# Lecture – 20

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- Time-Harmonic Fields

# Time – Harmonic Fields

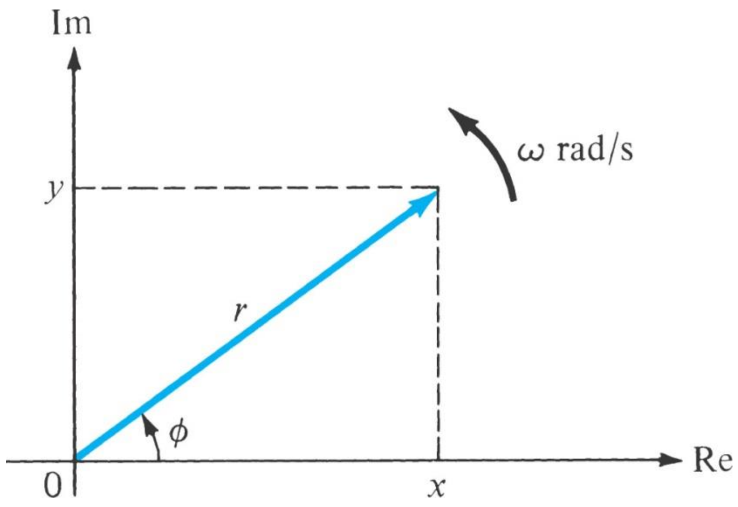
- So far, our time dependence of EM fields have been arbitrary.
- Let us consider the specific scenario where the fields are time-harmonic $\leftrightarrow$ Generally, time-varying electric and magnetic fields and their sources ($\rho_v$ and $\vec{J}$) depend on spatial coordinates $(x, y, z)$ and the time variable $t$. However, if their time variation is sinusoidal with angular frequency $\omega$, then these quantities can be represented by a phasor that depends on $(x, y, z)$ only.
- Time-harmonic field is one that varies periodically or sinusoidally with time.
- Sinusoidal analysis is of practical value.
- This can be extended to most waveforms by Fourier analysis.
- Sinusoids are easily expressed in phasors, which are more convenient to work with.

# Time – Harmonic Fields (contd.)

- A phasor is a complex number that contains the amplitude and phase information of a sinusoidal oscillation.

$$z = x + jy = r\angle\phi \quad \Rightarrow \quad z = re^{j\phi} = r\left(\cos\phi + j\sin\phi\right)$$

- The two forms of representing $z$ are illustrated below:




- To introduce the time element, we let:

$$\phi = \omega t + \theta$$

where, θ may be a function of time or space coordinates or constant

$$re^{j\phi} = re^{j\theta}re^{j\omega t}$$

$$\text{Re}(re^{j\phi}) = r\cos(\omega t + \theta)$$

$$\text{Im}(re^{j\phi}) = r\sin(\omega t + \theta)$$

# Time – Harmonic Fields (contd.)

- Thus a sinusoidal current $I(t) = I_0\cos(\omega t + \theta)$ equals the real part of $I_0 e^{j\theta} e^{j\omega t}$.
- The current $I'(t) = I_0\sin(\omega t + \theta)$, which is the imaginary part of $I_0 e^{j\theta} e^{j\omega t}$ can also be represented as the real part of $I_0 e^{j\theta} e^{j\omega t} e^{-j90^\circ}$.
- However, **be consistent** while representing the real and imaginary part of a quantity.
- The complex term $I_0 e^{j\theta}$ which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called the phasor current, denoted by $I_s$.

$$I_s = I_0 e^{j\theta} = I_0\angle\theta$$

- Therefore $I(t) = I_0\cos(\omega t + \theta)$ can be expressed as:

$$I(t) = \mathrm{Re}\left(I_s e^{j\omega t}\right)$$

# Time – Harmonic Fields (contd.)

- In general, a phasor could be a scalar or a vector.
- If a vector $\vec{A}(x, y, z, t)$ is a time-harmonic field, then the phasor form of $\vec{A}$ is $\vec{A}_s(x, y, z)$; the two quantities are related as:

$$\vec{A} = \text{Re}\left(\vec{A}_s e^{j\omega t}\right)$$

- For example, if $\vec{A} = A_0 \cos(\omega t - \beta x)\,\hat{a}_y$, then we can express $\vec{A}$ as:

$$\vec{A} = \text{Re}\left(A_0 e^{-j\beta x}\hat{a}_y e^{j\omega t}\right) = \text{Re}\left(\vec{A}_s e^{j\omega t}\right)$$

Where: $\vec{A}_s = A_0 e^{-j\beta x}\hat{a}_y$

- Notice that: $\dfrac{\partial \vec{A}}{\partial t} \to j\omega \vec{A}_s$
- Similarly: $\int \vec{A}\,\partial t \to \dfrac{\vec{A}_s}{j\omega}$

# Time – Harmonic Fields (contd.)

- Time-Harmonic Maxwell's equations assuming time factor $e^{j\omega t}$

| Differential Form | Integral Form |
|---|---|
| $\nabla . \vec{D}_s = \rho_{vs}$ | $\oint_S \vec{D}_s . \overline{ds} = \int_v \rho_{vs} dv$ |
| $\nabla . \vec{B}_s = 0$ | $\oint_S \vec{B}_s . \overline{ds} = 0$ |
| $\nabla \times \vec{E}_s = -j\omega \vec{B}_s$ | $\oint_L \vec{E}_s . \overline{dl} =- -j\omega \int_S \vec{B}_s . \overline{ds}$ |
| $\nabla \times \vec{H}_s = \vec{J}_s + j\omega \vec{D}_s$ | $\oint_L \vec{H}_s . \overline{dl} = \int_S (\vec{J}_s + j\omega \vec{D}_s) . \overline{ds}$ |

# Example – 1

- Convert the complex number in polar form: $z_1 = \dfrac{j(3-j4)^*}{(-1+j6)(2+j)^2}$

## Method – 1 (working in rectangular form):

$$z_1 = \frac{j(3-j4)^*}{(-1+j6)(2+j)^2} \Rightarrow z_1 = \frac{-4+j3}{-27+j14}$$

$$z_1 = 0.1622 - j0.027 \Rightarrow \therefore z_1 = 0.1644\angle -9.46^\circ$$

## Method – 2 (working in polar form):

- Convert $j,\ (3-j4)^*,\ (-1+j6),\ and\,(2+j)^2$ in polar form.

# Example – 2

- Convert the complex number in polar form:

$$z = -\left[\frac{1+j}{4-j8}\right]^{1/2}$$

$$z = 0.3976\angle 54.2°$$

# Example – 3

- Simplify the complex number: $$z = j^3\left[\frac{1+j}{2-j}\right]^2$$

$$z = 0.24 + j0.32$$

# Example – 4

- Express the following phasors in their instantaneous forms:

(a) $\vec{A}_s = (4 - 3j)e^{-j\beta x}\hat{a}_y$

(b) $\vec{B}_s = \frac{20}{\rho}e^{-j2z}\hat{a}_\rho$

(c) $\vec{C}_s = \frac{10}{r^2}(1 + 2j)e^{-j\varphi}sin\theta\hat{a}_\varphi$

**(a)** $\vec{A}_s = 5e^{-j(\beta x + 36.87°)}\hat{a}_y$ $\Rightarrow$ $\therefore \vec{A} = \text{Re}\left[\vec{A}_s e^{j\omega t}\right] = 5\cos(\omega t - \beta x - 36.87°)\hat{a}_y$

**(b)** $\vec{B} = \text{Re}\left[\vec{B}_s e^{j\omega t}\right]$ $\Rightarrow$ $\therefore \vec{B} = \text{Re}\left[\frac{20}{\rho}e^{-j2z}e^{j\omega t}\hat{a}_\rho\right] = \frac{20}{\rho}\cos(\omega t - 2z)\hat{a}_\rho$

# Example – 4 (contd.)

**(c)** $\vec{C}_s = \frac{10}{r^2}(1+2j)e^{-j\varphi}\sin\theta\hat{a}_\varphi \Rightarrow \vec{C}_s = \frac{10}{r^2}(2.236e^{j63.43°})e^{-j\varphi}\sin\theta\hat{a}_\varphi$

$$\Rightarrow C = \text{Re}\left[\vec{C}_s e^{j\omega t}\right] = \text{Re}\left[\frac{22.36}{r^2}e^{j(\omega t-\varphi+63.43°)}\sin\theta\hat{a}_\varphi\right]$$

$$\therefore C = \frac{22.36}{r^2}\cos(\omega t-\varphi+63.43°)\sin\theta\hat{a}_\varphi$$

# Example – 5

- Given $\vec{A} = 4sin\omega t\hat{a}_x + 3cos\omega t\hat{a}_y$ and $\vec{B}_s = j10ze^{-jz}\hat{a}_x$, express $\vec{A}$ in phasor form and $\vec{B}_s$ in instantaneous form.

$$\vec{A} = 4\cos(\omega t - 90^\circ)\hat{a}_x + 3\cos\omega t\hat{a}_y \Rightarrow \vec{A} = \text{Re}\left[4e^{j(\omega t - 90^\circ)}\hat{a}_x + 3e^{j\omega t}\hat{a}_y\right]$$

$$\therefore \vec{A} = \text{Re}\left[\left(4e^{-j90^\circ}\hat{a}_x + 3\hat{a}_y\right)e^{j\omega t}\right] \Rightarrow \therefore \vec{A}_s = 4e^{-j90^\circ}\hat{a}_x + 3\hat{a}_y = -j4\hat{a}_x + 3\hat{a}_y$$

$$\vec{B}_s = 10ze^{-jz}\hat{a}_x \Rightarrow \vec{B}_s = 10ze^{j90^\circ}e^{-jz}\hat{a}_x$$

$$\therefore \vec{B} = \text{Re}\left[\vec{B}_s e^{j\omega t}\right] = \text{Re}\left[10ze^{j(\omega t - z + 90^\circ)}\hat{a}_x\right]$$

$$\therefore \vec{B} = 10z\cos(\omega t - z + 90^\circ)\hat{a}_x = -10z\sin(\omega t - z)\hat{a}_x$$

# Example – 6

- The electric field phasor of an EM wave in free space is:

$$\vec{E}_s(y) = 10e^{-j4y}\hat{a}_x \quad V/m$$

Find (a) $\omega$ such that $\vec{E}_s$ satisfies Maxwell's equations.,
(b) the corresponding magnetic field $\vec{H}_s$.

**(a)** $$\nabla \times \vec{E}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{sx}(y) & 0 & 0 \end{vmatrix} = -\frac{\partial E_{sx}}{\partial y} = j40e^{-j4y}\hat{a}_z$$

**But:** $$\nabla \times \vec{E}_s = -j\mu_0\omega\vec{H}_s \quad \Rightarrow \quad \vec{H}_s = -\frac{40}{\mu_0\omega}e^{-j4y}\hat{a}_z$$

# Example – 6 (contd.)

**We know:** $\nabla . \vec{H}_s = 0$

$$\nabla \times \vec{H}_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_{sz}(y) \end{vmatrix} = \frac{\partial H_{sz}}{\partial y} \hat{a}_x = \frac{j160}{\mu_0 \omega} e^{-j4y} \hat{a}_x$$

**But:** $\nabla \times \vec{H}_s = j\omega\varepsilon_0 \vec{E}_s$ → $\vec{E}_s = \frac{160}{\mu_0 \varepsilon_0 \omega^2} e^{-j4y} \hat{a}_x$

$$\Rightarrow \vec{E}_s = \frac{160}{\mu_0 \varepsilon_0 \omega^2} e^{-j4y} \hat{a}_x = 10 e^{-j4y} \hat{a}_x$$

→ $\therefore \omega = 12 \times 10^8 \, rad/s$

**(b)**

$$\vec{H}_s = -\frac{40}{\mu_0 \omega} e^{-j4y} \hat{a}_z = -\frac{40}{4\pi \times 10^{-7} \times 12 \times 10^8} e^{-j4y} \hat{a}_z$$

$$\therefore \vec{H}_s = -\frac{40}{\mu_0 \omega} e^{-j4y} \hat{a}_z = -26.53 e^{-j4y} \hat{a}_z \text{ mA/m}$$

# Example – 7

- In air, $\vec{E} = \frac{sin\theta}{r}\cos(6\times 10^7 t - \beta r)\,\hat{a}_\varphi \frac{V}{m}$, Find $\beta$ and $\vec{H}$.

**You will need:**

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{bmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{bmatrix}$$

**Let us first determine:**

$$\nabla \times \vec{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\varphi \sin\theta)\hat{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(rE_\varphi)\hat{a}_\theta$$

$$\nabla \times \vec{E} = \frac{2\cos\theta}{r^2}\cos(\omega t - \beta r)\hat{a}_r - \frac{\beta}{r}\sin\theta\sin(\omega t - \beta r)\hat{a}_\theta$$

# Example – 7 (contd.)

**We know:** $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

$$\vec{H} = -\frac{2\cos\theta}{\mu\omega r^2}\sin(\omega t - \beta r)\hat{a}_r - \frac{\beta}{\mu\omega r}\sin\theta\cos(\omega t - \beta r)\hat{a}_\theta$$

**Now:** $\beta = \frac{\omega}{c} = \frac{6\times10^7}{3\times10^8} = 0.2 rad/m$

$$\therefore \vec{H} = -\frac{1}{12\pi r^2}\cos\theta\sin(6\times10^7 t - 0.2r)\hat{a}_r - \frac{1}{120\pi r}\sin\theta\cos(6\times10^7 t - 0.2r)\hat{a}_\theta$$