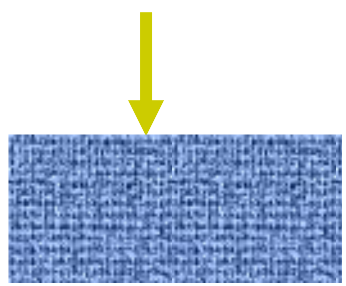
Lecture – 23 Date: 11.04.2015

- Incidence, Reflection, and Transmission of Plane Waves

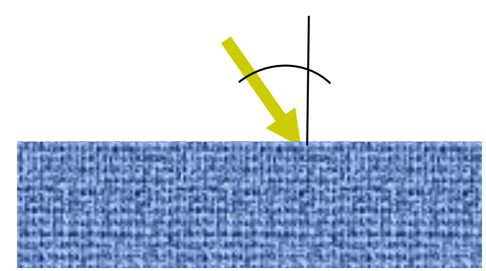
Wave Incidence

- For many applications, [such as fiber optics, wire line transmission, wireless transmission], it's necessary to know what happens to a wave when it meets a different medium.
    - How much is *transmitted*?
    - How much is *reflected* back?

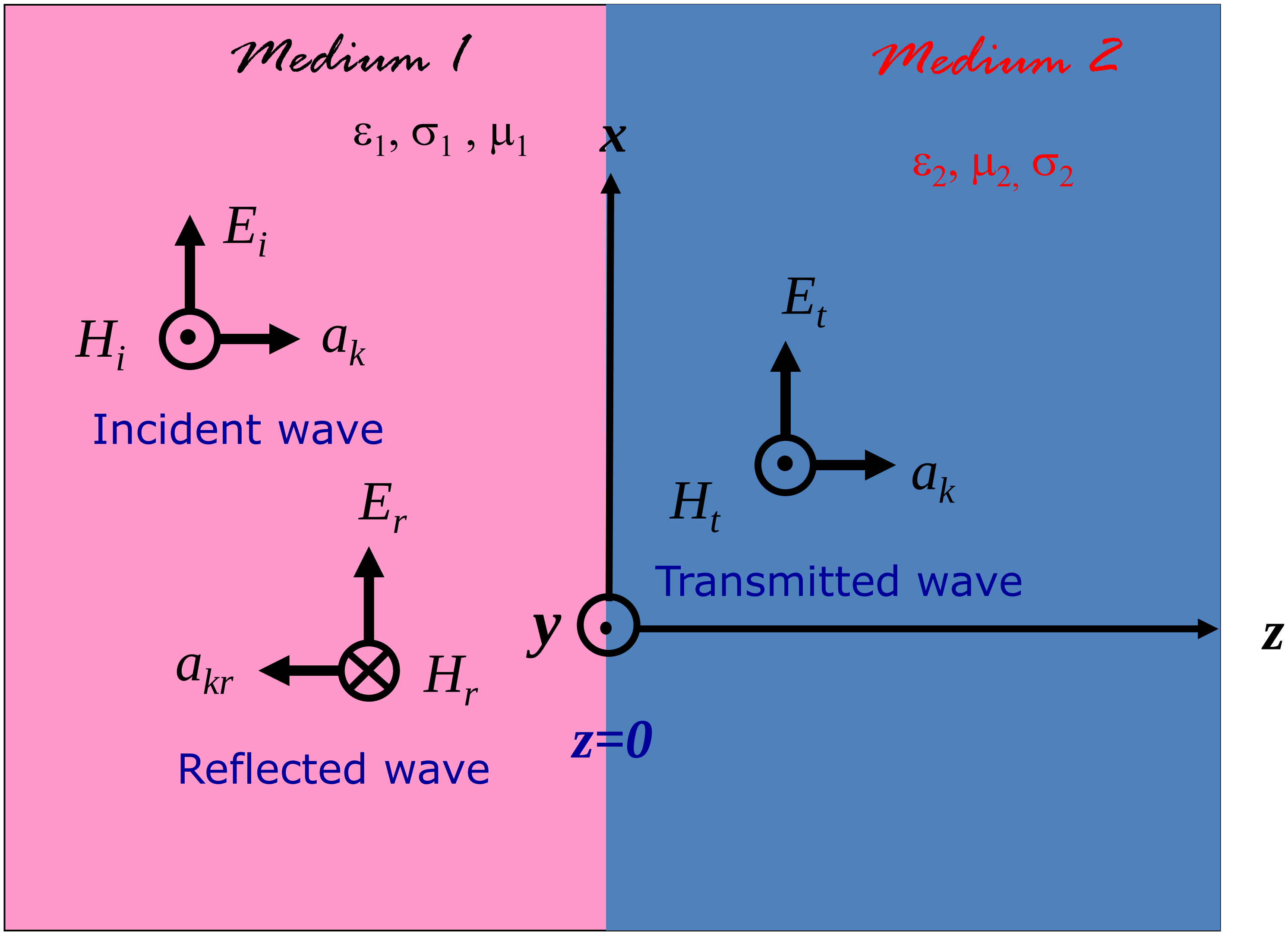
➢ **Normal incidence:** *Wave arrives at 0º from normal*



➢ **Oblique incidence:** *Wave arrives at another angle*



Reflection at Normal Incidence

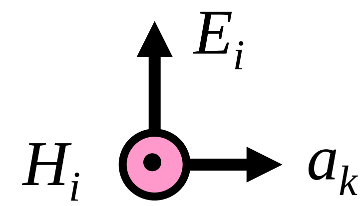


Reflection at Normal Incidence (contd.)

Incident wave

$$\vec{E}_{is}(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x$$

$$\vec{H}_{is}(z) = H_{io} e^{-\gamma_1 z} \hat{a}_y = \frac{E_{io}}{\eta_1} e^{-\gamma_1 z} \hat{a}_y$$

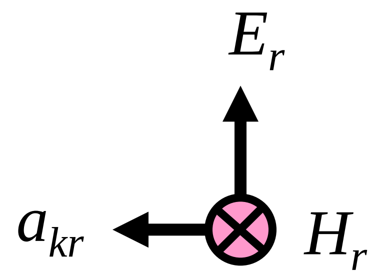


Incident wave

Reflected wave

$$\vec{E}_{rs}(z) = E_{ro} e^{\gamma_1 z} \hat{a}_x$$

$$\vec{H}_{rs}(z) = H_{ro} e^{\gamma_1 z} \left(-\hat{a}_y\right) = -\frac{E_{io}}{\eta_1} e^{\gamma_1 z} \hat{a}_y$$

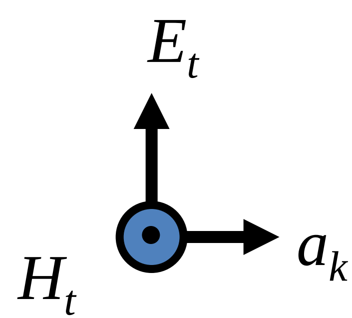


Reflected wave

Transmitted wave

$$\vec{E}_{ts}(z) = E_{to} e^{-\gamma_2 z} \hat{a}_x$$

$$\vec{H}_{ts}(z) = H_{to} e^{-\gamma_2 z} \hat{a}_y = \frac{E_{to}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y$$



Transmitted wave

Reflection at Normal Incidence (contd.)

- The total waves in medium 1: $\vec{E}_1 = \vec{E}_i + \vec{E}_r$ $\vec{H}_1 = \vec{H}_i + \vec{H}_r$
- The total waves in medium 2: $\vec{E}_2 = \vec{E}_t$ $\vec{H}_2 = \vec{H}_t$
- At the interface $z = 0$, the boundary conditions require that the tangential components of $\vec{E}$ and $\vec{H}$ fields must be continuous.
- Since the waves are transverse, $\vec{E}$ and $\vec{H}$ fields are entirely tangential to the surface.
- Therefore, at $z = 0$: $\vec{E}_{1tan} = \vec{E}_{2tan}$ and $\vec{H}_{1tan} = \vec{H}_{2tan}$ imply that -

$$\vec{E}_{is}(0) + \vec{E}_{rs}(0) = \vec{E}_{ts}(0) \quad \Rightarrow \quad E_{io} + E_{ro} = E_{to}$$

$$\vec{H}_{is}(0) + \vec{H}_{rs}(0) = \vec{H}_{ts}(0) \quad \Rightarrow \quad \frac{1}{\eta_1}\left(E_{io} + E_{ro}\right) = \frac{E_{to}}{\eta_2}$$

Reflection at Normal Incidence (contd.)

- Simplification results in:

$$E_{ro} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{io}$$

$$E_{to} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{io}$$

These expressions aid us in the definitions of $reflection\ coefficient\ \Gamma$ and $transmission\ coefficient\ \tau$.

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \Rightarrow E_{ro} = \Gamma E_{io}$$

$$\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1} \Rightarrow E_{to} = \tau E_{io}$$

- It is important to note that:
  - $1 + \Gamma = \tau$
  - Both Γ and τ are dimensionless and may be complex
  - $0 \leq |\Gamma| \leq 1$

Reflection at Normal Incidence (contd.)

- Therefore the total fields in the two medium are:

$$\vec{E}_{1s} = E_{io}\left[e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right]\hat{a}_x$$

$$\vec{H}_{1s} = \frac{E_{io}}{\eta_1}\left[e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}\right]\hat{a}_y$$

$$\vec{E}_{2s} = E_{io}\tau e^{-\gamma_2 z}\hat{a}_x$$

$$\vec{H}_{2s} = \frac{E_{io}}{\eta_2}\tau e^{-\gamma_2 z}\hat{a}_y$$

- Special Cases:
  - $\eta_1 = \eta_2$ $\Gamma = 0$ $\tau = 1$ (total transmission, no reflection)
  - $\eta_1 = 0$ $\Gamma = 1$ $\tau = 2$ (total reflection, no inversion of $\vec{E}$)
  - $\eta_2 = 0$ $\Gamma = -1$ $\tau = 0$ (total reflection, inversion of $\vec{E}$)

Reflection at Normal Incidence (contd.)

Special Case – I

- Medium 1: perfect dielectric (lossless): $\sigma_1 = 0$, $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$, $\alpha_1 = 0, \gamma = j\beta_1$
- Medium 2: perfect conductor: $\sigma_2 = \infty$, $\eta_2 = 0$, $\alpha_2 = \beta_2 = \infty$

$\eta_2 = 0$ $\Gamma = -1$ $\tau = 0$ (total reflection, inversion of $\vec{E}$

$$\vec{E}_{1s} = \vec{E}_{is} + \vec{E}_{rs}$$

$$\vec{E}_{1s} = -E_{io}\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\hat{a}_x$$

$$\vec{E}_{1s} = -2jE_{io}\sin\beta_1 z\hat{a}_x$$

$$\vec{H}_{1s} = \frac{E_{io}}{\eta_1}\left[e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}\right]\hat{a}_y$$

$$\vec{H}_{1s} = \frac{2E_{io}}{\eta_1}\cos\beta_1 z\hat{a}_y$$

$$\vec{E}_{2s} = E_{io}\tau e^{-\gamma_2 z}\hat{a}_x = 0$$

$$\vec{H}_{2s} = \frac{E_{io}}{\eta_2}\tau e^{-\gamma_2 z}\hat{a}_y = 0$$

Reflection at Normal Incidence (contd.)

- The instantaneous electric field: $\vec{E}_1(z,t) = \mathrm{Re}\left(\vec{E}_{1s} e^{j\omega t}\right) = 2E_{io} \sin \beta_1 z \sin \omega t \hat{a}_x$

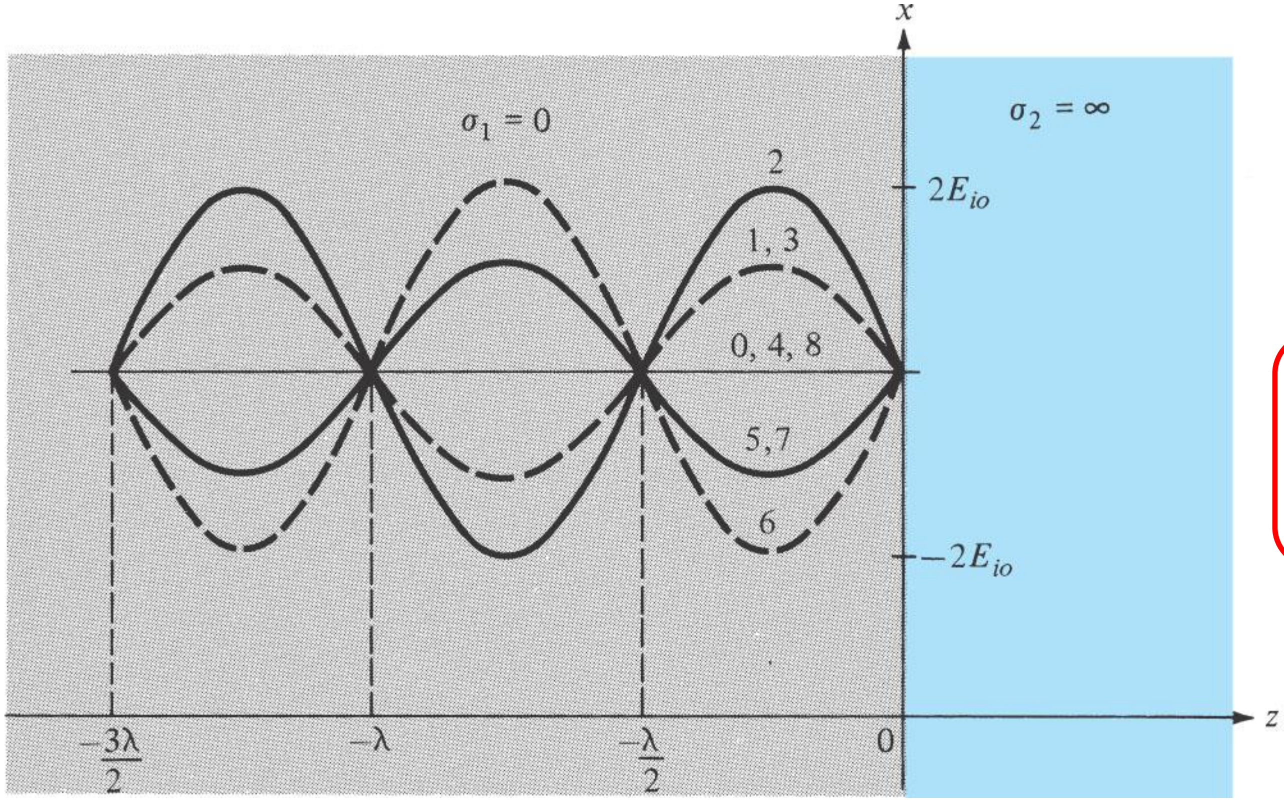
- Similar steps result in: $\vec{H}_1(z,t) = \frac{2E_{io}}{\eta_1} \cos \beta_1 z \cos \omega t \hat{a}_y$

Note that the position dependence of the instantaneous electric and magnetic fields is not a function of time → standing wave!!!

It is expected considering that there is total reflection and in a lossless dielectric the waves consist of two travelling waves ($\vec{E}_i$ and $\vec{E}_r$) of equal amplitudes but in opposite directions.

Reflection at Normal Incidence (contd.)

$$\vec{E}_1(z,t) = \text{Re}\left(\vec{E}_{1s}e^{j\omega t}\right) = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x$$



The wave doesn't travel but oscillate

Standing waves $\vec{E} = 2E_{io}sin\beta_1 zsin\omega t\hat{a}_x$. The curves 0, 1, 2, 3, 4, . . ., are, respectively, at times $t = 0$, $T/8$, $T/4$, $3T/8$, $T/2$, . . . ; $\lambda = 2\pi/\beta_1$.

Reflection at Normal Incidence (contd.)

- The locations of the minimums (nulls) and maximums (peaks) in the standing wave electric field pattern are found by:

$$\left|\vec{E}_1(z,t)\right|_{\min} = 0 \quad \xrightarrow{\text{when}} \quad \sin\beta_1 z = 0 \quad \Rightarrow \quad \beta_1(-z) = n\pi$$

$$z = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}$$

$$\left|\vec{E}_1(z,t)\right|_{\max} = 2E_{io} \quad \xrightarrow{\text{when}} \quad \sin\beta_1 z = 1 \quad \Rightarrow \quad \beta_1(-z) = (2n+1)\frac{\pi}{2}$$

$$z = -\frac{(2n+1)\pi}{2\left(\frac{2\pi}{\lambda_1}\right)} = -\frac{(2n+1)}{4}\lambda_1$$

Reflection at Normal Incidence (contd.)

Special Case – II: Two Perfect Dielectrics

- Medium 1: perfect dielectric (lossless): $\sigma_1 = 0$, $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$, $\alpha_1 = 0, \gamma = j\beta_1$
- Medium 2: perfect dielectric (lossless): $\sigma_2 = 0$, $\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$, $\alpha_2 = 0, \gamma = j\beta_2$

If $\eta_2 > \eta_1$: $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ **$0 < \Gamma < 1$**; $\tau = \frac{2\eta_2}{\eta_2 + \eta_1}$ **$1 < \tau < 2$**

If $\eta_2 < \eta_1$: $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ **$-1 < \Gamma < 0$**; $\tau = \frac{2\eta_2}{\eta_2 + \eta_1}$ **$0 < \tau < 1$**

Reflection at Normal Incidence (contd.)

- Therefore:

$$\vec{E}_{1s} = E_{io}(e^{-j\gamma_1 z} + \Gamma e^{+j\gamma_1 z})\hat{a}_x = E_{io}e^{-j\beta_1 z}(1+\Gamma e^{+2j\beta_1 z})\hat{a}_x$$

$$\vec{H}_{1s} = \frac{E_{io}}{\eta_1}(e^{-j\gamma_1 z} - \Gamma e^{+j\gamma_1 z})\hat{a}_y = \frac{E_{io}}{\eta_1}(1-\Gamma e^{+2j\beta_1 z})\hat{a}_y$$

$$\vec{E}_{2s} = E_{io}\tau e^{-j\gamma_2 z}\hat{a}_x = E_{io}\tau e^{-j\beta_2 z}\hat{a}_x$$

$$\vec{H}_{2s} = \frac{E_{io}}{\eta_2}e^{-j\gamma_2 z}\hat{a}_y = \frac{E_{io}}{\eta_2}e^{-j\beta_2 z}\hat{a}_y$$

Standing wave exists only in medium 1.

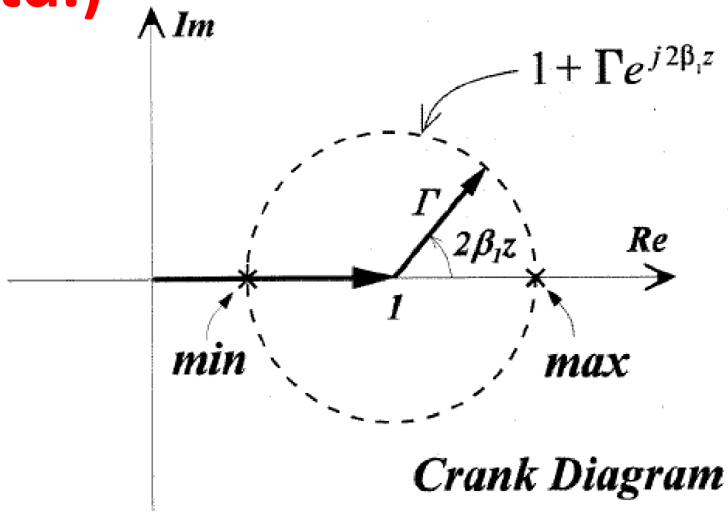
- The magnitude of the electric field in medium 1 can be analyzed to determine the locations of the maximum and minimum values of the electric field standing wave pattern.

$$\left|\vec{E}_{1s}\right| = E_{io}\left|(1+\Gamma e^{+2j\beta_1 z})\right|$$

$$\left|(1+\Gamma e^{+2j\beta_1 z})\right| = \left|1\angle 0^\circ + \Gamma\angle 2\beta_1 z\right|$$

This can be described in the complex plane using *crank diagram*

Reflection at Normal Incidence (contd.)



- The distance from the origin to the respective point on the circle in the crank diagram represents the magnitude of:

$$1+\Gamma e^{+2j\beta_1 z}$$

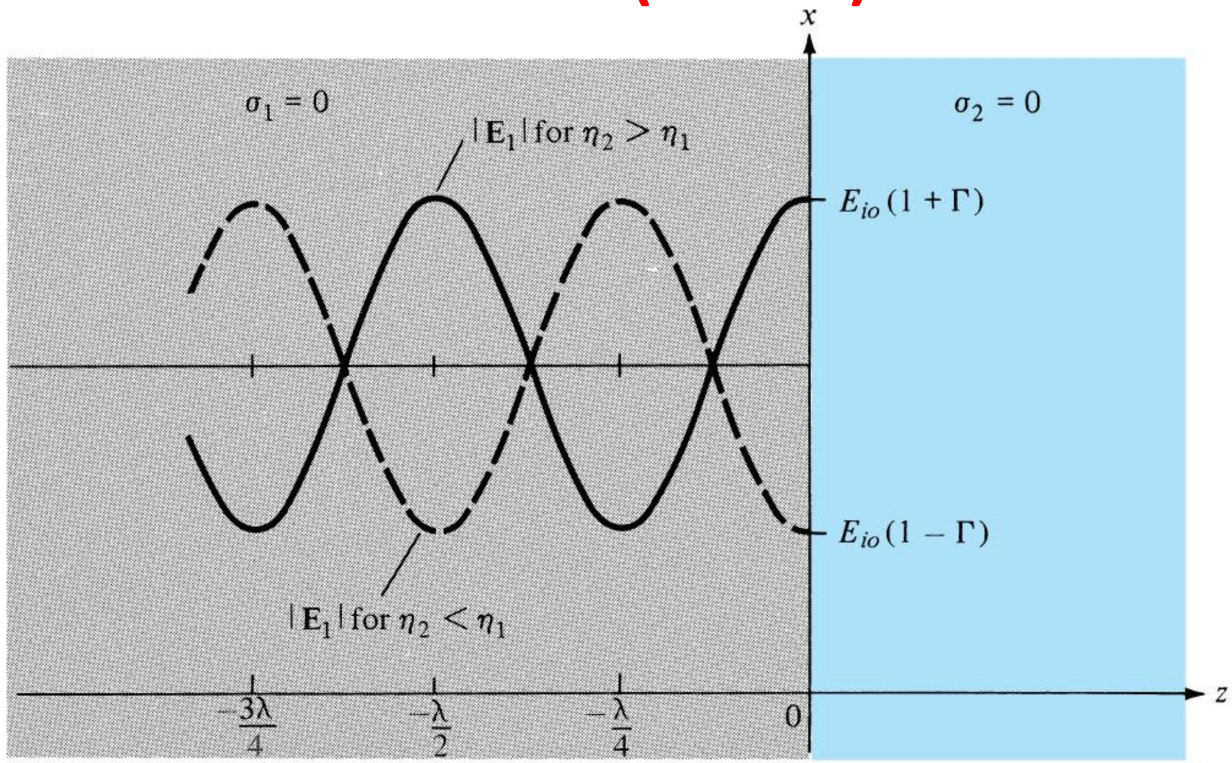
- If $\eta_2 > \eta_1$, (Γ is positive), then the maximum and minimum of the function are:

$$\left|\left(1+\Gamma e^{+2j\beta_1 z}\right)\right|_{\max} = 1+\Gamma \quad \text{when} \quad 2\beta_1(-z) = n(2\pi) \Rightarrow z = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}$$

$$\left|\left(1+\Gamma e^{+2j\beta_1 z}\right)\right|_{\min} = 1-\Gamma \quad \text{when} \quad 2\beta_1(-z) = (2n+1)\pi \Rightarrow z = -\frac{(2n+1)\pi}{\beta_1} = -\frac{(2n+1)}{4}\lambda_1$$

$$\left|\vec{E}_{1s}\right|_{\max} = E_{io}\left(1+|\Gamma|\right) \qquad \left|\vec{E}_{1s}\right|_{\min} = E_{io}\left(1-|\Gamma|\right)$$

Reflection at Normal Incidence (contd.)



- If $\eta_2 > \eta_1$ then Γ is negative.
- The positions of the maximums and minimums are reversed, but the equations for the maximum and minimum electric field magnitude in terms of $|\Gamma|$ are the same.

$$\left|\vec{E}_{1s}\right|_{max} = E_{io}\left(1+|\Gamma|\right)$$

$$\left|\vec{E}_{1s}\right|_{min} = E_{io}\left(1-|\Gamma|\right)$$

Reflection at Normal Incidence (contd.)

- The standing wave ratio (s) in a medium where standing waves exist is defined as the ratio of the maximum electric field magnitude to the minimum electric field magnitude.

$$s = \frac{\left|\vec{E}_{1s}\right|_{\max}}{\left|\vec{E}_{1s}\right|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

- The standing wave ratio (purely real) ranges from a minimum value of 1 (no reflection, $|\Gamma| = 0$) to ∞ (total reflection, $|\Gamma| = 1$).
- The standing wave ratio is sometimes defined in dB as:

$$s(dB) = 20\log_{10} s$$

Example – 1

- A uniform plane wave in air is normally incident on an infinite lossless dielectric material having $\varepsilon = 3\varepsilon_0$ and $\mu = \mu_0$. If the incident wave is $\vec{E}_{is} = 10\cos(\omega t - z)\hat{a}_y\ V/m$, find (a) ω and λ of the waves in both the mediums, (b) $\vec{H}_{is}$, (c) Γ and τ, (d) the total electric field and time-average power in both mediums.

Example – 1 (contd.)

Medium 1 [z < 0] : Air

$(\mu_1 = \mu_0, \varepsilon_1 = \varepsilon_0, \sigma_1 = 0)$

Medium 2 [z > 0] : Dielectric

$(\mu_2 = \mu_0, \varepsilon_2 = 3\varepsilon_0, \sigma_2 = 0)$

$$\alpha_1 = 0, \beta_1 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c}$$

$$\gamma_1 = j\beta_1$$

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0$$

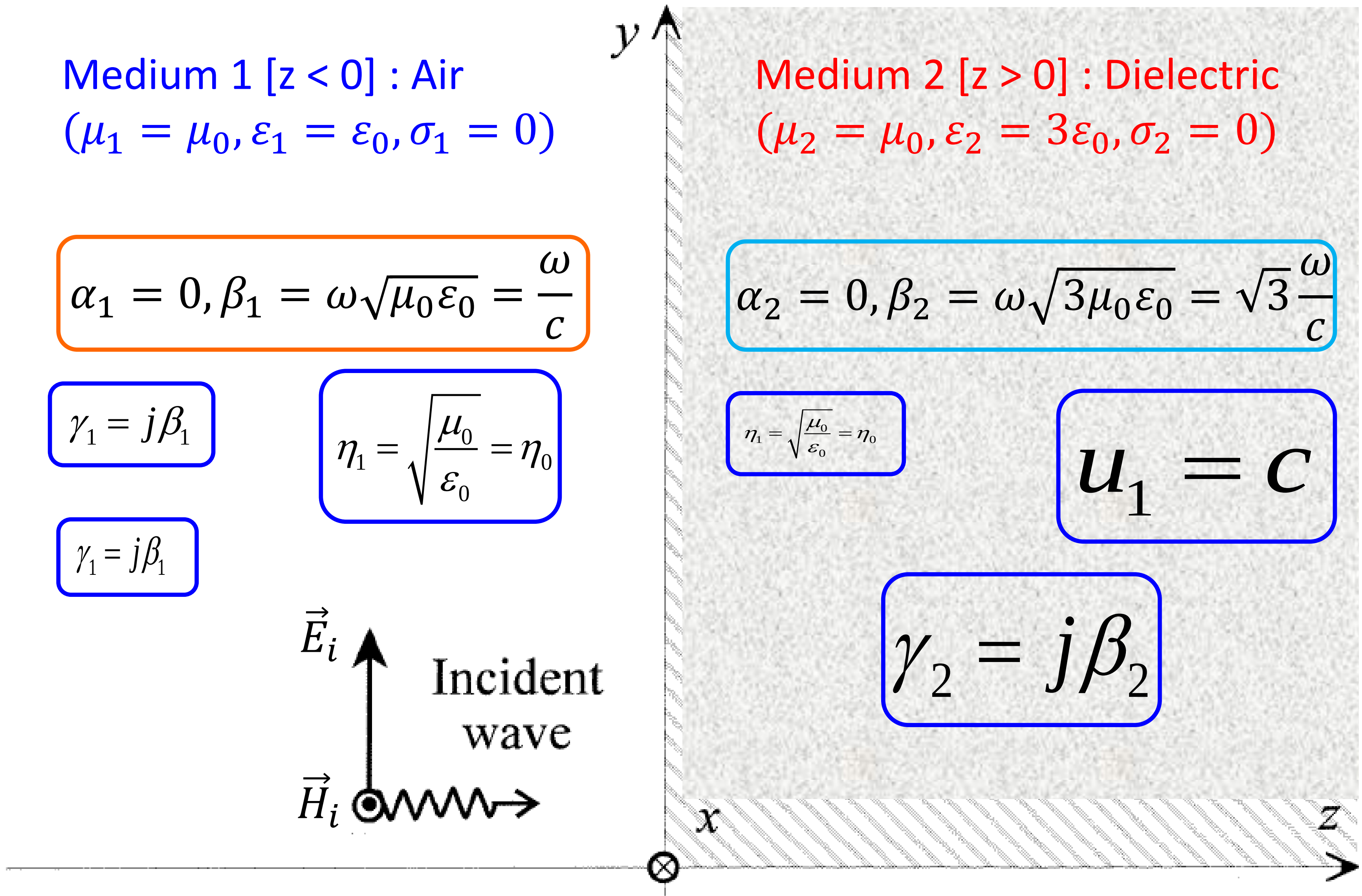
$$\gamma_1 = j\beta_1$$

$$\alpha_2 = 0, \beta_2 = \omega\sqrt{3\mu_0\varepsilon_0} = \sqrt{3}\frac{\omega}{c}$$

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0$$

$$u_1 = c$$

$$\gamma_2 = j\beta_2$$



Example – 1 (contd.)

(a) $\vec{E}_{is} = 10\cos(\omega t - z)\hat{a}_y$ → $\vec{E}_{is} = 10e^{-jz}\hat{a}_y = E_o e^{-j\beta_1 z}\hat{a}_y$ → $E_{io} = 10 \quad \beta_1 = 1$

$$\beta_1 = \frac{2\pi}{\lambda_1} = \frac{\omega}{u_1} = \frac{\omega}{c} = 1 \text{ rad/m}$$

$$\beta_2 = \frac{2\pi}{\lambda_2} = \frac{\omega}{u_2} = \sqrt{3}\frac{\omega}{c} = \sqrt{3}\beta_1 \quad \text{rad/m}$$

$$\lambda_1 = \frac{2\pi}{\beta_1} = 2\pi = 6.28 \text{ m}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = 3.63 \text{ m}$$

$\omega = \beta_1 u_1 = \beta_2 u_2 = 3\times10^8$ rad/sec → $47.8\ MHz$

(b) $$\vec{H}_{is} = \frac{E_{io}}{\eta_0}e^{-j\beta_1 z}(-\hat{a}_x) = -\frac{10}{377}e^{-jz}\hat{a}_x = -0.0266e^{-jz}\hat{a}_x$$

$$\vec{H}_{is} = \text{Re}\left\{-0.0266e^{-jz}\hat{a}_x\right\} = -0.0266\cos(\omega t - z)\hat{a}_x \text{ A/m}$$

Example – 1 (contd.)

(c)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \Rightarrow \Gamma = \frac{\frac{\eta_0}{\sqrt{3}} - \eta_0}{\frac{\eta_0}{\sqrt{3}} + \eta_0} \Rightarrow \Gamma = -0.268 \qquad \tau = 1 + \Gamma = 0.732$$

(d)

$$\vec{E}_{1s} = E_{io}(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z})\hat{a}_y \Rightarrow \vec{E}_1 = [10\cos(\omega t - z) - 2.68\cos(\omega t + z)]\hat{a}_y \quad \text{V/m}$$

$$\vec{E}_{2s} = \tau E_o e^{-j\beta_2 z}\hat{a}_y \Rightarrow \vec{E}_2 = \left[7.32\cos(\omega t - \sqrt{3}z)\right]\hat{a}_y \quad \text{V/m}$$

The time average power density in medium 1 is due to the +z directed incident wave and the –z directed reflected wave. The time-average power density in medium 2 is due to the +z directed transmitted wave.

$$\vec{P}_{ave,1} = \frac{\left|\vec{E}_{is}\right|^2}{2\eta_1}\hat{a}_z + \frac{\left|\vec{E}_{rs}\right|^2}{2\eta_1}(-\hat{a}_z)$$