Lecture – 19

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- Electromagnetic Fields (Contd.)
- Displacement Current
- Maxwell's Equations
- Time Varying Potentials

Moving Conductor in a Time-Varying $\vec{B}$

- For a general case of a single turn conducting loop moving in time-varying magnetic field, the induced emf is the sum of a $transformer\ emf$ and $motional\ emf$.

$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

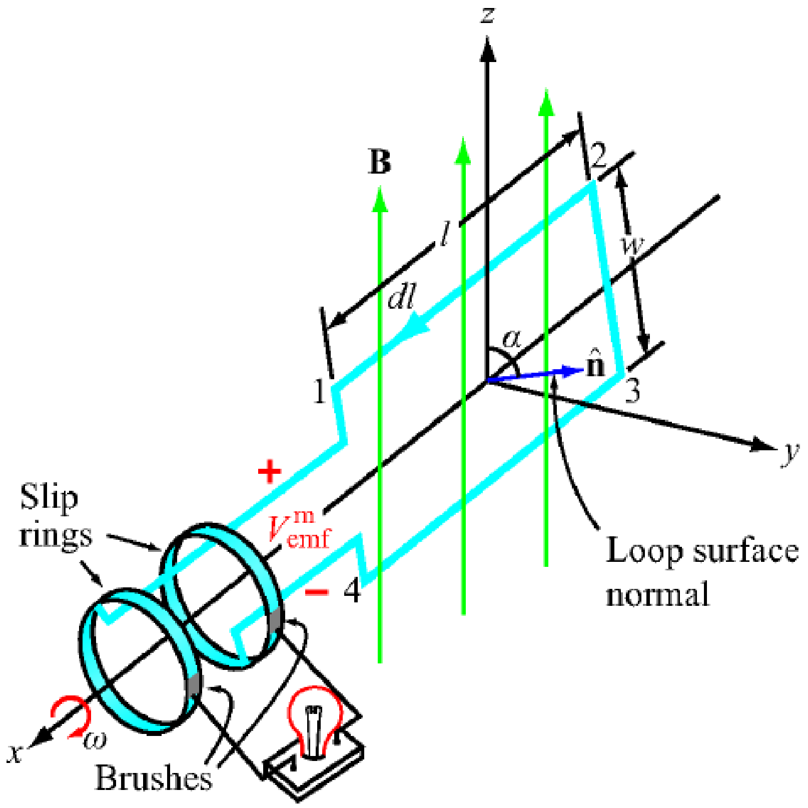
$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t} . \overline{ds} + \oint_C \left(\vec{u} \times \vec{B}\right) . \overline{dl}$$

- induced emf also equals: $V_{emf} = -\frac{d\Psi}{dt} = -\frac{d}{dt}\int_S \vec{B} . \overline{ds}$

Both expressions are equivalent and choice between these two depends on the type of problem.

Example – 1

- Find the induced voltage when the rotating loop of the electromagnetic generator, shown in figure, is in a magnetic field $\vec{B} = \hat{a}_z B_0 cos\omega t$. Assume that $\alpha = 0$ at $t = 0$.



Displacement Current

- You can recall that the Ampere's law in differential form is given by:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- Integration of the above expression gives: $$\int_S \left(\nabla \times \vec{H}\right).\overline{ds} = \int_S \vec{J}.\overline{ds} + \int_S \frac{\partial \vec{D}}{\partial t}.\overline{ds}$$

- Simplification gives: $$\oint_C \overrightarrow{H}.\overline{dl} = I_c + \int_S \frac{\partial \vec{D}}{\partial t}.\overline{ds}$$

Conduction Current

- The second term has the unit of current because its proportional to the time derivative of the electric flux density $\vec{D}$ called the electric displacement.
- This term is therefore called the *Displacement Current*, I_d.

Displacement Current (contd.)

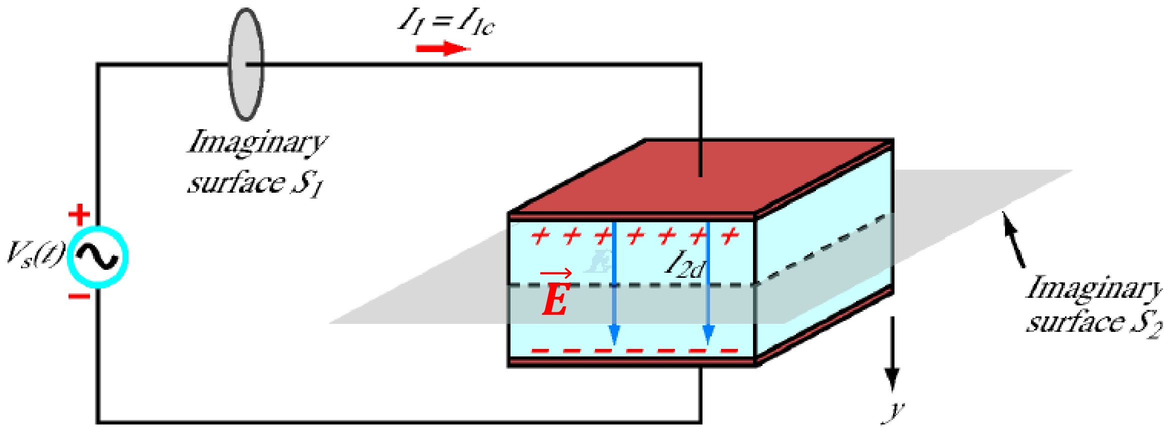
$$I_d = \int_S \vec{J}_d . \overline{ds} = \int_S \frac{\partial \vec{D}}{\partial t} . \overline{ds}$$

$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is called displacement current density

- Therefore: $\oint_C \vec{H} . \overline{dl} = I_c + I_d = I$ ← I is the total current
- In electrostatics, $\frac{\partial \vec{D}}{\partial t} = 0$ and therefore $I_d = 0$ and $I = I_c$.
- The concept of displacement current was introduced by James Clerk Maxwell when he formulated the unified theory of electricity and magnetism under time-varying conditions.

Displacement Current (contd.)

- Let us consider the following parallel-plate capacitor to understand the physical meaning of *displacement current*.



- Let us find I_c and I_d through each of the two imaginary surfaces: (1) cross section of the conducting wire, S_1; (2) cross section of the capacitor, S_2.
- The simple circuit consists of a capacitor and an ac source given by:

$$V_s(t) = V_0 \cos \omega t$$

- We know from Maxwell's hypothesis that the total current flowing through any surface consists, in general, of a conduction current and a displacement current.

Displacement Current (contd.)

- In the perfect conducting wire, $\vec{E} = \vec{D} = 0$; hence, $I_{1d} = 0$.
- As for I_{1c}, we know: $$I_{1c} = C\frac{dV_c}{dt} = C\frac{d}{dt}\left(V_0 \cos\omega t\right) = -CV_0\omega \sin\omega t$$
- With no displacement current in the wire, the total current in the wire is: $$I_1 = I_{1c} = -CV_0\omega \sin\omega t$$
- Now in the perfect dielectric with permittivity ε between the capacitor plates, $\sigma = 0$.
- Therefore, $I_{2c} = 0$ because no conduction current exists there.
- To determine I_{2d}, we need to determine $\vec{E}$ in the dielectric spacing: $$\vec{E} = \hat{a}_y \frac{V_C}{d} = \hat{a}_y \frac{V_0}{d}\cos\omega t$$

d is the spacing between the plates, and $\hat{a}_y$ is the direction from the higher potential plate to the lower potential plate at $t = 0$.

Displacement Current (contd.)

- Therefore displacement current in the dielectric is:

$$I_{2d} = \int_A \frac{\partial \vec{D}}{\partial t} . \overline{ds} \Rightarrow I_{2d} = \int_A \left[\frac{\partial}{\partial t} \left(\hat{a}_y \frac{\varepsilon V_0}{d} \cos \omega t \right) \right] . \left(\hat{a}_y ds \right)$$

$$\therefore I_{2d} = -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t = -C V_0 \omega \sin \omega t$$

- It is apparent that the expression for displacement current in the dielectric is identical to the conduction current in the wire.
- The fact that these two are equal ensures the continuity of the total current flowing through the circuit.
- Even though the displacement current doesn't transport free charges, it nonetheless behaves like a real current.
- Caution, in this example we considered the wire as perfect conductor whereas the dielectric as perfect as well.
- In practice, none of them are perfect and therefore the total current at all the time is sum of conductions and displacement currents.

Example – 2

- The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\epsilon_r = 1$ is given by $I_c = 2sin\omega t\ (mA)$. If $\omega = 10^9 \frac{rad}{s}$, find the displacement current.

Solution

The conduction current is:

$$I_c = JA = \sigma EA$$

where A is the cross section of the wire.

Therefore:

$$E = \frac{I_c}{\sigma A} = \frac{2\times10^{-3}\sin\omega t}{2\times10^{7}\times A} = \frac{1\times10^{-10}}{A}\sin\omega t$$

Example – 2 (contd.)

We know:

$$I_d = J_d A \;\Rightarrow\; I_d = A\frac{\partial D}{\partial t} \;\Rightarrow\; I_d = \varepsilon A\frac{\partial E}{\partial t}$$

$$I_d = \varepsilon A\frac{\partial}{\partial t}\left(\frac{1\times10^{-10}}{A}\sin\omega t\right)$$

$$I_d = \varepsilon\omega\times10^{-10}\cos\omega t$$

$$\therefore I_d = 0.885\times10^{-12}\cos\omega t \; (A)$$

Example – 3

- (a) Show that the ratio of the amplitudes of the conduction current density and displacement current density is $\frac{\sigma}{\omega\epsilon}$ for the applied field $E = E_m cos\omega t$, assume $\mu = \mu_0$. (b) What is this amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$.

Solution

(a)

$$J_c = \sigma E = \sigma E_m \cos \omega t$$

$$J_d = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t} = -\varepsilon \omega E_m \sin \omega t$$

Therefore the ratio is:

$$\left|\frac{J_c}{J_d}\right| = \left|\frac{\sigma}{\varepsilon\omega}\right| = \frac{\sigma}{\varepsilon\omega}$$

(b)

$$\left|\frac{J_c}{J_d}\right| = \frac{\sigma\tau}{\varepsilon}$$

Maxwell's Equations

- Generalized forms of Maxwell's equations:

Differential Form	Integral Form	Remarks
$\nabla . \vec{D} = \rho_v$	$\oint_S \vec{D} . \overline{ds} = \int_v \rho_v dv$	Gauss's Law
$\nabla . \vec{B} = 0$	$\oint_S \vec{B} . \overline{ds} = 0$	Nonexistence of isolated magnetic charge
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E} . \overline{dl} = -\frac{\partial}{\partial t} \int_S \vec{B} . \overline{ds}$	Faraday's Law
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H} . \overline{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) . \overline{ds}$	Ampere's Circuital Law

Maxwell's Equations (contd.)

- Other equations that go hand-in-hand with Maxwell's equations is the Lorentz force equation:

$$\vec{F} = Q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

- Continuity equation is another that is closely associated with Maxwell's equations:

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

- The concept of linearity, isotropy, and homogeneity of a material applies to time-varying fields as well.
- In a linear, homogeneous, and isotropic medium:

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P} \qquad \vec{B} = \mu \vec{H} = \mu_0\left(\vec{H} + \vec{M}\right) \qquad \vec{J} = \sigma \vec{E} + \rho_v \vec{u}$$

- The boundary conditions remain valid for time-varying fields as well.

$$\vec{E}_{1t} - \vec{E}_{2t} = 0 \qquad \left(\vec{E}_1 - \vec{E}_2\right) \times \hat{a}_n = 0 \qquad \vec{H}_{1t} - \vec{H}_{2t} = \vec{K} \qquad \left(\vec{H}_1 - \vec{H}_2\right) \times \hat{a}_n = \vec{K}$$

$$\vec{D}_{1n} - \vec{D}_{2n} = \rho_s \qquad \left(\vec{D}_1 - \vec{D}_2\right).\hat{a}_n = \rho_s \qquad \vec{B}_{1n} - \vec{B}_{2n} = 0 \qquad \left(\vec{B}_1 - \vec{B}_2\right).\hat{a}_n = 0$$

- However, for a perfect conductor in a time-varying field:

$$\vec{E} = 0, \qquad \vec{H} = 0, \qquad \vec{J} = 0 \quad \Rightarrow \quad \vec{B}_n = 0 \qquad \vec{E}_t = 0$$

Example – 4

- Electric field intensity throughout an enclosed region of free space is $E_y = A(sin20x)(sinbz)\{sin(12 \times 10^9 t)\}\frac{V}{m}$. Beginning with the $\nabla \times \vec{E}$ relationship, use Maxwell's equation to find a numerical value for b, assuming $b > 0$.

Time-Varying Potentials

- For the static EM fields, the electric scalar potential was expressed as:

$$V = \int_v \frac{\rho_v dv}{4\pi\varepsilon R}$$

- Whereas, the magnetic vector potential was expressed as:

$$\vec{A} = \int_v \frac{\mu \vec{J} dv}{4\pi R}$$

- Let us examine, what happens to these potentials when the field vary with time.
- Recall that, $\vec{A}$ was defined from the fact that $\nabla . \vec{B} = 0$, which still holds for time-varying case. Therefore:

$$\vec{B} = \nabla \times \vec{A}$$

- We know from Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Therefore:

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right) \Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Time-Varying Potentials (contd.)

- We know, that the curl of the gradient of a scalar field is zero: $\nabla \times -\nabla V = 0$, therefore:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \quad \Rightarrow \quad \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Thus we can determine $\vec{E}$ and $\vec{B}$ provided V and $\vec{A}$ are known.

- However, determination of V and $\vec{A}$ require expressions that are suitable for time varying fields.
- We know that $\nabla . \vec{D} = \rho_v$ is valid for time-varying conditions. We can write:

$$\nabla.\vec{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}\left(\nabla.\vec{A}\right) \quad \Rightarrow \quad \nabla^2 V + \frac{\partial}{\partial t}\left(\nabla.\vec{A}\right) = -\frac{\rho_v}{\varepsilon}$$

Time-Varying Potentials (contd.)

- Furthermore: $\nabla \times \nabla \times \vec{A} = \nabla \times \vec{B} = \mu \vec{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)$ ⟸ $\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial E}{\partial t}$

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} - \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\nabla^2 \vec{A} - \nabla\left(\nabla . \vec{A}\right) = -\mu \vec{J} + \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

- We know that a vector field is uniquely defined when its curl and divergence are specified. The curl of $\vec{A}$ has been specified as $\vec{B}$, therefore the divergence for $\vec{A}$ can be expressed as:

$$\nabla . \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

This expression relates V and $\vec{A}$ and is called *Loretnz condition for potentials.*

Time-Varying Potentials (contd.)

- Therefore:

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

EM Wave Equations

- *Lorentz condition* uncouples and also creates symmetry between V and $\vec{A}$ and therefore aid the analysis of wave equations.
- Actually, V and $\vec{A}$ satisfy *Poisson's equations* for time-varying potentials.
- From these expressions, it can be deduced that the solutions for V and $\vec{A}$ are:

$$V = \int_v \frac{[\rho_v]dv}{4\pi\varepsilon R}$$

$$\vec{A} = \int_v \frac{\mu[\vec{J}]dv}{4\pi R}$$

Where $[\rho_v]$ and $[\vec{J}]$ are the retarded values. The respective V and $\vec{A}$ are called the *retarded electric scalar potential* and the *retarded magnetic vector potential*.

Time-Varying Potentials (contd.)

- It means that the time t in $\rho_v(x, y, z, t)$ or $\vec{J}(x, y, z, t)$ is replaced by retarded time t' given by:

$$t' = t - \frac{R}{u}$$

- Where, $R = |\bar{r} - \overline{r'}|$ is the distance between the source point $\overline{r'}$ and the observation point $\bar{r}$.

- Whereas: $$u = \frac{1}{\sqrt{\varepsilon\mu}}$$

u is the velocity of wave propagation. In free space, $u = c \cong 3 \times 10^8\ m/s$ is the speed of light in vacuum.

Example – 4

- Show that another form of Faraday's law is: $\vec{E} = -\dfrac{\partial \vec{A}}{\partial t}$

  where $\vec{A}$ is the magnetic vector potential.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right) = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$\therefore \vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

Example – 5

- Assuming source free region, derive the diffusion equation:

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$