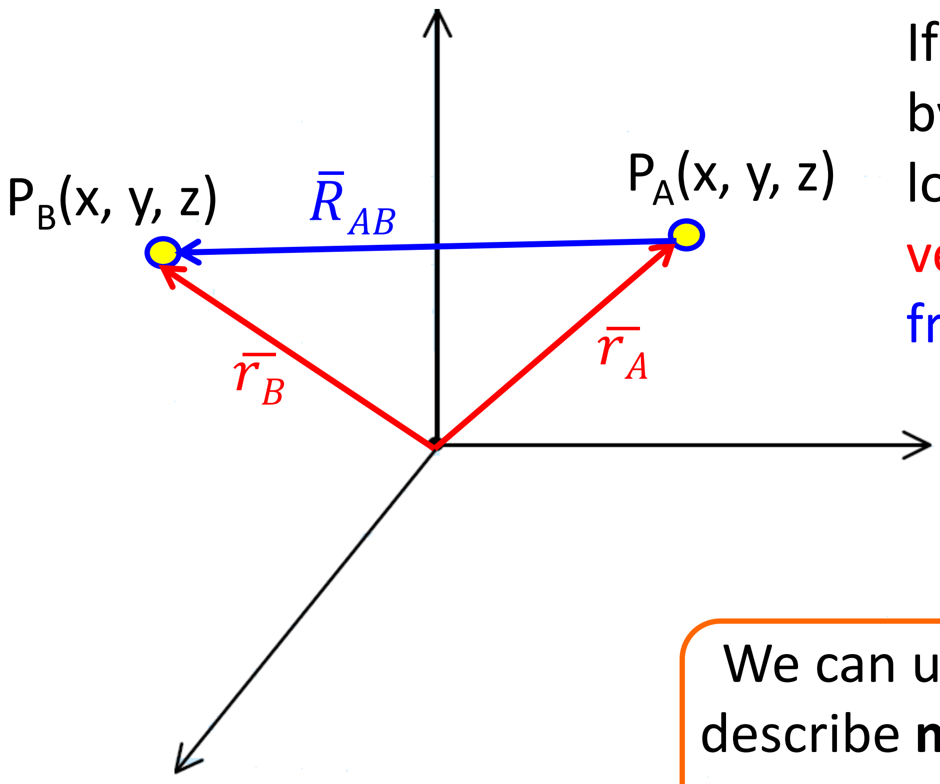
Q: Why must we **always** use Cartesian base vectors? You said that we could express **any** vector using spherical or base vectors. Doesn’t this **also** apply to position vectors?



A: The reason we **only** use Cartesian base vectors for constructing a position vector is that Cartesian base vectors are the only base vectors whose directions are **fixed**—independent of position in space!

Applications of the Position Vector

- Position vectors are **particularly useful** when we need to determine the directed distance between **two** arbitrary points in space.



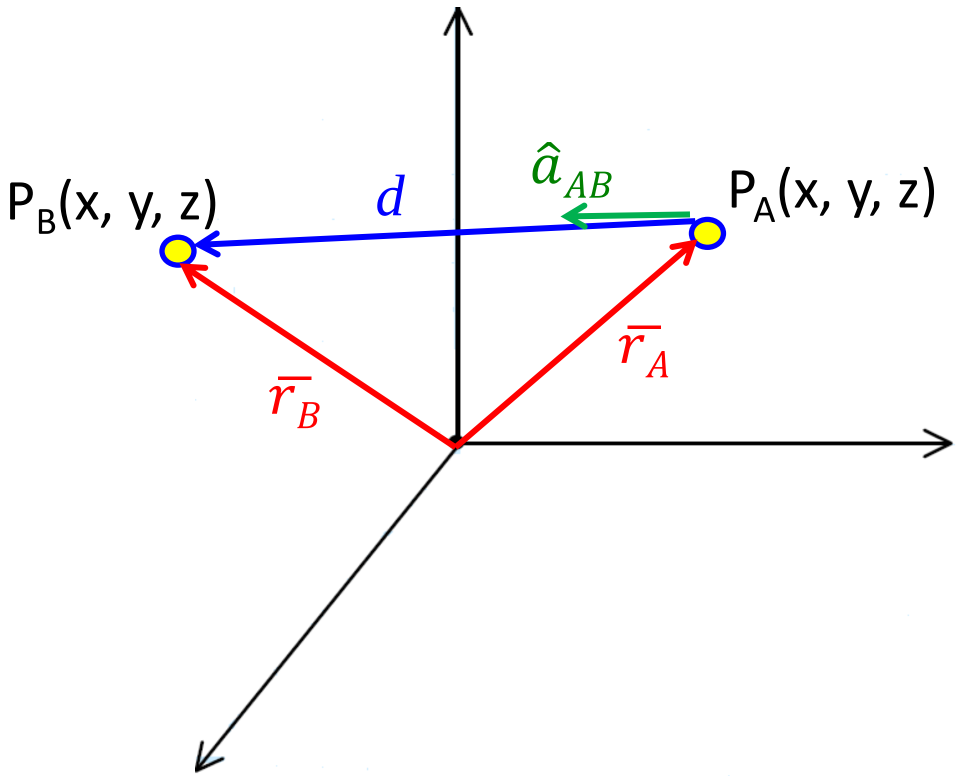
If the location of **point P_A** is denoted by position vector $\bar{r}_A$, and the location of **point P_B** by position vector $\bar{r}_B$, then the **directed distance** from point P_A to point P_B, is:

$$\bar{R}_{AB} = \bar{r}_B - \bar{r}_A$$

We can use this **directed distance** $\bar{R}_{AB}$ to describe **much** about the relative locations of point P_A and P_B!

Application of the Position Vector

- For example, the physical **distance** between these two points is simply the magnitude of this directed distance.
- Likewise, we can specify the **direction** toward point P_B, with **respect** to point P_A, by defining the **unit vector** $\hat{a}_{AB}$:

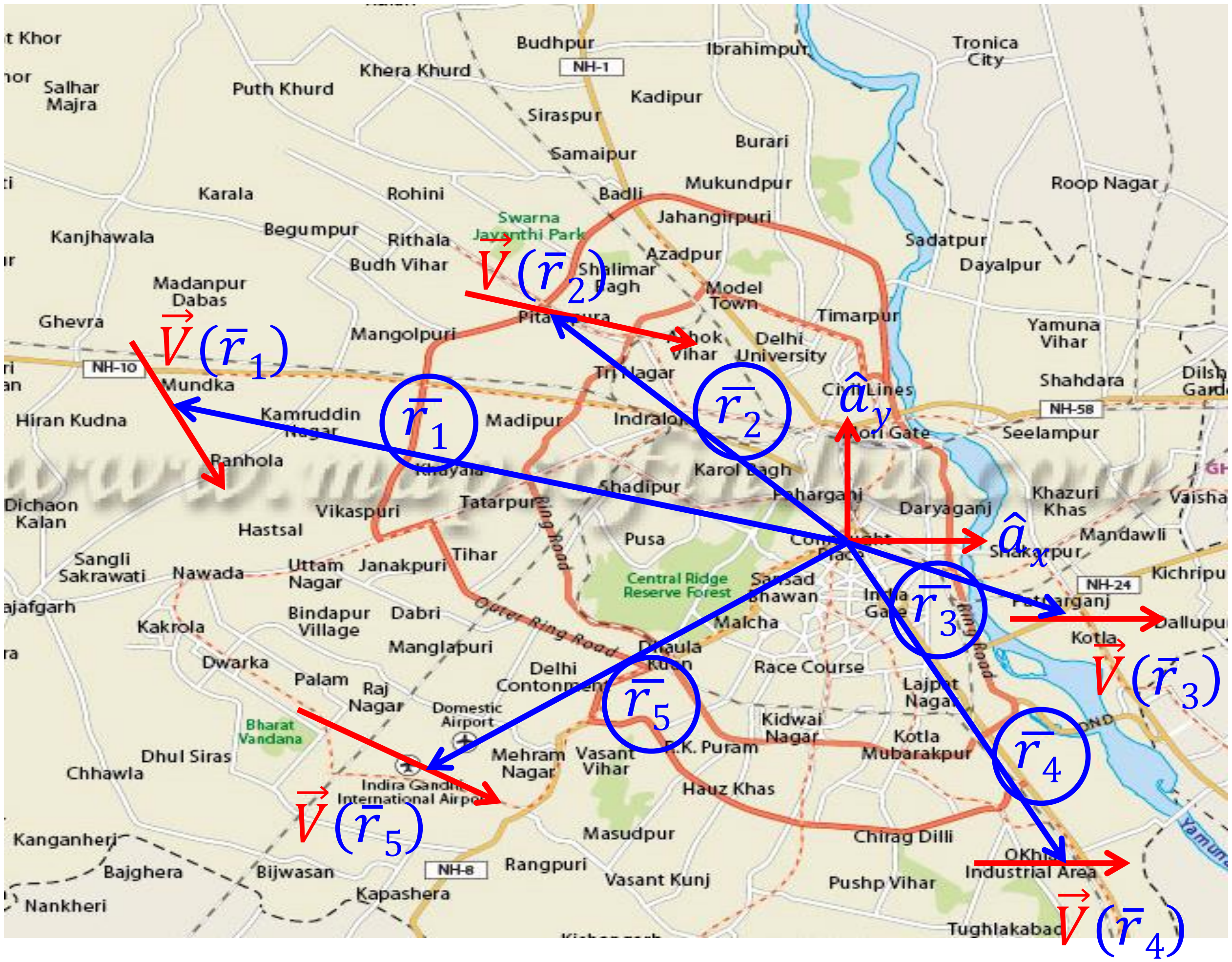


$$d = \left|\bar{R}_{AB}\right| = \left|\bar{r}_B - \bar{r}_A\right|$$

$$\hat{a}_{AB} = \frac{\bar{R}_{AB}}{\left|\bar{R}_{AB}\right|} = \frac{\bar{r}_B - \bar{r}_A}{\left|\bar{r}_B - \bar{r}_A\right|}$$

Vector Field Notation

- **Consider the vector field** $\vec{V}(\bar{r})$, which describes the **wind velocity** across the state of Delhi.



In this map, the **origin** has been placed at Connaught Place. The **locations** of Delhi locality can thus be identified using **position vectors** (units in kms)

Vector Field Notation (contd.)

$\bar{r}_1 = -400\hat{a}_x + 20\hat{a}_y$	→	The **location** of Mundka
$\bar{r}_2 = -90\hat{a}_x + 70\hat{a}_y$	→	The **location** of Pitampura
$\bar{r}_3 = 30\hat{a}_x - 5\hat{a}_y$	→	**The location** of Patparganj
$\bar{r}_4 = 40\hat{a}_x - 90\hat{a}_y$	→	The **location** of Okhla Industrial Estate
$\bar{r}_5 = -130\hat{a}_x - 70\hat{a}_y$	→	The **location** of IGI Airport locality

- Evaluating the **vector field** $\vec{V}(\bar{r})$ at these locations provides the wind velocity **at** each Delhi locality (units of kmph).

$\vec{V}(\bar{r}_1) = 15\hat{a}_x - 17\hat{a}_y$	→	The **wind velocity** in Mundka
$\vec{V}(\bar{r}_2) = 15\hat{a}_x - 9\hat{a}_y$	→	The **wind velocity** in Pitampura
$\vec{V}(\bar{r}_3) = 11\hat{a}_x$	→	**The wind velocity** in Patparganj
$\vec{V}(\bar{r}_4) = 7\hat{a}_x$	→	The **wind velocity** in Okhla Industrial Estate
$\vec{V}(\bar{r}_5) = 9\hat{a}_x - 4\hat{a}_y$	→	The **wind velocity** in IGI Airport locality

Vector Field Notation (contd.)

- From **vector field** $\vec{A}(\bar{r})$, we can find the **magnitude** and **direction** of the discrete vector $\vec{A}$ that is **located** at the **point** defined by position vector $\bar{r}$.
- This **discrete vector** $\vec{A}$ does **not** "extend" from the origin to the point described by position vector $\bar{r}$. Rather, the discrete vector $\vec{A}$ describes a quantity **at that point**, and that point only. The magnitude of vector $\vec{A}$ does **not** have units of distance! The **length** of the arrow that represents vector $\vec{A}$ is merely symbolic—its length has **no** direct physical meaning.
- On the other hand, the position vector $\bar{r}$, being a directed distance, **does** extend from the origin to a specific **point** in space. The magnitude of a position vector $\bar{r}$ **is** distance—the length of the **position vector** arrow **has** a direct physical meaning!
- Additionally, we should again note that a vector field need not be static. A **dynamic** vector field is likewise a function of **time**, and thus can be described with the notation:

$$\vec{A}(\bar{r}, t)$$

The Contour *C*

- In this class, we will limit ourselves to studying only those contours that are formed when we change the location of a point by varying **just one** coordinate parameter. In other words, the other two coordinate parameters will remain **fixed**.

Mathematically, therefore, a **contour** is described by:

2 equalities (e.g., **x =2, y =-4; r =3, ϕ =π/4**) **AND**

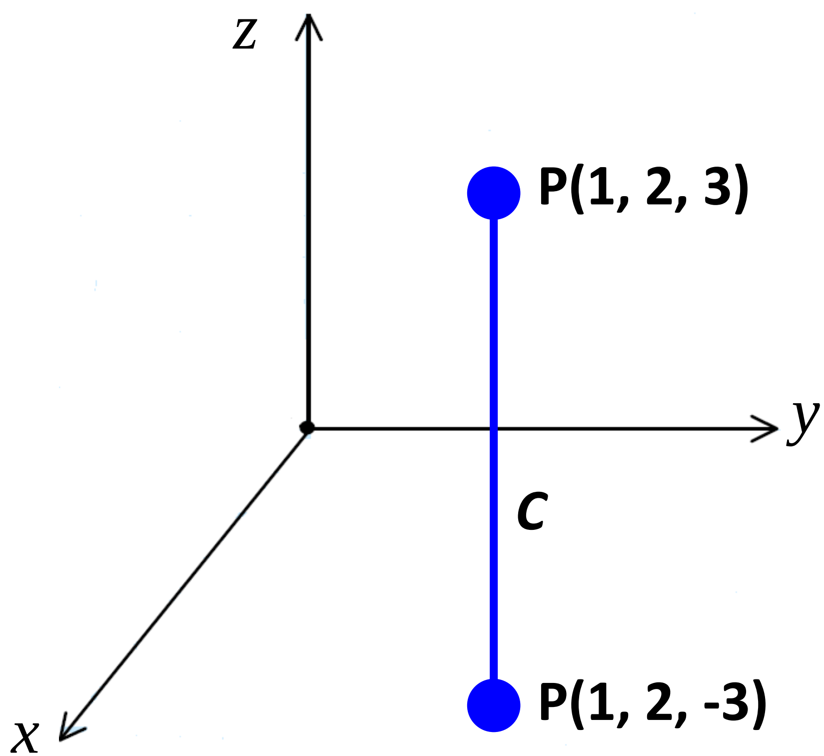
1 inequality (e.g., **-1 < z < 5; 0 < θ < π/2**)

- **Likewise,** we will need to explicitly determine the **differential displacement vector** $\overline{dl}$ for each contour.

Recall we have studied **seven** coordinate parameters (x, y, z, ρ, ϕ, r, θ). As a result, we can form **seven** different contours C!

Cartesian Contours

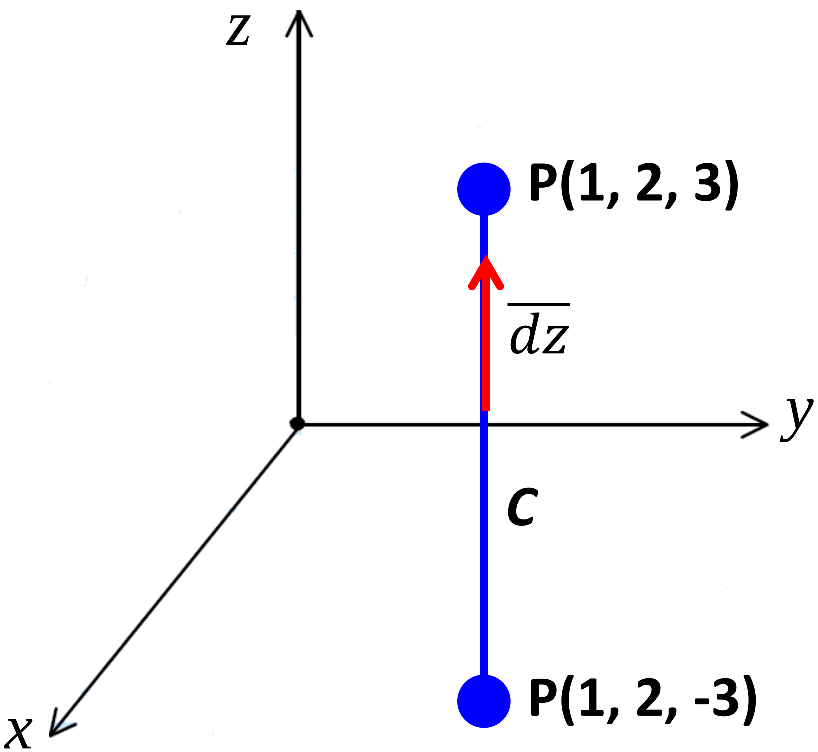
- Say we move a point from P(x =1, y =2, z =-3) to P(x =1, y =2, z=3) by changing **only the coordinate variable z from z =-3 to z=3**. In other words, the coordinate values x and y remain **constant** at x = 1 and y = 2.
- We form a contour that is a **line segment**, **parallel** to the z-axis!



Note that **every** point along this segment has coordinate values x =1 and y =2. As we move along the contour, **the only coordinate value that changes is** z.

Cartesian Contours (contd.)

- Therefore, the **differential** directed distance associated with a change in position from z to z +dz, is $\overline{dl} = \overline{dz} = \hat{a}_z dz$



Similarly, a line segment parallel to the x-axis (or y-axis) can be formed by changing coordinate parameter x (or y), with a resulting differential displacement vector of $\overline{dl} = \overline{dx} = \hat{a}_x dx$ (or $\overline{dl} = \overline{dy} = \hat{a}_y dy$).

Cartesian Contours (contd.)

The three **Cartesian contours** are therefore:

1. Line segment parallel to the z-axis

$$x = c_x \qquad y = c_y \qquad c_{z1} \le z \le c_{z2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_z dz$$

2. Line segment parallel to the y-axis

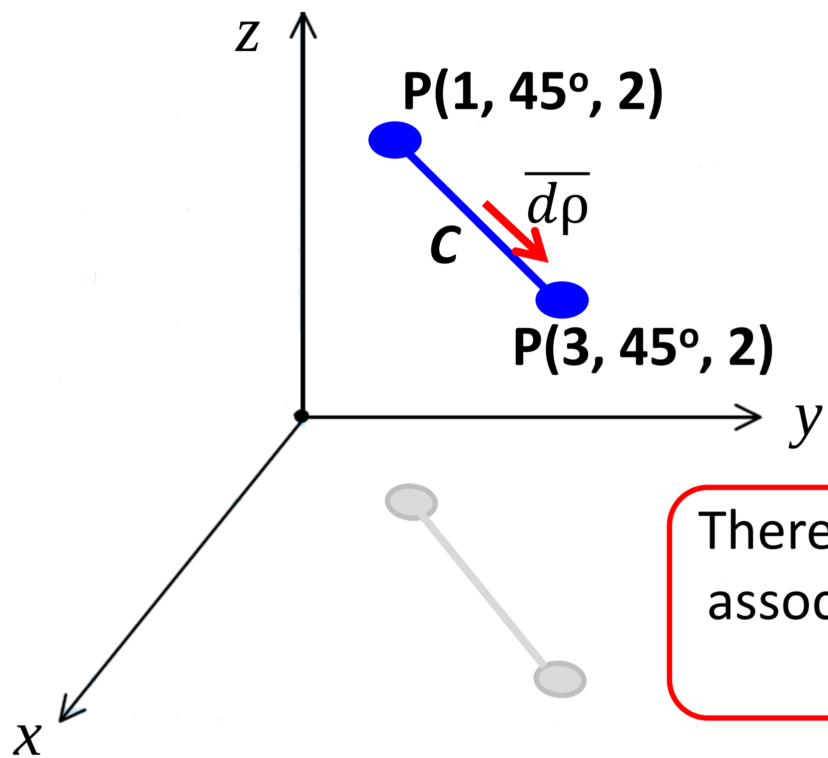
$$x = c_x \qquad z = c_z \qquad c_{y1} \le y \le c_{y2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_y dy$$

3. Line segment parallel to the x-axis

$$y = c_y \qquad z = c_z \qquad c_{x1} \le x \le c_{x2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_x dx$$

Cylindrical Contours

- Say we move a point from P(ρ =1, ϕ = 45°, z =2) to P(ρ =3, ϕ = 45°, z =2) by changing only the coordinate variable ***ρ*** from ρ =1 to ρ =3. In other words, the coordinate values ϕ and z remain **constant** at ϕ = 45° and z =2.
- We form a contour that is a **line segment**, **parallel** to the x-y plane (i.e., perpendicular to the z-axis).

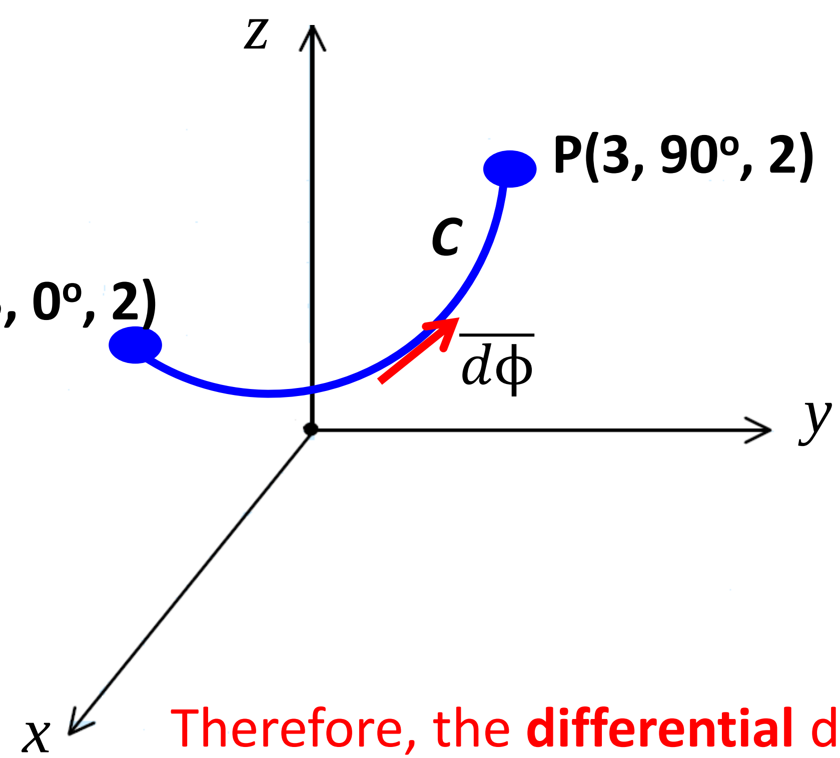


Note that **every** point along this segment has coordinate values ϕ = 45° and z =2. As we move along the contour, the **only** coordinate value that changes is ρ.

Therefore, the **differential** directed distance associated with a change in position from ρ to ρ+dρ, is $\overline{dl} = \overline{d\rho} = \hat{a}_\rho d\rho$

Cylindrical Contours (contd.)

- Alternatively, say we move a point from P(ρ =3, ϕ = 0º, z =2) to P(ρ =3, ϕ = 90º, z =2) by changing **only** the coordinate variable ϕ from ϕ = 0º to ϕ = 90º. In other words, the coordinate values ρ and z remain **constant** at ρ =3 and z =2. We form a contour that is a **circular arc**, parallel to the x-y plane.



Note: if we move from ϕ = 0º to ϕ = 360º, a complete **circle** is formed around the z-axis.

Every point along the arc has coordinate values ρ = 3 and z =2. As we move along the contour, the **only** coordinate value that changes is ϕ.

Therefore, the **differential** directed distance associated with a change in position from ϕ to ϕ+dϕ is $\overline{dl} = \overline{d\varphi} = \hat{a}_{\varphi}\rho d\phi$

Cylindrical Contours (contd.)

The three cylindrical contours are therefore described as:

1. Line segment parallel to the z-axis

$$\rho = c_\rho \quad \phi = c_\phi \quad c_{z1} \le z \le c_{z2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_z dz$$

2. Circular arc parallel to the xy-plane

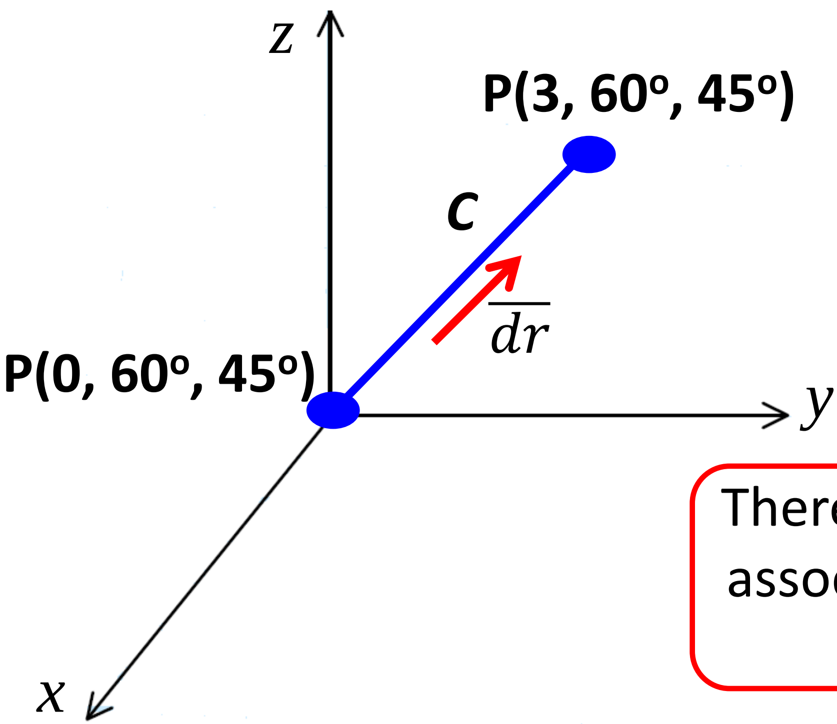
$$\rho = c_\rho \quad z = c_z \quad c_{\phi 1} \le \phi \le c_{\phi 2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_\phi \rho d\phi$$

3. Line segment parallel to the xy plane

$$\phi = c_\phi \quad z = c_z \quad c_{\rho 1} \le \rho \le c_{\rho 2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_\rho d\rho$$

Spherical Contours

- Say we move a point from P(r =0, θ = 60º, ϕ = 45º) to P(r =3, θ = 60º, ϕ = 45º) by changing **only** the coordinate variable r from r=0 to r =3. In other words, the coordinate values θ and ϕ remain **constant** at θ = 60º and ϕ = 45º.
- We form a contour that is a **line segment**, emerging from the origin.

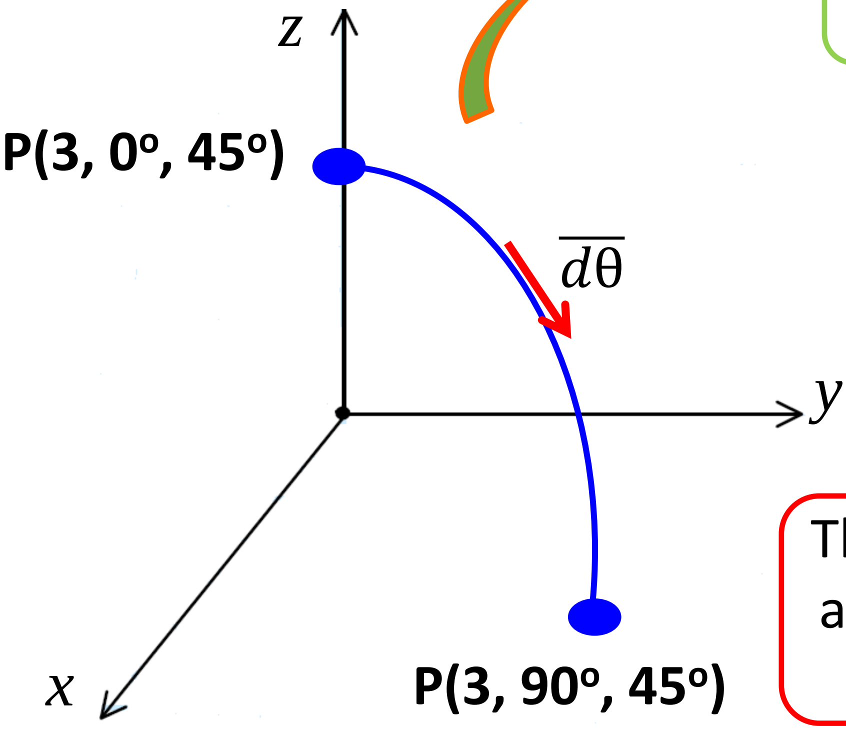


Every point along the line segment has coordinate values θ = 60º and ϕ = 45º. As we move along the contour, the **only** coordinate value that changes is r.

Therefore, the **differential** directed distance associated with a change in position from r to $r+dr$, is $\overline{dl} = \overline{dr} = \hat{a}_r$dr

Spherical Contours (contd.)

- Alternatively, say we move a point from P(r =3, θ = 0^o, ϕ = 45^o) to P(r =3, θ = 90^o, ϕ = 45^o) by changing **only** the coordinate variable θ from θ = 0^o to θ =90^o. In other words, the coordinate values r and ϕ remain **constant** at r = 3 and ϕ = 45^o



We form a **circular arc**, whose plane includes the z-axis.

Every point along the arc has coordinate values r = 3 and ϕ = 45^o. As we move along the contour, the **only** coordinate value that changes is θ.

Therefore, the **differential** directed distance associated with a change in position from θ to θ+dθ, is $\overline{dl} = \overline{d\theta} = \hat{a}_\theta r d\theta$

Spherical Contours (contd.)

- Finally, we could fix coordinates r and θ and vary coordinate ϕ only—but we **already** did this in cylindrical coordinates! We **again** find that a **circular arc** is generated, an arc that is parallel to the x-y plane.

The three spherical contours are therefore:

1. Circular arc parallel to the xy-plane

$$r = c_r \quad \theta = c_\theta \quad c_{\phi 1} \leq \phi \leq c_{\phi 2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_\phi r \sin\theta \, d\phi$$

2. Circular arc in a plane that includes z-axis

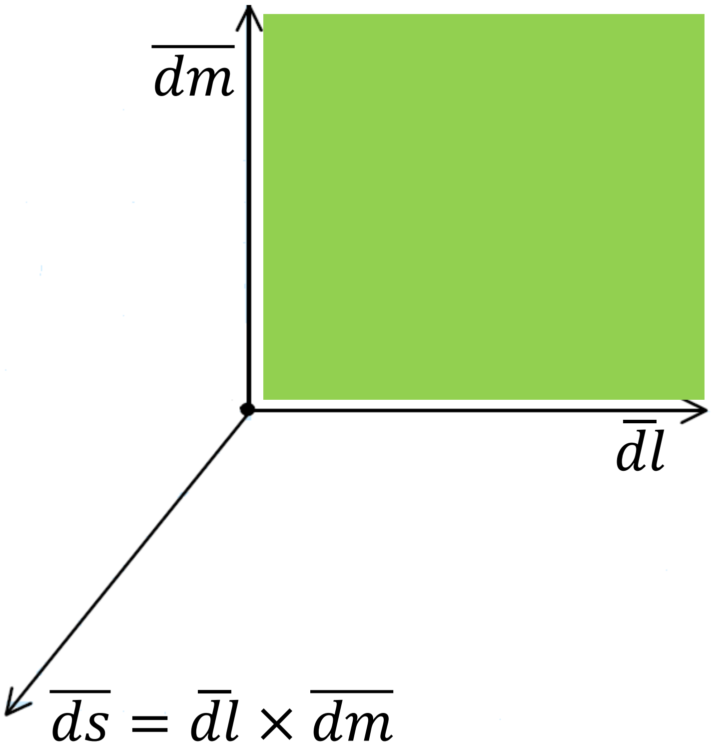
$$r = c_r \quad \phi = c_\phi \quad c_{\theta 1} \leq \theta \leq c_{\theta 2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_\theta r \, d\theta$$

3. Line segment directed towards the origin

$$\theta = c_\theta \quad \phi = c_\phi \quad c_{r1} \leq r \leq c_{r2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_r dr$$

The Differential Surface Vector for Coordinate Systems

- Given that $\overline{ds} = \overline{dl} \times \overline{dm}$, we can determine the differential surface vectors for each of the **three** coordinate systems.



Cartesian

$$\overline{ds_x} = \overline{dy} \times \overline{dz} = \hat{a}_x dydz$$

$$\overline{ds_y} = \overline{dz} \times \overline{dx} = \hat{a}_y dxdz$$

$$\overline{ds_z} = \overline{dx} \times \overline{dy} = \hat{a}_z dxdy$$

It is apparent that these differential surface vectors define a small patch of area on the surface of **flat plane**.

The Differential Surface Vector for Coordinate Systems

Cylindrical

$$\overline{ds_\rho} = \overline{d\phi} \times \overline{dz} = \hat{a}_\rho \rho d\phi dz$$

$$\overline{ds_\phi} = \overline{dz} \times \overline{d\rho} = \hat{a}_\phi d\rho dz$$

$$\overline{ds_z} = \overline{d\rho} \times \overline{d\phi} = \hat{a}_z \rho d\rho d\phi$$

We shall find that $\overline{ds_\rho}$ describes a small patch of area on the surface of a **cylinder**, $\overline{ds_\phi}$ describes a small patch of area on the surface of a **plane**, and $\overline{ds_z}$ again describes a small patch of area on the surface of a flat **plane**.

Spherical

$$\overline{ds_r} = \overline{d\theta} \times \overline{d\phi} = \hat{a}_r r^2 \sin\theta d\theta d\phi$$

$$\overline{ds_\theta} = \overline{d\phi} \times \overline{dr} = \hat{a}_\theta r \sin\theta dr d\phi$$

$$\overline{ds_\phi} = \overline{dr} \times \overline{d\theta} = \hat{a}_\phi r dr d\theta$$

We shall find that $\overline{ds_r}$ describes a small patch of area on the surface of a **sphere**, $\overline{ds_\theta}$ describes a small patch of area on the surface of a **cone**, and $\overline{ds_\phi}$ again describes a small patch of area on the surface of a **plane**.

The Surface S

- Although **S** represents **any** surface, no matter how **complex** or **convoluted**, we will study only **basic** surfaces. In other words, $\overline{ds}$ will correspond to one of the differential surface vectors from Cartesian, cylindrical, or spherical coordinate systems.

- In this class, we will limit ourselves to studying only those surfaces that are formed when we change the location of a point by varying **two** coordinate parameters. In other words, the other coordinate parameters will remain **fixed**.

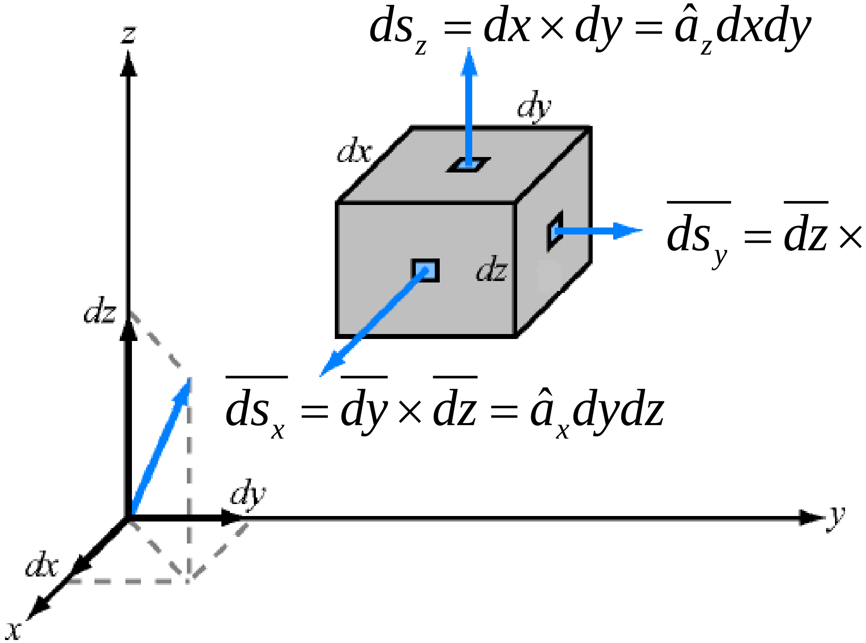
Mathematically, therefore, a surface is described by:

1 equality (e.g., **x=5** **OR** **r = 3**) **AND** **2 inequalities** (e.g., **-1 < y < 5** **and** **-2 < z < 7** **OR** **0 < θ < π/2** **and** **0 < ϕ < π**)

- Therefore, we will need to **explicitly** determine the **differential surface vector** $\overline{ds}$ for each contour.

Cartesian Coordinate Surfaces

$$\overline{ds_z} = \overline{dx} \times \overline{dy} = \hat{a}_z dxdy$$



$$\overline{ds_y} = \overline{dz} \times \overline{dx} = \hat{a}_y dxdz$$

$$\overline{ds_x} = \overline{dy} \times \overline{dz} = \hat{a}_x dydz$$

1. Flat plane parallel to y-z plane.

$$x = c_x \qquad c_{y1} \le y \le c_{y2} \qquad c_{z1} \le z \le c_{z2}$$

$$\overline{ds} = \pm\overline{ds_x} = \pm\hat{a}_x dydz$$

2. Flat plane parallel to x-z plane.

$$y = c_y \qquad c_{x1} \le x \le c_{x2} \qquad c_{z1} \le z \le c_{z2}$$

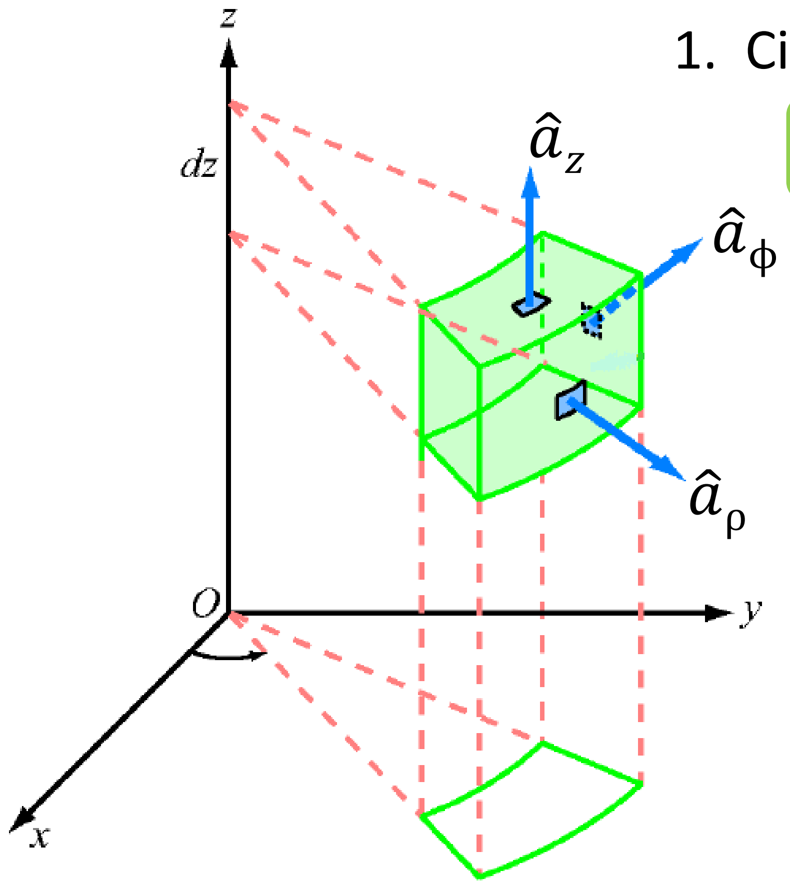
$$\overline{ds} = \pm\overline{ds_y} = \pm\hat{a}_y dxdz$$

3. Flat plane parallel to x-y plane.

$$z = c_z \qquad c_{x1} \le x \le c_{x2} \qquad c_{y1} \le y \le c_{y2}$$

$$\overline{ds} = \pm\overline{ds_y} = \pm\hat{a}_z dxdz$$

Cylindrical Coordinate Surfaces



1. Circular cylinder centered around the z-axis.

$$\rho = c_\rho \qquad c_{\phi 1} \le \phi \le c_{\phi 2} \qquad c_{z1} \le z \le c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\rho} = \pm \hat{a}_\rho \rho \, d\phi \, dz$$

2. Vertical plane extending from the z-axis

$$\phi = c_\phi \qquad c_{\rho 1} \le \rho \le c_{\rho 2} \qquad c_{z1} \le z \le c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi \, d\rho \, dz$$

3. Flat plane parallel to x-y plane.

$$z = c_z \qquad c_{\rho 1} \le \rho \le c_{\rho 2} \qquad c_{\phi 1} \le \phi \le c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_z} = \hat{a}_z \rho \, d\phi \, d\rho$$

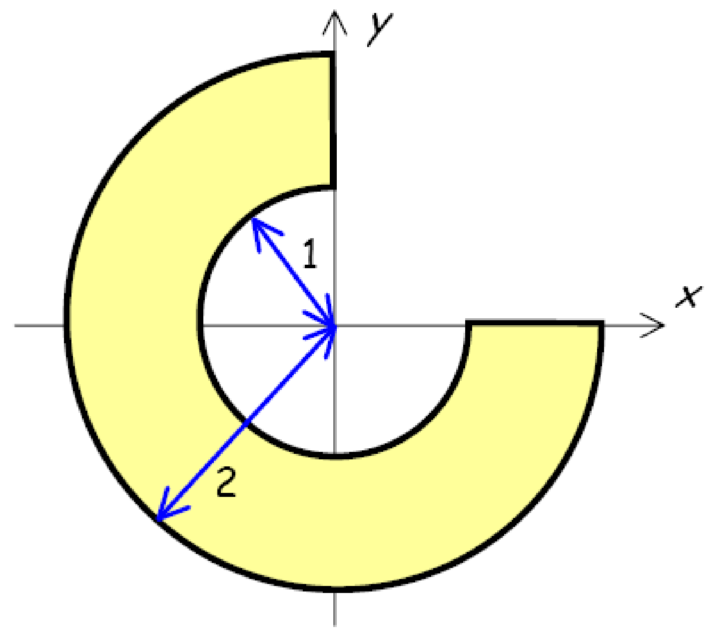
Cylindrical Coordinate Surfaces

Now let's see if **you've** been paying attention! Determine the two **inequalities** that define this flat surface.

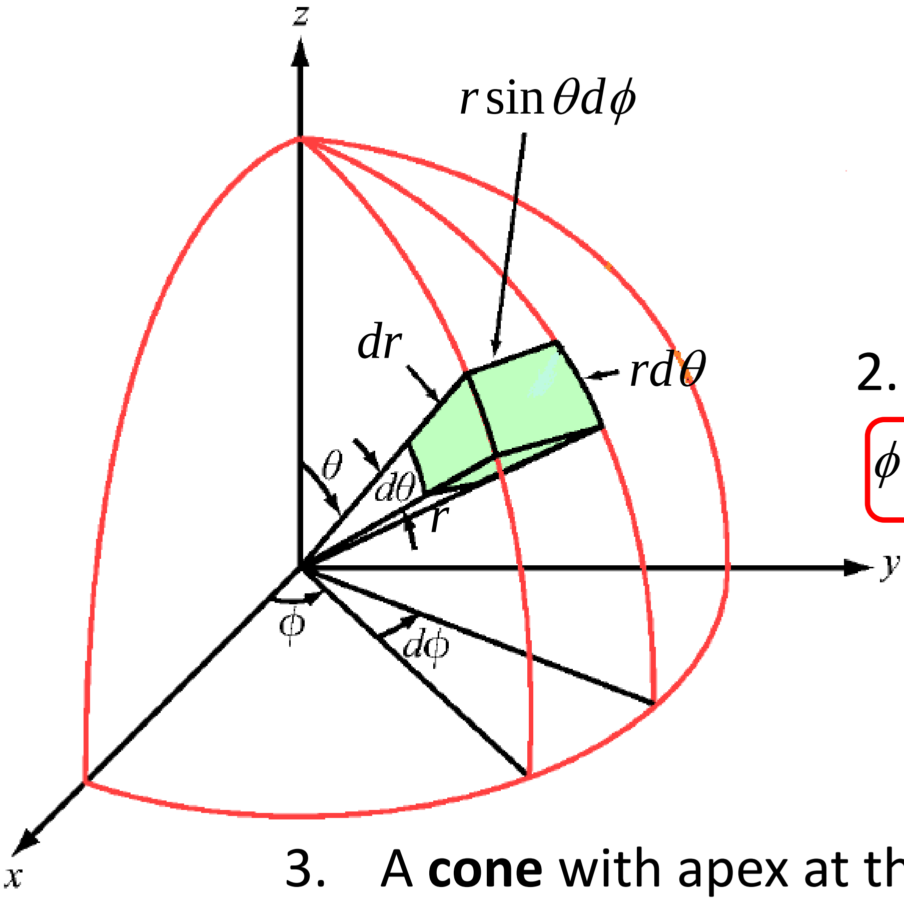
$$z = 0$$

$$\leq \rho \leq$$

$$\leq \phi \leq$$



Spherical Coordinate Surfaces



1. **Sphere** centered at the origin.

$$r = c_r \qquad c_{\theta 1} \le \theta \le c_{\theta 2} \qquad c_{\phi 1} \le \phi \le c_{\phi 2}$$

$$\overline{ds} = \pm\overline{ds_r} = \pm\hat{a}_r r^2 \sin\theta d\theta d\phi$$

2. Vertical **plane** extending from the z-axis

$$\phi = c_\phi \qquad c_{\theta 1} \le \theta \le c_{\theta 2} \qquad c_{r1} \le r \le c_{r2}$$

$$\overline{ds} = \pm\overline{ds_z} = \pm\hat{a}_\phi r dr d\theta$$

3. A **cone** with apex at the origin and aligned with the z-axis

$$\theta = c_\theta \qquad c_{r1} \le r \le c_{r2} \qquad c_{\phi 1} \le \phi \le c_{\phi 2}$$

$$\overline{ds} = \pm\overline{ds_\theta} = \pm\hat{a}_\theta r\sin\theta d\phi dr$$

The Volume V

- As we might expect from our knowledge about how to specify a **point** P (3 equalities), a **contour** C (2 equalities and 1 inequality), and a **surface** S (1 equality and 2 inequalities), a **volume** v is defined by **3 inequalities**.

Cartesian

The inequalities: $c_{x1} \leq x \leq c_{x2}$ $c_{y1} \leq y \leq c_{y2}$ $c_{z1} \leq z \leq c_{z2}$

define a **rectangular volume**, whose sides are parallel to the x-y, y-z, and x-z planes.

- The differential volume **dv** used for constructing this Cartesian volume is:

$$dv = dxdydz$$

$$\therefore v = \int_{c_{x1}}^{c_{x2}} \int_{c_{y1}}^{c_{y2}} \int_{c_{z1}}^{c_{z2}} dxdydz$$

The Volume V

Cylindrical

The inequalities: $c_{\rho 1} \leq \rho \leq c_{\rho 2}$ $c_{\phi 1} \leq \phi \leq c_{\phi 2}$ $c_{z1} \leq z \leq c_{z2}$

defines a **cylinder,** or some **subsection** thereof (e.g. a **tube**!).

- The differential volume **dv** is used for constructing this cylindrical volume is:

$$dv = \rho d\rho d\phi dz$$

$$\therefore v = \int_{c_{\rho 1}}^{c_{\rho 2}} \int_{c_{\phi 1}}^{c_{\phi 2}} \int_{c_{z1}}^{c_{z2}} \rho d\rho d\phi dz$$

Spherical

The inequalities: $c_{r1} \leq r \leq c_{r2}$ $c_{\theta 1} \leq \theta \leq c_{\theta 2}$ $c_{\phi 1} \leq \phi \leq c_{\phi 2}$

defines a **sphere**, or some subsection thereof (e.g., an "**orange slice**" !).

- The differential volume **dv** used for constructing this spherical volume is:

$$dv = r^2 \sin\theta \, dr d\theta d\phi$$

$$\therefore v = \int_{c_{r1}}^{c_{r2}} \int_{c_{\theta 1}}^{c_{\theta 2}} \int_{c_{\phi 1}}^{c_{\phi 2}} \rho d\rho d\phi dz$$

Example: The Volume Integral

Let's evaluate the **volume** integral: $\iiint_v g(\bar{r})dv$

where g($\bar{r}$) =1 and the volume *v* is a **sphere** with radius R.

In other words, the volume *v* is described for:

$$0 \le r \le R$$
$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$

- Therefore we use for the **differential** volume **dv**:

$$dv = \overline{dr}.\overline{d\theta} \times \overline{d\phi} = r^2 \sin\theta dr d\theta d\phi$$

- Therefore: $\iiint_v g(\bar{r})dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta dr d\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{R} r^2 dr = (2\pi)(2)\left(\frac{R^3}{3}\right)$

$$\therefore \iiint_v g(\bar{r})dv = \frac{4\pi R^3}{3}$$

Example: The Volume Integral

Q: So what's the volume integral even good for?

A: Generally speaking, the scalar function $g(\bar{r})$ will be a density function, with units of **things/unit volume**. Integrating $g(\bar{r})$ with the volume integral provides us the **number of things** within the space v!

For example, let's say $g(\bar{r})$ describes the **density** of a big **swarm of insects**, using units of **insects/m³** (i.e., insects are the **things**).

Note that $g(\bar{r})$ must indeed be a **function** of position, as the density of insects changes at different locations throughout the swarm.

Example: The Volume Integral

- Now, say we want to know the total number of insects within the swarm, which occupies some space v. We can determine this by simply applying the volume integral!

$$\text{number of insects in swarm} = \iiint_v g(\bar{r})\,dv$$

where space v completely encloses the insect swarm.