Lecture – 22 Date: 09.04.2015

- Plane Wave in Good Conductors (contd.)
- Power and Poynting Vector
- Wave Polarization

Plane Waves in Good Conductors

- For a good conductor: $\sigma \simeq \infty, \ \varepsilon = \varepsilon_0, \quad \mu = \mu_0\mu_r.$
- Therefore: $\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$ → $\alpha = \beta = \sqrt{\pi f \mu\sigma}$

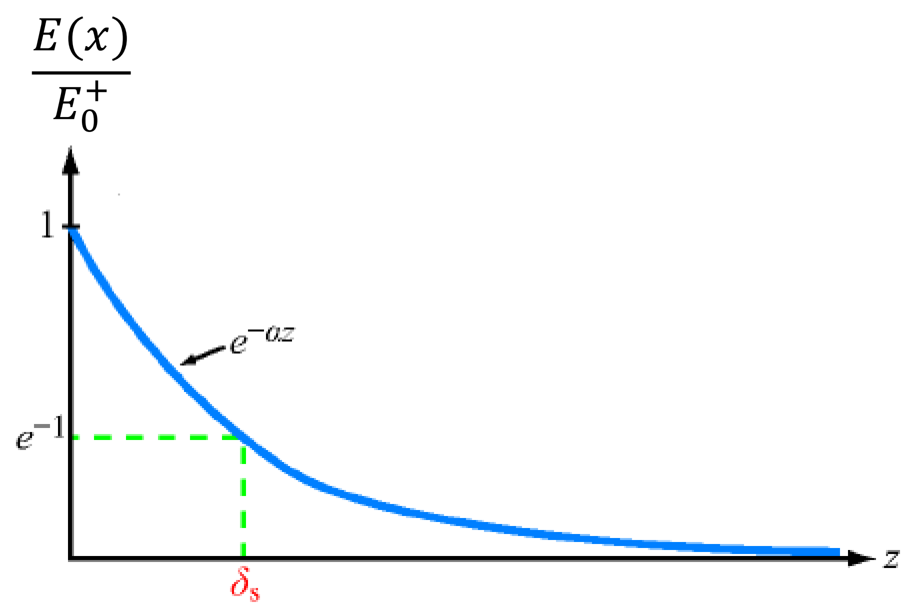
$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$\lambda = \frac{2\pi}{\beta}$$

- Furthermore: $\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45°$ ← Thus $\vec{E}$ leads $\vec{H}$ by 45°
- If: $\vec{E} = E_0^+ e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_x$
- Then: $\vec{H} = \frac{E_0^+}{\sqrt{\omega\mu/\sigma}} e^{-\alpha z}\cos(\omega t - \beta z - 45°)\hat{a}_y$

Plane Waves in Good Conductors (contd.)

- The amplitude of $\vec{E}$ or $\vec{H}$ is attenuated by the factor $e^{-\alpha z}$ as it travels along the medium.
- The rate of attenuation in a good conductor is characterized by distance called $skin\ depth\ (\delta) \leftrightarrow$ a distance over which plane wave is attenuated by a factor e^{-1} (about 37% of the original value) in a good conductor.



$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

For a partially conducting medium, the skin depth can be considerably large.

- For a good conductor: $$\eta = \frac{1}{\sigma\delta}\sqrt{2}e^{j\pi/4} = \frac{1+j}{\sigma\delta}$$

- For good conductors, $\alpha = \beta = \frac{1}{\delta}$, therefore: $$\vec{E} = E_0^+ e^{-z/\delta}\cos(\omega t - \frac{z}{\delta})\hat{a}_x$$

Example – 1

- A plane wave $\vec{E} = E_0 \cos(\omega t - \beta z)\hat{a}_x$ is incident on a good conductor at $z \geq 0$. Find the current density in the conductor.

- Since, $\vec{J} = \sigma \vec{E}$, the wave equation changes to:

$$\boxed{\nabla^2 \vec{J}_s - \gamma^2 \vec{J}_s = 0}$$

- Furthermore, the incident $\vec{E}$ has only an x-component that varies with z. Therefore, $\vec{J} = J_x(z,t)\hat{a}_x$ and:

$$\boxed{\frac{d^2}{dz^2} J_{sx} - \gamma^2 J_{sx} = 0}$$

The solution is:

$$\boxed{J_{sx} = Ae^{-\gamma z} + Be^{+\gamma z}}$$

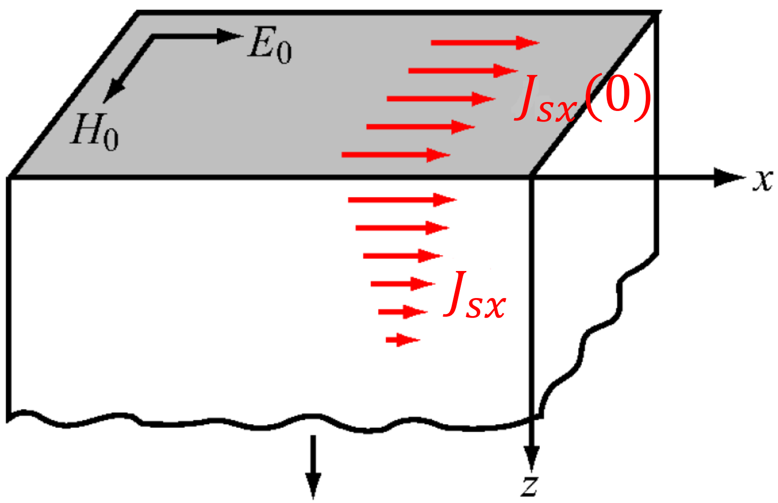
Example – 1 (contd.)

- B is zero considering that wave is propagating in $+z\ direction$.
- Furthermore, in a good conductor $\sigma \gg \omega\varepsilon$ so that $\alpha = \beta = \frac{1}{\delta}$. Therefore,

$$\gamma = \alpha + j\beta = \alpha(1+j) = \frac{(1+j)}{\delta}$$

- Therefore: $J_{sx} = Ae^{-z(1+j)/\delta}$ $\Rightarrow$ $J_{sx} = J_{sx}(0)e^{-z(1+j)/\delta}$

Where, $J_{sx}(0)$ is the current density on the conductor surface.



This depicts the scenario

Example – 2

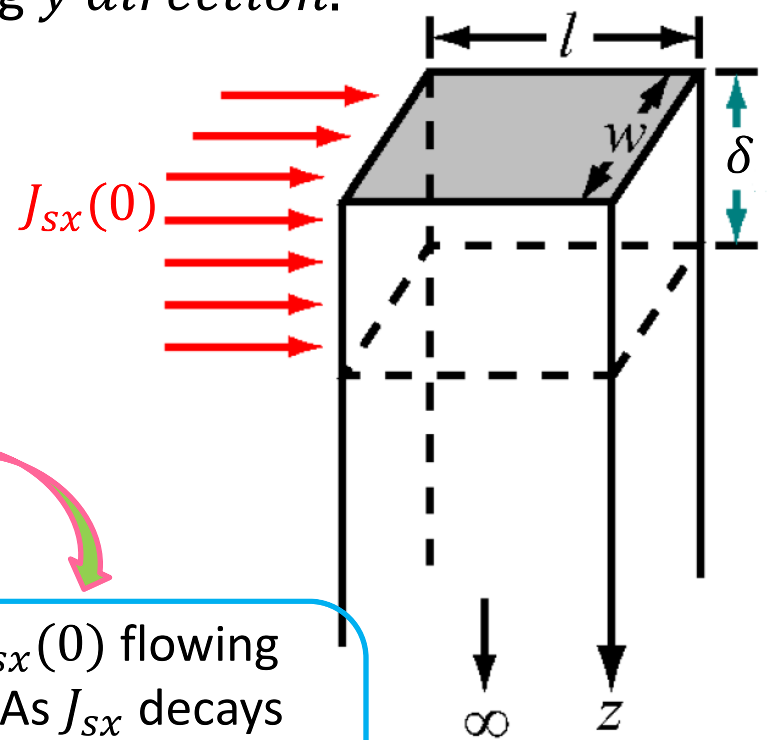
- Given the current density of previous problem $J_{sx} = J_{sx}(0)e^{-z(1+j)/\delta}$, find the magnitude of total current through a strip of the conductor of infinite depth along $z\ direction$ and width w along $y\ direction$.

$$I_s = \int_0^w \int_0^\infty J_{sx}\,dy\,dz$$

$$I_s = J_{sx}(0)\int_0^w dy \int_0^\infty e^{-z(1+j)/\delta}\,dz$$

$$I_s = \frac{J_{sx}(0)w\delta}{1+j}$$

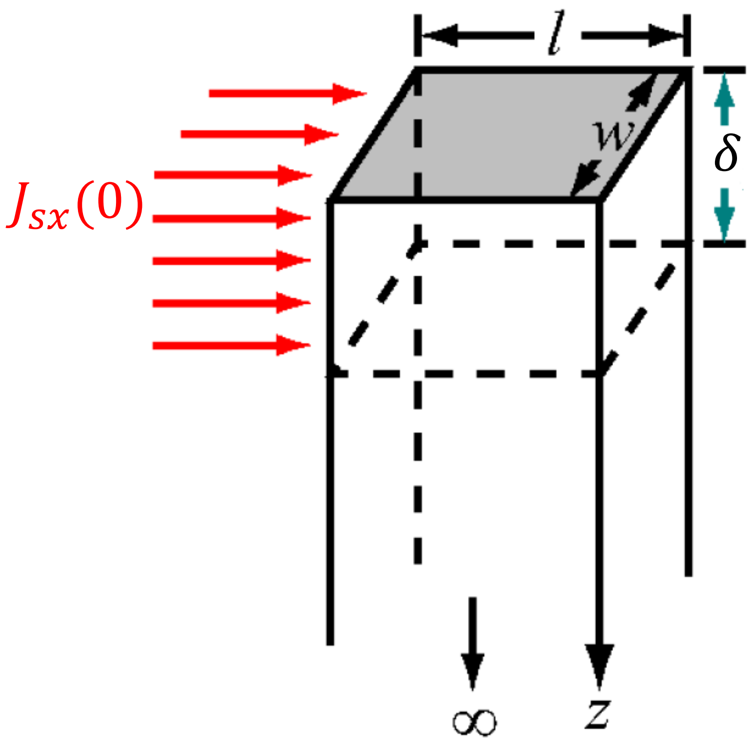
$$|I_s| = \frac{J_{sx}(0)w\delta}{\sqrt{2}}$$



It actually resembles a uniform current density $J_{sx}(0)$ flowing through a thin surface width w and depth δ. $\rightarrow$ As J_{sx} decays exponentially with depth z, a conductor of finite thickness d can be considered electrically equivalent to one of infinite depth as long as d exceeds a few skin depth (δ).

Example – 3

- In the previous example, what is the voltage across a length l at the surface. What is the impedance of the conductor in consideration?



$$\boxed{V_s = E_0 l = \frac{J_{sx}(0)}{\sigma} l}$$

$$\boxed{Z = \frac{V_s}{I_s} = \frac{J_{sx}(0)}{\sigma} l \times \frac{1+j}{J_{sx}(0) w\delta}}$$

$$\boxed{Z = \frac{V_s}{I_s} = \frac{1+j}{\sigma\delta} \times \frac{l}{w}}$$

$$\boxed{Z = Z_s \frac{l}{w}}$$

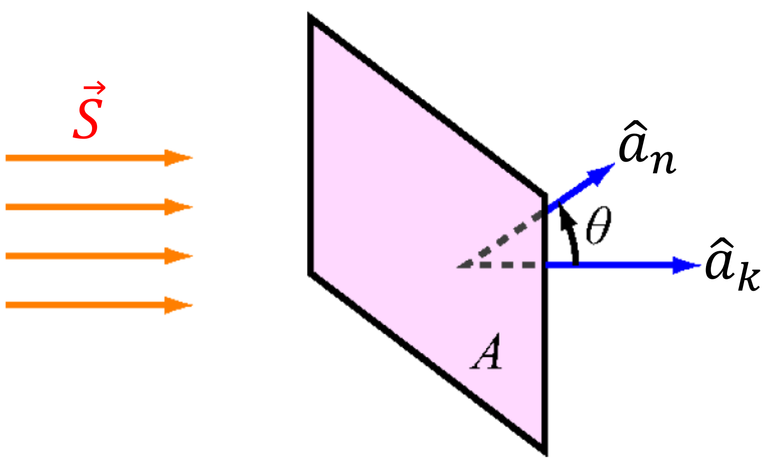
Z_s is surface impedance and the real part of this is called ac resistance.

Power and Poynting Vector

- For any wave with an electric field $\vec{E}$ and magnetic field $\vec{H}$, the direction of wave propagation is also the direction of power per unit area (or power density) carried by the wave. It is represented by $Poynting\ Vector\ \vec{S}$.

$$\vec{S} = \vec{E} \times \vec{H} \qquad W/m^2$$

Instantaneous Poynting Vector – direction and density of power flow at a point



- The total power flowing through this aperture is:

$$P = \int_A \vec{S}.\hat{a}_n dA = SAcos\theta$$

Power and Poynting Vector (contd.)

- Except for the fact that units of $\vec{S}$ are per unit area, the $Poynting\ Vector$ is the vector analogue of the scalar expression for the instantaneous power $P(z,t)$ flowing through a transmission line:

$$P(z,t) = v(z,t)i(z,t)$$

From LC we can recall

$$P_{av}(z) = \frac{1}{2}\mathrm{Re}\left[V_s(z)I_s^*(z)\right]$$

- In a similar manner, power density (W/m^2) associated with a time-harmonic EM field in terms of $\vec{E}$ and $\vec{H}$ phasors is:

$$\vec{P}_{ave} = \frac{1}{2}\mathrm{Re}\left[\vec{E}_s \times \vec{H}_s^*\right]$$

- The total time-average real power passing through a given surface A is:

$$\vec{P}_{ave} = \frac{1}{2}\mathrm{Re}\left[\vec{E}_s \times \vec{H}_s^*\right]$$

Example – 4

- Determine expressions for the time-average power density for an EM plane wave in terms of electric field only and magnetic field only given (a) a lossy medium, (b) a lossless medium.

(a)

$$\vec{P}_{ave}=\frac{1}{2}\mathrm{Re}\left[\vec{E}_s\times\vec{H}_s^*\right] \Rightarrow \vec{P}_{ave}=\frac{1}{2}\mathrm{Re}\left[E_sH_s^*\hat{a}_k\right] \Rightarrow \vec{P}_{ave}=\frac{\hat{a}_k}{2}\mathrm{Re}\left[E_sH_s^*\right]$$

$$E_s=H_s\eta \qquad \eta=|\eta|e^{j\theta_\eta} \Rightarrow H_s^*=\frac{E_s^*}{\eta^*}=\frac{E_s^*}{|\eta|e^{-j\theta_\eta}}$$

$$\vec{P}_{ave}=\frac{\hat{a}_k}{2}\mathrm{Re}\left[\frac{E_sE_s^*}{|\eta|e^{-j\theta_\eta}}\right] \Rightarrow \vec{P}_{ave}=\frac{\hat{a}_k}{2}\mathrm{Re}\left[\frac{E_sE_s^*}{|\eta|e^{-j\theta_\eta}}\right]=\frac{|E_s|^2}{2|\eta|}\cos\theta_\eta\hat{a}_k$$

Example – 4 (contd.)

$$\vec{P}_{ave} = \frac{\hat{a}_k}{2}\mathrm{Re}\left[H_s|\eta|e^{j\theta_\eta}H_s^*\right]$$

$$\vec{P}_{ave} = \frac{\hat{a}_k}{2}\mathrm{Re}\left[\frac{E_s E_s^*}{|\eta|e^{-j\theta_\eta}}\right] = \frac{|\eta||H_s|^2}{2}\cos\theta_\eta\hat{a}_k$$

(b) Lossless Medium → η − $real,\ \theta_\eta = 0$

$$\vec{P}_{ave} = \frac{|E_s|^2}{2|\eta|}\hat{a}_k$$

$$\vec{P}_{ave} = \frac{|\eta||H_s|^2}{2}\hat{a}_k$$

General Relations Between $\vec{E}$ and $\vec{H}$

- We learnt that if $\hat{a}_E$, $\hat{a}_H$ and $\hat{a}_k$ are unit vectors along $\vec{E}$, $\vec{H}$ and the direction of propagation, then:

$$\hat{a}_k \times \hat{a}_E = \hat{a}_H \qquad \hat{a}_k \times \hat{a}_H = -\hat{a}_E \qquad \hat{a}_E \times \hat{a}_H = \hat{a}_k$$

- In general it can be deduced that:

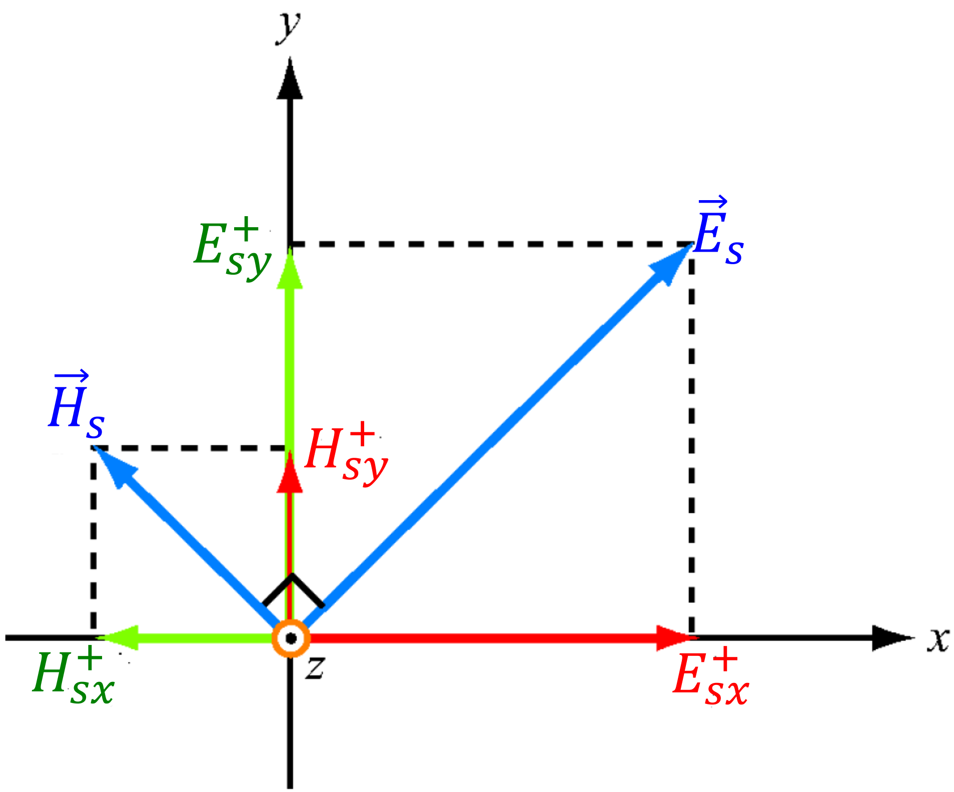
$$\vec{H}_s = \frac{1}{\eta}\hat{a}_k \times \vec{E}_s \qquad \vec{E}_s = -\eta\hat{a}_k \times \vec{H}_s$$

- Furthermore, a uniform plane wave travelling in the $+\hat{a}_z$ direction may have both $x-$ and $y-components$.
- In such a scenario: $\vec{E}_s = \hat{a}_x\vec{E}_{sx}^{+}(z) + \hat{a}_y\vec{E}_{sy}^{+}(z)$
- The associated magnetic field will be: $\vec{H}_s = \hat{a}_x\vec{H}_{sx}^{+}(z) + \hat{a}_y\vec{H}_{sy}^{+}(z)$
- The exact expression of magnetic field in terms of electric field will be:

$$\vec{H}_s = \frac{1}{\eta}\hat{a}_z \times \vec{E}_s = -\hat{a}_x\frac{\vec{E}_{sy}^{+}(z)}{\eta} + \hat{a}_y\frac{\vec{E}_{sx}^{+}(z)}{\eta}$$

- Thus: $\vec{H}_{sx}^{+} = -\dfrac{\vec{E}_{sy}^{+}(z)}{\eta}$ $\qquad \vec{H}_{sy}^{+}(z) = \dfrac{\vec{E}_{sx}^{+}(z)}{\eta}$

General Relations Between $\vec{E}$ and $\vec{H}$ (contd.)



In general, a TEM wave may have an electric field in any direction in the plane orthogonal to the direction of wave travel, and the associated magnetic field is also in the same plane with appropriate magnitude and direction.

Wave Polarization

- Very important concept considering its use in energy transmission of waves and its applications in the design of components such as Antenna.
- The $polarization$ of a uniform plane wave describes the locus traced by the tip of the $\vec{E}$ vector (in the phase orthogonal to the direction of propagation) at a given point in space as a function of time.
- In the most general case, the locus of tip of $\vec{E}$ is an ellipse, and wave is said to be $elliptically\ polarized$.
- Under certain conditions, the ellipse may degenerate into a circle or a straight line, in which case the $polarization\ state$ is called $circular$ or $linear$ respectively.
- We know that the $z-components$ of the electric and magnetic fields of a $z-propagating$ plane wave are both zero.
- Hence, in the most general case, the electric field phasor may consist of an $x-component$ and a $y-component$.

$$\boxed{\vec{E}_s = \hat{a}_x \vec{E}_{sx}^{+}(z) + \hat{a}_y \vec{E}_{sy}^{+}(z)}$$

Wave Polarization (contd.)

- With: $E_{sx}(z) = E_{x0}e^{-j\beta z}$ $E_{sy}(z) = E_{y0}e^{-j\beta z}$

Where, E_{x0} and E_{y0} are the amplitudes of $E_{sx}(z)$ and $E_{sy}(z)$ respectively.

- The amplitudes E_{x0} and E_{y0} are, in general, complex quantities → each characterized by phase and magnitude.
- The phase of a wave is defined relative to a reference state, such as $z = 0$, and $t = 0$ or any other combination of z and t.
- Essentially, the polarization of wave depends on phase of E_{y0} relative to that of E_{x0} and not the absolute phases of E_{x0} and E_{y0}.
- Therefore, for convenience, let us assign a phase of zero to E_{x0} and denote the phase of E_{y0}, relative to that of E_{x0}, as δ_p.
- Accordingly: $E_{x0} = A_x$ $E_{y0} = A_y e^{j\delta_p}$

Wave Polarization (contd.)

$E_{x0} = A_x \qquad E_{y0} = A_y e^{j\delta_p}$

Where, $A_x = |E_{x0}| \geq 0$ and $A_y = |E_{y0}| \geq 0$ are the magnitudes of E_{x0} and E_{y0} respectively.

Thus by definition, A_x and A_y may not assume negative values.

- Therefore, the electric field phasor is:

$$\vec{E}_s(z) = E_{sx}(z)\hat{a}_x + E_{sy}(z)\hat{a}_y \Rightarrow \vec{E}_s(z) = \left(\hat{a}_x A_x + \hat{a}_y A_y e^{j\delta_p}\right)e^{-j\beta z}$$

- The corresponding instantaneous field is:

$$\vec{E}(z,t) = \mathrm{Re}\left\{\vec{E}_s(z)e^{j\omega t}\right\} \Rightarrow \vec{E}(z,t) = \hat{a}_x A_x \cos\left(\omega t - \beta z\right) + \hat{a}_y A_y \cos\left(\omega t - \beta z + \delta_p\right)$$

- An electric field at a given point in space is characterized by its magnitude and direction.
- The magnitude of $\vec{E}(z,t)$ is: $\left|\vec{E}(z,t)\right| = \left[A_x^2 \cos^2(\omega t - \beta z) + A_y^2 \cos^2(\omega t - \beta z + \delta_p)\right]^{1/2}$

Wave Polarization (contd.)

- At a specific position z, the direction of $\vec{E}(z,t)$ is characterized by its *inclination angle* ψ with respect to the x-axis:

$$\psi(z,t) = \tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

Linear Polarization

- A wave is said to be linearly polarize if for a fixed z, the tip of $\vec{E}(z,t)$ traces a straight line segment as a function of time $\leftrightarrow$ happens when $E_x(z,t)$ and $E_y(z,t)$ are $in-phase$ $(\delta_p = 0)$ or out $-$ of $-$ $phase$ $(\delta_p = \pi)$.

- Under these conditions:

$$\vec{E}(0,t) = \left(\hat{a}_x A_x + \hat{a}_y A_y\right)\cos\left(\omega t - \beta z\right)$$

In-phase

$$\vec{E}(0,t) = \left(\hat{a}_x A_x - \hat{a}_y A_y\right)\cos\left(\omega t - \beta z\right)$$

Out-of-phase

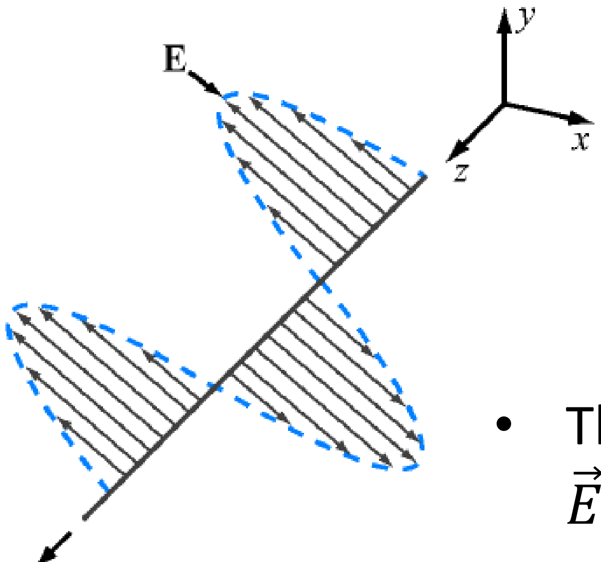
- Let us assume out $-$ of $-$ $phase$ case:

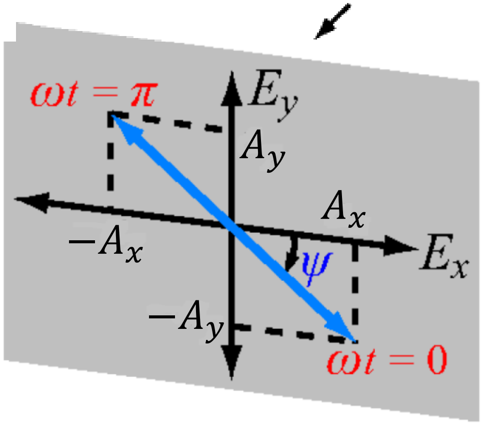
$$\left|\vec{E}(z,t)\right| = \left[A_x^2 + A_y^2\right]^{1/2}\left|\cos(\omega t - \beta z)\right|$$

$$\psi(z,t) = \tan^{-1}\left(\frac{-A_y}{A_x}\right)$$

Linear Polarization (contd.)

- We note that ψ is independent of both z and t.
- Following figure displays the line segment traced by the tip of $\vec{E}$ at $z = 0$ over half a cycle.





- The trace would be the same over any other value of z as well.
- At $z = 0, t = 0$:

$$\boxed{\left|\vec{E}(0,0)\right| = \left[A_x^2 + A_y^2\right]^{1/2}}$$

- The length of the vector representing $\vec{E}(0,t)$ decreases to zero at $\omega t = \pi/2$.
- The vector then reverses direction and increases in magnitude to:

$$\boxed{\left|\vec{E}(0,t)\right| = \left[A_x^2 + A_y^2\right]^{1/2}}$$

In the second quadrant of the $x - y$ $plane$ at $\omega t = \pi$.

Linear Polarization (contd.)

- Since ψ is independent of both z and t, $\vec{E}(z,t)$ maintains a direction along the line making an angle ψ with the $x-axis$, while oscillating back and forth across the origin.
- If $A_y = 0$, then $\psi = 0$ or $180°$, and the wave is $x - polarized$.
- If $A_x = 0$, then $\psi = 90°$ or $-90°$, and the wave is y $- polarized$.

Circular Polarization

- Let us consider the special case when $A_x = A_y$ and $\delta_p = \pm\pi/2$.
- For reasons that will become evident shortly, the wave polarization is called $left-hand$ polarized when $\delta_p = \pi/2$, and $right-hand$ polarized when $\delta_p = -\pi/2$.

Left-Hand Circular Polarization (LHCP)

- For $A_x = A_y = A$ and $\delta_p = \pi/2$, the electric field phasor and instantaneous electric field become:

$$\boxed{\vec{E}_s(z) = \left(\hat{a}_x A + \hat{a}_y A e^{j\pi/2}\right)e^{-j\beta z} = A\left(\hat{a}_x + j\hat{a}_y\right)e^{-j\beta z}}$$

Left-Hand Circular Polarization (LHCP) (contd.)

$$\boxed{\vec{E}(z,t) = \hat{a}_x A\cos(\omega t - \beta z) + \hat{a}_y A\cos\left(\omega t - \beta z + \frac{\pi}{2}\right) = \hat{a}_x A\cos(\omega t - \beta z) - \hat{a}_y A\sin(\omega t - \beta z)}$$

- The corresponding magnitude and inclination angle are:

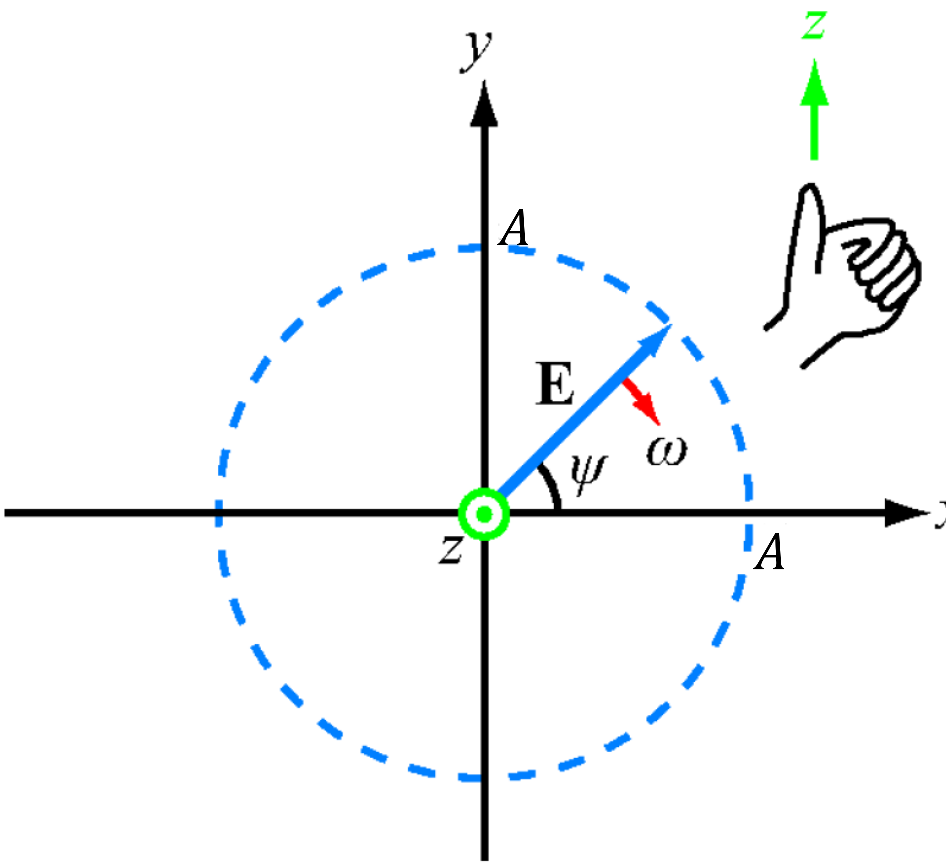
$$\boxed{\left|\vec{E}(z,t)\right| = A}$$

$$\boxed{\psi(z,t) = -(\omega t - \beta z)}$$

- Apparently the magnitude of $\vec{E}$ is independent of both z and t, whereas ψ depends on both variables → these functional dependencies are converse of those for the $linear\ polarization$ case.
- At $z = 0$, the inclination angle $\psi = -\omega t$.
- The negative sign implies that the inclination angle decreases with the increase in time.

Left-Hand Circular (LHC) Polarization (contd.)

- As seen in the figure, the tip of $\vec{E}(t)$ traces a circle in $x - y\, plane$ and rotates in clockwise direction as a function of time (when viewing the wave approaching).
- Such a wave is called $left - hand\ circularly$ polarized.



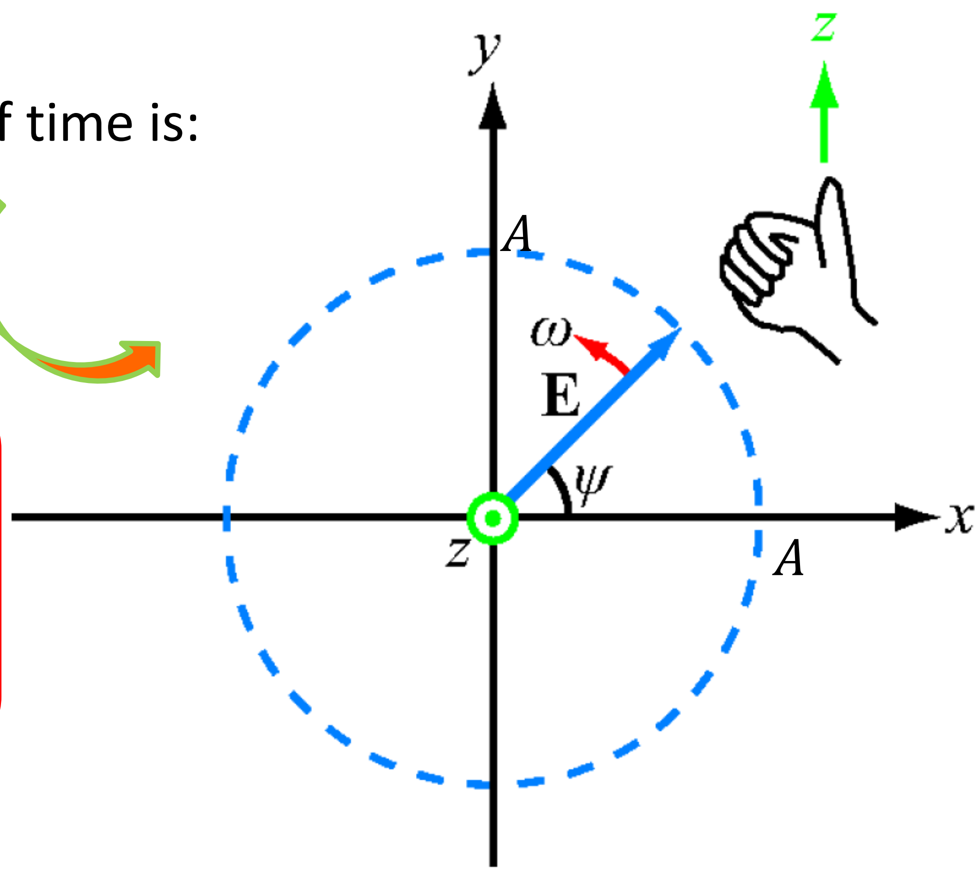
When the thumb of the left hand points along the direction of propagation (the $z - direction$ in this case), the other four fingers point in the direction of rotation of $\vec{E}$.

Right-Hand Circular Polarization (RHCP)

- For $A_x = A_y = A$ and $\delta_p = -\pi/2$, we get: $\boxed{\left|\vec{E}(z,t)\right| = A}$ $\boxed{\psi(z,t) = (\omega t - \beta z)}$

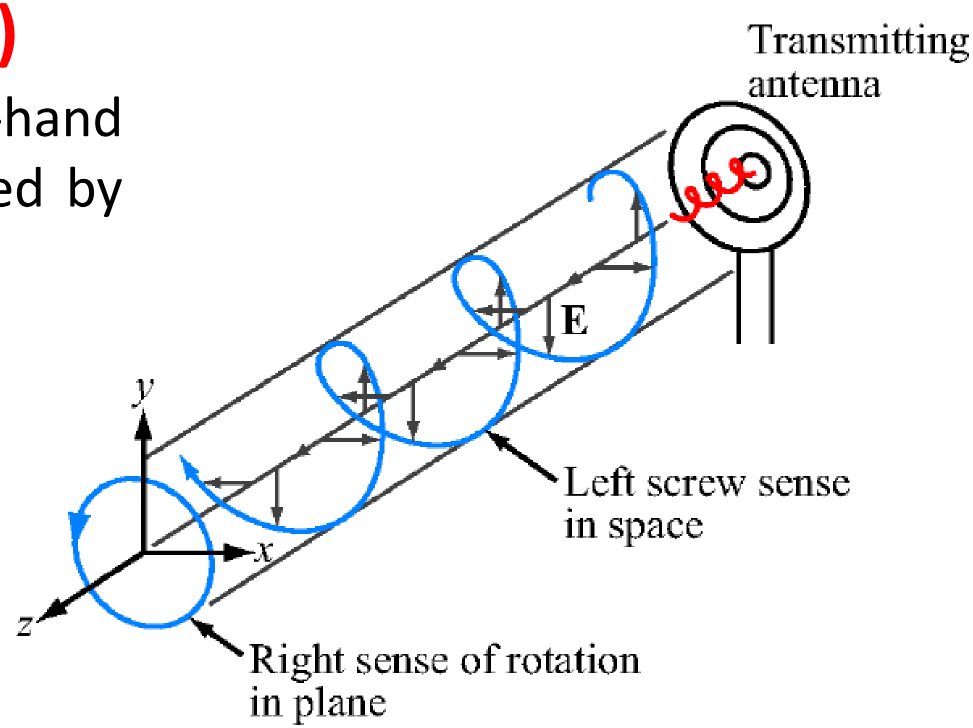
- The trace of $\vec{E}(t)$ as a function of time is:

For $RHCP$, the fingers of the right hand point in the direction of rotation of $\vec{E}$ when the thumb point in the direction of propagation.



Circular Polarization (contd.)

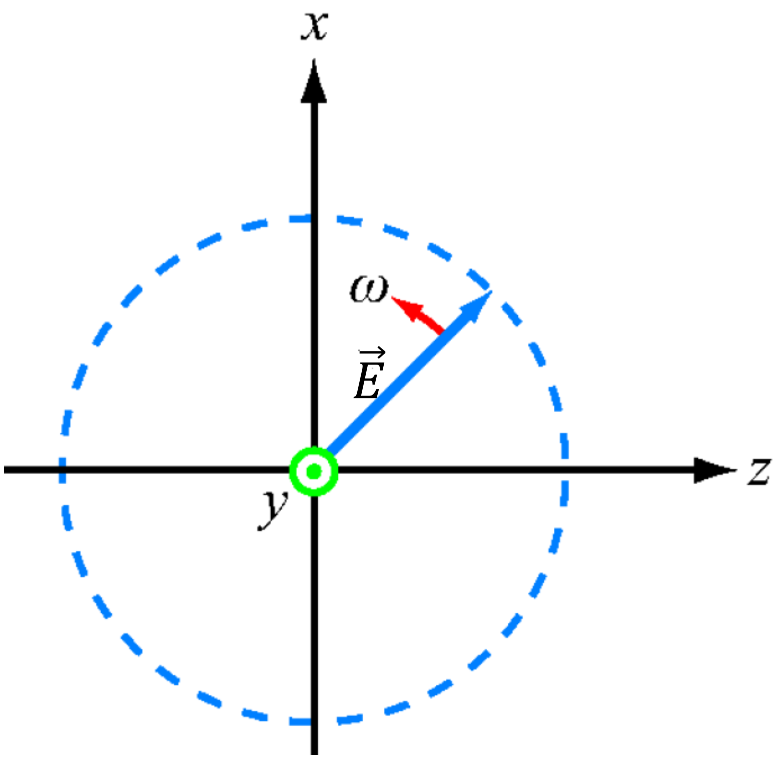
- This figure depicts a right-hand circularly polarized wave radiated by a helical antenna.



Polarization handedness is defined in terms of the rotation of $\vec{E}$ as a function of time in a fixed plane orthogonal to the direction of propagation, which is opposite to the direction of rotation of $\vec{E}$ as a function of distance at a fixed point in time.

Example – 5

- An RHC polarized plane wave with electric field magnitude of $3\ mV/m$ is traveling in the $+y - direction$ in a dielectric medium with $\varepsilon = 4\varepsilon_0, \mu = \mu_0$ and $\sigma = 0$. If the frequency is 100MHz, obtain the expression for $\vec{E}(y,t)$ and $\vec{H}(y,t)$.



- Let us assign the z-component of $\vec{E}_s(y)$ a phase angle of zero and the x-component a phase shift of $\delta_p = -\frac{\pi}{2}$.
- Then:

$$\vec{E}_s(y) = \left(\hat{a}_x E_{sx} + \hat{a}_z E_{sz}\right) e^{-j\beta y}$$

$$\vec{E}_s(y) = 3\left(\hat{a}_x e^{-j\pi/2} + \hat{a}_z\right) e^{-j\beta y} \quad \text{mV/m}$$

$$\therefore \vec{E}_s(y) = 3\left(-j\hat{a}_x + \hat{a}_z\right) e^{-j\beta y} \quad \text{mV/m}$$

Example – 5 (contd.)

- Similarly: $\vec{H}_s(y) = \frac{1}{\eta}\left[\hat{a}_y \times \vec{E}_s(y)\right]$ $\Rightarrow$ $\vec{H}_s(y) = \frac{3}{\eta}\left(\hat{a}_x + j\hat{a}_z\right)e^{-j\beta y}$ mA/m

- The instantaneous fields are:

$$\vec{E}(y,t) = \mathrm{Re}\left[\vec{E}_s(y)e^{j\omega t}\right] = \mathrm{Re}\left[3\left(-j\hat{a}_x + \hat{a}_z\right)e^{-j\beta y}e^{j\omega t}\right] \text{ mV/m}$$

$$\therefore \vec{E}(y,t) = 3\left[\hat{a}_x \sin(\omega t - \beta y) + \hat{a}_z \cos(\omega t - \beta y)\right] \text{ mV/m}$$

$$\vec{H}(y,t) = \mathrm{Re}\left[\vec{H}_s(y)e^{j\omega t}\right] = \mathrm{Re}\left[\frac{3}{\eta}\left(\hat{a}_x + j\hat{a}_z\right)e^{-j\beta y}e^{j\omega t}\right] \text{ mA/m}$$

$$\therefore \vec{H}(y,t) = \frac{3}{\eta}\left[\hat{a}_x \cos(\omega t - \beta y) - \hat{a}_z \sin(\omega t - \beta y)\right] \text{ mA/m}$$