Lecture – 13

Date: 19.02.2015

- Applications of Ampere's Law
- Magnetic Flux Density
- Magnetic Vector Potential

Applications of Ampere's Law (contd.)

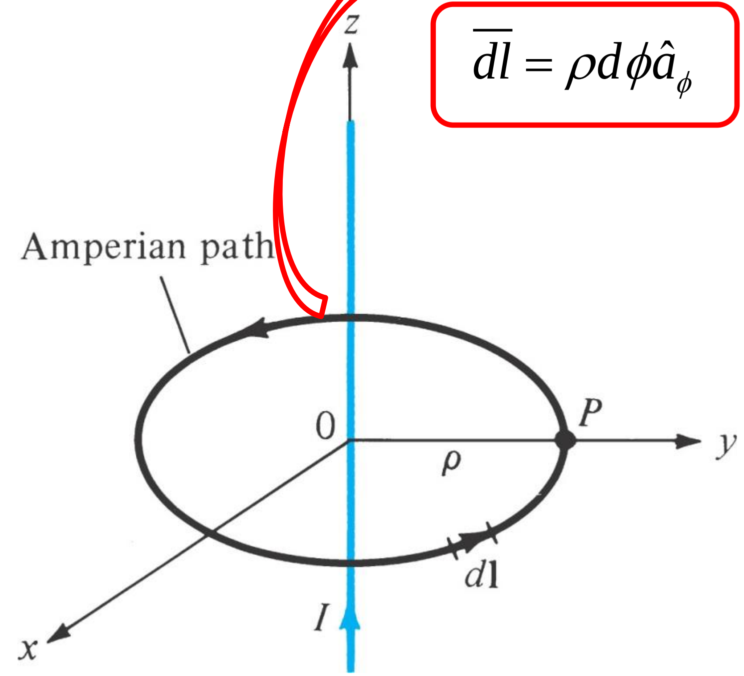
Infinite Line Current

- Let us consider an infinitely long filamentary current along the z-axis.
- To determine $\vec{H}$ at point P, let us form a closed path to pass through P.
- This path is called Amperian path (analogous to Gaussian surface).
- From Ampere's law we can write:

$$\oint_C \vec{H}.\overline{dl} = I = \int H_\phi \hat{a}_\phi . \rho d\phi \hat{a}_\phi$$

On this path:

$$\overline{dl} = \rho d\phi \hat{a}_\phi$$



As $\vec{H}$ is parallel to $\overline{dl}$

$$\Rightarrow I = H_\phi \int \rho d\phi$$

For fixed ρ

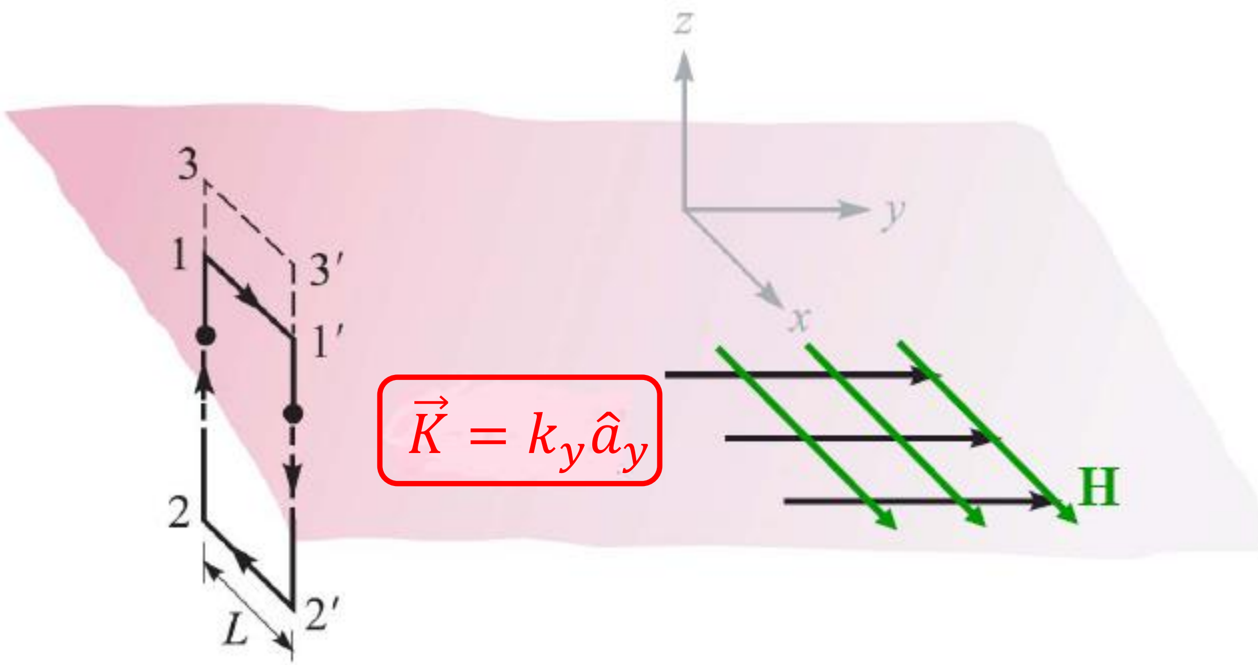
$$\Rightarrow I = H_\phi \left(2\pi\rho\right)$$

$$\therefore \vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

Applications of Ampere's Law (contd.)

Infinite Sheet of Current

- Let us consider an infinite current sheet in the $z = 0$ plane.
- The sheet has a uniform current density $\vec{K} = k_y \hat{a}_y$ A/m as shown.

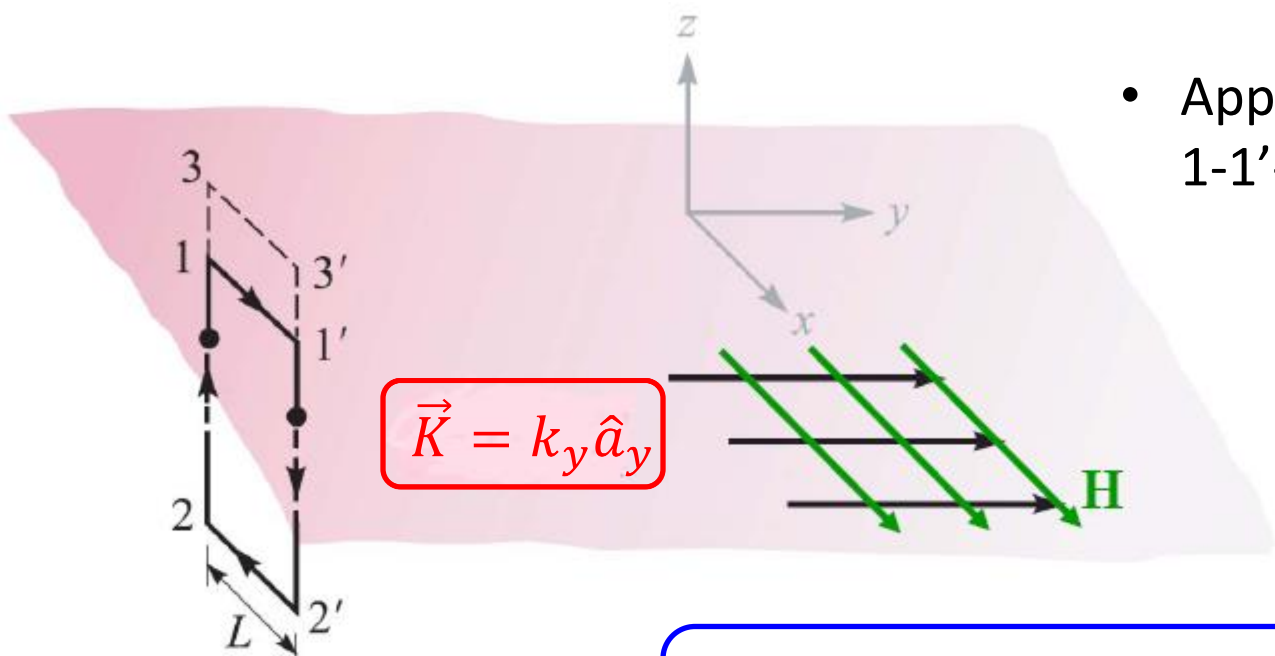


- Consider the sheet as a finite number of filaments cascaded together
- Field doesn't vary with x and y as the source doesn't vary with x and y
- $H_y = 0$, since current is along $y - axis$ [field is perpendicular to current]

Applications of Ampere’s Law (contd.)

Infinite Sheet of Current

- $H_z = 0$, as two symmetric filamentary elements along $x - axis$ will cancel the $z - components$.
- Resultant fields will be along $x - axis$ and doesn’t vary with x and y.



- Apply Ampere’s law along 1-1’-2’-2-1

$$\oint_C \vec{H}.\overline{dl} = I_{encl}$$

$$\int_{1}^{1'} H_{x1}dx + \int_{1'}^{2'} (-H_{z1})dz + \int_{2'}^{2} (-H_{x2})dx + \int_{2}^{1} H_{z1}dz = I_{encl}$$

Applications of Ampere's Law (contd.)

Infinite Sheet of Current

Doesn't vary with x

$$\int_{1}^{1'} H_{x1} dx + \underbrace{\int_{1'}^{2'} (-H_{z1}) dz}_{} + \int_{2'}^{2} (-H_{x2}) dx + \underbrace{\int_{2}^{1} H_{z1} dz}_{} = I_{encl}$$

Zero contribution from segments 1'-2' and 1-2 ($H_z = 0$)

$$\boxed{\Rightarrow H_{x1} L - H_{x2} L = k_y L}$$

$$\boxed{\therefore H_{x1} - H_{x2} = k_y}$$

- Similarly application of Ampere's law along 3-3'-2'-2-3 results into

$$\boxed{\therefore H_{x3} - H_{x2} = k_y}$$

Applications of Ampere's Law (contd.)

Infinite Sheet of Current

$$\therefore H_{x1} - H_{x2} = k_y \qquad\qquad \therefore H_{x3} - H_{x2} = k_y$$

- Simplification gives: $$H_{x1} = H_{x3} = \frac{k_y}{2} \qquad\qquad H_{x2} = -\frac{k_y}{2}$$

Therefore, it can be said that the field is same for all positive z and similarly the same for all negative z

- Because of symmetry, the magnetic field intensity on one side of the current sheet is negative of that on the other.

$$H_x = \frac{k_y}{2} \qquad (z > 0)$$

$$H_x = -\frac{k_y}{2} \qquad (z < 0)$$

Applications of Ampere's Law (contd.)

Infinite Sheet of Current

- If $\hat{a}_N$ is the unit vector normal (outward) to the current sheet, the result may be expressed as:

$$\boxed{\overrightarrow{H} = \frac{1}{2}\vec{k} \times \hat{a}_N}$$

- Magnetic field doesn't depend on the distance from the infinite current sheet → analogous to $\vec{D}\ field$ of an infinite charge sheet.

$$\boxed{\overrightarrow{H} = \frac{1}{2}\vec{k} \times \hat{a}_N} \qquad \boxed{\overrightarrow{D} = \frac{1}{2}\rho_s \hat{a}_N}$$

- If a second sheet of current flowing in the opposite direction, $\vec{K} = -k_y \hat{a}_y$, is placed at $z = h$, then the field in the region between the sheets is:

$$\boxed{\overrightarrow{H} = \vec{k} \times \hat{a}_N \qquad (\mathbf{0} < \boldsymbol{z} < \boldsymbol{h})}$$

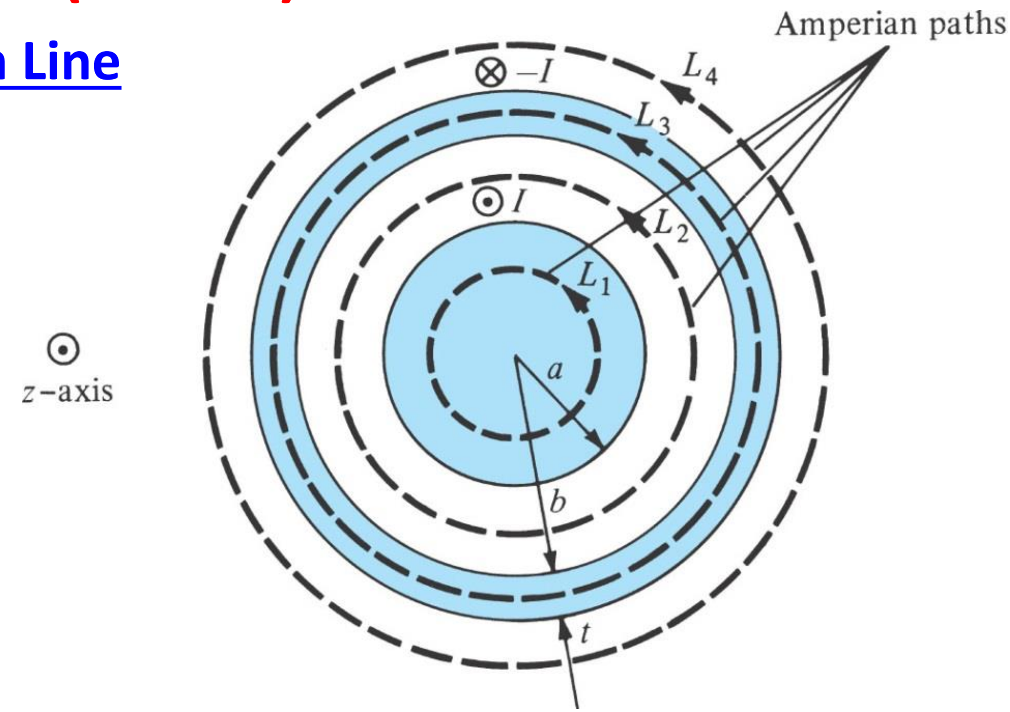
- and is zero elsewhere:

$$\boxed{\overrightarrow{H} = 0 \qquad (\boldsymbol{z} < \mathbf{0}, \qquad \boldsymbol{z} > \boldsymbol{h})}$$

Applications of Ampere's Law (contd.)

Infinitely Long Coaxial Transmission Line

- Let us consider coaxial transmission line with two concentric cylinders having their axes along the z-axis, where the z-axis is out of page.
- The inner conductor has radius a and carries current I, while the outer conductor has inner radius b and thickness t and carries return current $-I$.
- Determine field $\vec{H}$ everywhere.



Since the current distribution is symmetric, we apply Ampere's law along the Amperian path for each of the four possible regions:

$0 \leq \rho \leq a$, $a \leq \rho \leq b$, $b \leq \rho \leq b + t$, $\rho \geq b + t$

Applications of Ampere's Law (contd.)

Infinitely Long Coaxial Transmission Line

- For region $0 \le \rho \le a$, we have:

$$\vec{J} = \frac{I}{\pi a^2}\hat{a}_z$$

$$\overline{dS} = \rho d\phi d\rho \hat{a}_z$$

$$I_{enc} = \int \vec{J}.\overline{dS} = \frac{I}{\pi a^2}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\rho d\phi d\rho$$

$$\therefore I_{enc} = \frac{I\rho^2}{a^2}$$

Therefore application of Ampere's law over path L_1 gives:

$$H_\phi \int_{L_1} dl = H_\phi (2\pi\rho) = \frac{I\rho^2}{a^2}$$

$$\therefore H_\phi = \frac{I\rho}{2\pi a^2}$$

- For region $a \le \rho \le b$, we have: $I_{enc} = I$

Therefore application of Ampere's law over path L_2 gives:

$$H_\phi \int_{L_2} dl = H_\phi (2\pi\rho) = I$$

$$\therefore H_\phi = \frac{I}{2\pi\rho}$$

Applications of Ampere's Law (contd.)

Infinitely Long Coaxial Transmission Line

- For region $b \leq \rho \leq b + t$, we get:

$$I_{enc} = I + \int \vec{J} . \overline{dS}$$

Here, $\vec{J}$ is the current density of the outer conductor and is along $-\hat{a}_z$

$$\vec{J} = -\frac{I}{\pi\left[(b+t)^2 - b^2\right]}\hat{a}_z$$

$$I_{enc} = I - \frac{I}{\pi\left[(b+t)^2 - b^2\right]}\int_{\phi=0}^{2\pi}\int_{\rho=b}^{\rho} \rho d\phi d\rho$$

$$I_{enc} = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

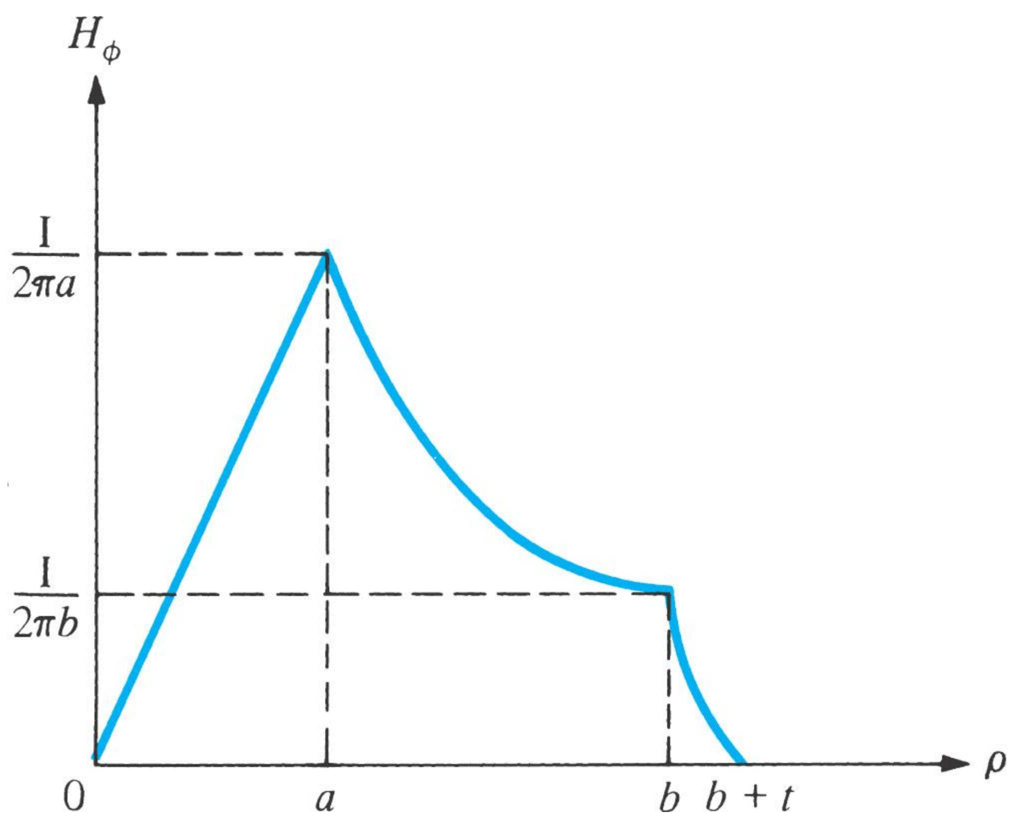
$$\Rightarrow H_\phi = \frac{I_{enc}}{2\pi\rho}$$

$$\therefore H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

Applications of Ampere's Law (contd.)

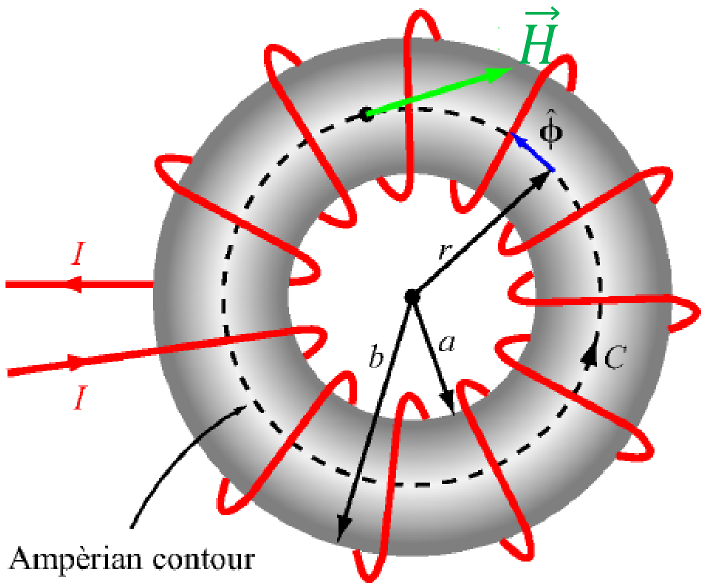
Infinitely Long Coaxial Transmission Line

- For region $\boldsymbol{\rho \geq b + t}$, we get: $\boxed{I_{enc} = I - I = 0}$ $\boxed{\therefore H_\phi = 0}$

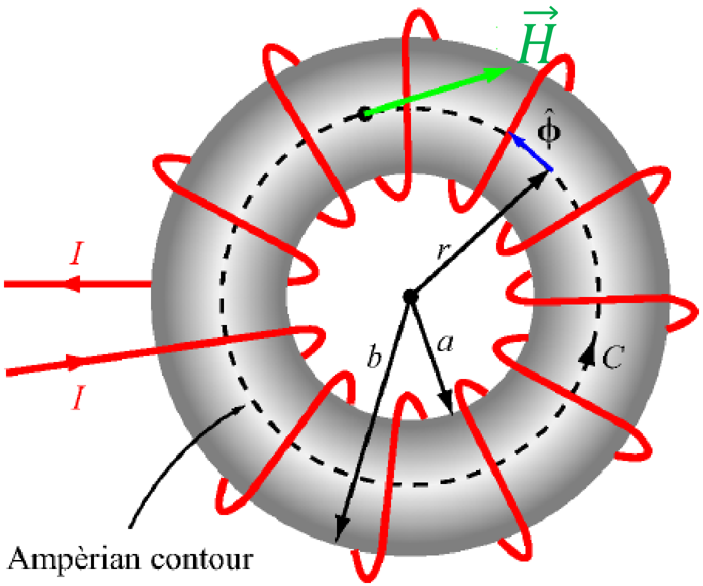


Example – 1

- A toroidal coil is a doughnut-shaped structure (called the core) wrapped in a closely spaced turns of wire (as shown in figure). For clarity, the turns have been shown as spaced far apart, but in practice they are wound in a closely spaced arrangement. The toroid is used to magnetically couple multiple circuits and to measure the magnetic properties of materials. For a toroid with N turns carrying a current I, determine the magnetic field $\vec{H}$ in each of the following three regions: $r < a$, $a < r < b$, and $r > b$, all in the azimuthal plane symmetry of the toroid.



Example – 1 (contd.)



- From Symmetry: It is apparent that $\vec{H}$ is uniform in the azimuthal direction.
- For circular Amperian path $r < a$, there will be no current through the surface of the contour.
- Similarly, for circular Amperian path $r > b$, there will be no current through the surface of the contour.
- Therefore, $\vec{H} = 0$ in the region external to the core.
- For region inside the core: Let us construct path of radius r.
- For each loop of radius r, we know that the field $\vec{H}$ at the center of the loop points along the axis of the loop, which in this case is the $\varphi - direction$.
- Now solve using Ampere's Circuital Law!!!

Magnetic Flux Density

- The magnetic flux density is similar to electric flux density $\vec{D}$.
- We know $\vec{E} = \varepsilon_0 \vec{E}$ in free space → similarly, the magnetic flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ as:

$$\boxed{\vec{B} = \mu_0 \vec{H}}$$

Where, μ_0 is a constant known as permeability of free space. The constant is in henrys per meter (H/m) and has the value:

$$\boxed{\mu_0 = 4\pi \times 10^{-7}\, H/m}$$

- The magnetic flux through a surface S is given by:

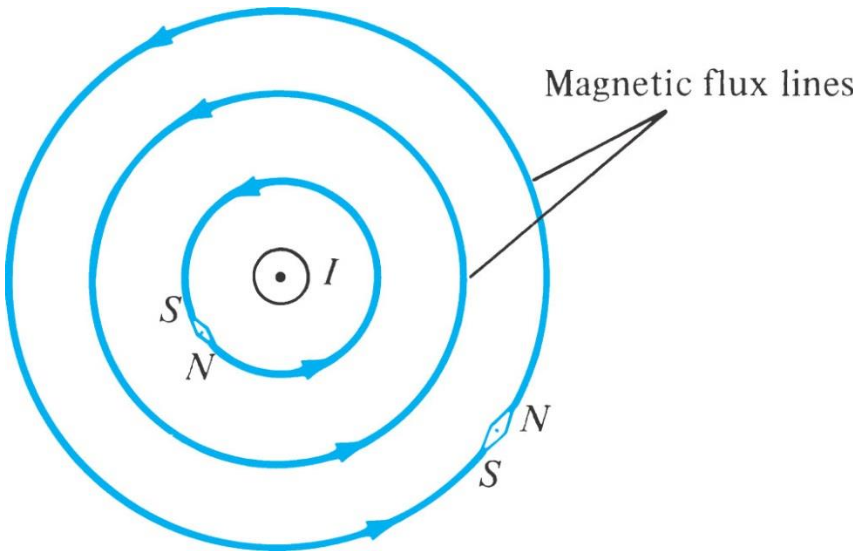
$$\boxed{\psi = \int_S \vec{B}.\overline{ds}}$$

Webers (Wb)

Magnetic Flux Density (contd.)

- Magnetic flux line is a path to which $\vec{B}$ is tangential at every point on the line.
- It is the line along which the needle of a magnetic compass will orient itself if placed in the presence of a magnetic field.

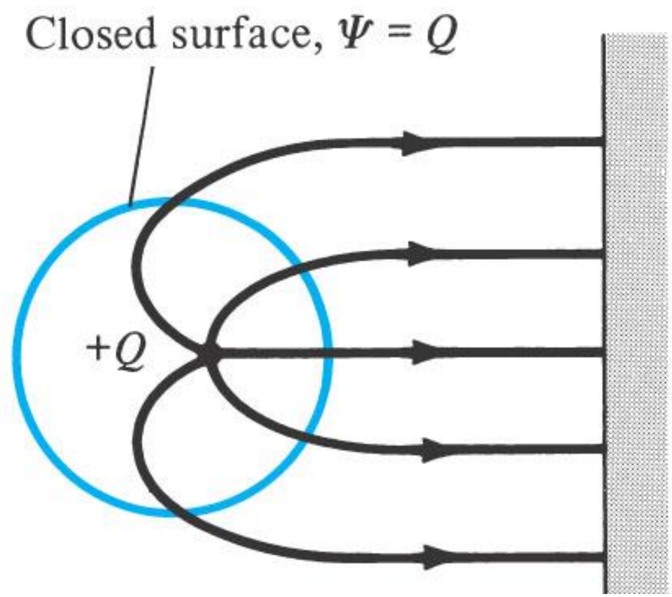
- For example, the magnetic flux lines due to a straight long wire is

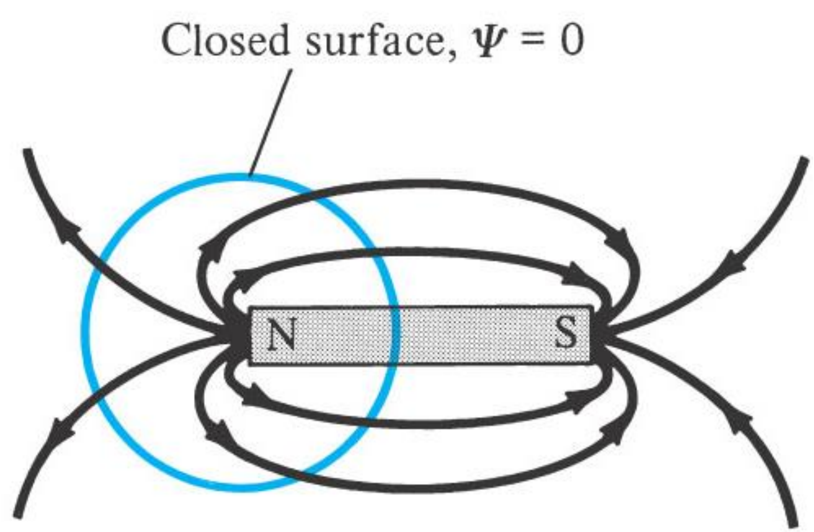


Note that each flux lines is closed and has no beginning or end. It is generally true that magnetic flux lines are closed and do not cross each other regardless of the current distribution.

Magnetic Flux Density (contd.)

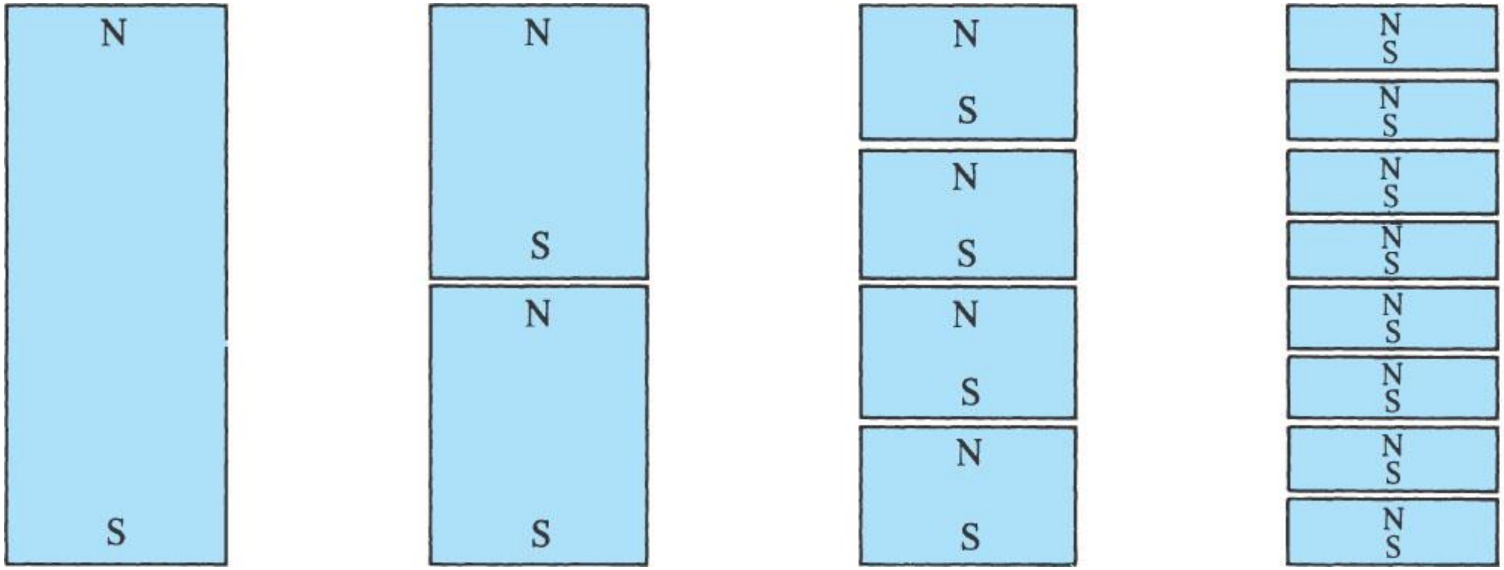
- In an electrostatic field, the flux passing through a closed surface is the same as charge enclosed ($\psi = \oint \vec{D}.\overline{ds} = Q$) → thus it is possible to have an isolated electric charge such that flux lines are not necessarily closed.
- Unlike electric flux lines, magnetic flux lines always close upon themselves → therefore, the total flux through a closed surface in a magnetic field must be zero ($\psi = \oint \vec{B}.\overline{ds} = 0$) → not possible to have isolated magnetic poles or magnetic charges.





Magnetic Flux Density (contd.)

- Thus, if we desire to have an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having north and south poles → we find it impossible to separate the north pole from the south pole.



Magnetic Flux Density (contd.)

$$\oint \vec{B}.\overline{ds} = 0$$

Law of conservation of magnetic flux or Gauss's law for magnetostatic fields

Divergence Theorem

$$\oint \vec{B}.\overline{ds} = \int_v \nabla.\vec{B}dv = 0$$

$$\nabla.\vec{B} = 0$$

Maxwell Equation

Magnetic fields have no source or sinks ↔ Magnetic field lines are always continuous

Maxwell's Equations for Static Fields

Differential Form	Integral Form	Remarks
$\nabla . \vec{D} = \rho_v$	$\oint_S \vec{D}.\overline{ds} = \int_v \rho_v dv$	Gauss's Law
$\nabla . \vec{B} = 0$	$\oint_S \vec{B}.\overline{ds} = 0$	None existence of magnetic monopole
$\nabla \times \vec{E} = 0$	$\oint_C \vec{E}.\overline{dl} = 0$	Conservative Nature of $\vec{E}$
$\nabla \times \vec{H} = \vec{J}$	$\oint_C \vec{H}.\overline{dl} = \int_S \vec{J}.\overline{ds}$	Ampere's Law

Magnetic Scalar and Vector Potentials

- We learnt, some electrostatic problems became simpler by relating electric field intensity $\vec{E}$ $\left(\vec{E} = -\nabla V\right)$.
- Similarly, one can define potential associated with $\vec{H}$ or $\vec{B}$.
- The idea is that $\vec{B}$ should be defined in such a way that divergence of $\vec{B}$ should be always zero.
- Actually, magnetic potential could be scalar denoted as V_m or vector denoted as $\vec{A}$.
- Let us use following two identities:

$$\boxed{\nabla \times (\nabla V) = 0}$$

This holds for any scalar V

$$\boxed{\nabla . (\nabla \times \vec{A}) = 0}$$

This holds for any vector $\vec{A}$

- We define the magnetic scalar potential as:

$$\vec{H} = -\nabla V_m \quad \Rightarrow \quad \vec{J} = \nabla \times \vec{H} = \nabla \times \left(-\nabla V_m\right)$$

Magnetic Scalar and Vector Potentials (contd.)

$$\vec{J} = \nabla \times \vec{H} = \nabla \times (-\nabla V_m) = 0$$

This Form

$$\nabla \times (\nabla V) = 0$$

Thus magnetic scalar potential is valid only in a region where $\vec{J} = 0$

Very useful term for defining parameters of a permanent magnet

V_m satisfies Laplace's equation

$$\nabla^2 V_m = 0$$

- Furthermore,

$$\underbrace{\boxed{\nabla . \vec{B} = 0} \qquad \boxed{\nabla . (\nabla \times \vec{A}) = 0}}_{\text{Gives definition of vector magnetic potential}}$$

Gives definition of vector magnetic potential

Magnetic Scalar and Vector Potentials (contd.)

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$

- We defined: $V = \int \frac{dQ}{4\pi\varepsilon_0 r}$

- Similarly we can define:

$$\vec{A} = \int_C \frac{\mu_0 I\,\overline{dl}}{4\pi R}$$ For line current

$$\vec{A} = \int_C \frac{\mu_0 \vec{K} ds}{4\pi R}$$ For surface current

$$\vec{A} = \int_v \frac{\mu_0 \vec{J} dv}{4\pi R}$$ For volume current

Magnetic Scalar and Vector Potentials (contd.)

- We can express flux alternatively as:

$$\psi = \int_S \vec{B}.\overline{ds} \Rightarrow \psi = \int_S \left(\nabla \times \vec{A}\right).\overline{ds} \xrightarrow{\text{Stoke's Theorem}} \psi = \int_C \vec{A}.\overline{dl}$$

Thus the magnetic flux through a given area can be found using the magnetic vector potential

The magnetic field can be determined through the use of either V_m or $\vec{A}$ → the choice is dependent on the type of problem → Obviously, V_m can be used only in source free region

The use of magnetic vector potential provides a powerful approach to solving EM problems, particularly those relating to antennas → For antennas, its more convenient to find $\vec{A}$ than finding $\vec{B}$

Example – 2

- Given the magnetic vector potential $\vec{A} = -\frac{\rho^2}{4}\hat{a}_z \frac{\mathrm{Wb}}{\mathrm{m}}$, calculate the total magnetic flux crossing the surface $\phi = \frac{\pi}{2}$, $1 \le \rho \le 2\, m$, $0 \le z \le 5\, m$.

Method-1: $\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_z}{\partial \rho}\hat{a}_\phi$ $\overline{dS} = d\rho dz \hat{a}_\phi$

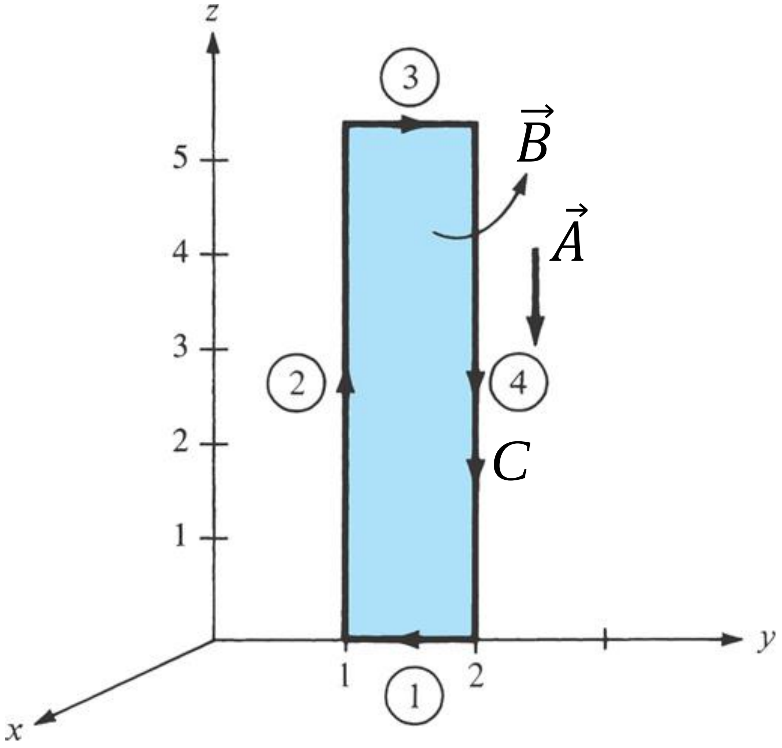
- Therefore: $\boxed{\psi = \int_S \vec{B}.\overline{ds}} \Rightarrow \boxed{\psi = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho d\rho dz} \Rightarrow \boxed{\psi = 3.75Wb}$

Example – 2 (contd.)

Method-2:

- We use: $\psi = \int_C \vec{A}.\overline{dl} = \psi_1 + \psi_2 + \psi_3 + \psi_4$

where, C is the path bounding surface S; ψ_1, ψ_2, ψ_3, and ψ_4 are respectively the evaluations of $\int \vec{A}.\overline{dl}$ along segments of C labeled 1 to 4.



- Since $\vec{A}$ has only z-component: $\psi_1 = \psi_3 = 0$

- Therefore: $\psi = \psi_2 + \psi_4 = -\frac{1}{4}\left[(1)^2 \int_0^5 dz + (2)^2 \int_5^0 dz \right]$ $\Rightarrow$ $\psi = 3.75 Wb$

Example – 3

- A current distribution gives rise to the vector magnetic potential $\vec{A} = x^2 y\hat{a}_x + y^2 x\hat{a}_y - 4xyz\hat{a}_z \frac{Wb}{m}$. Calculate the following:

  (a) $\vec{B}$ at $(-1,\ 2,\ 5)$

  (b) The flux through the surface defined by $\mathrm{z} = 1, 0 \le \mathrm{x} \le 1, -1 \le \mathrm{y} \le 4$

(a) $\vec{B} = \nabla \times \vec{A} \quad = (-4xz - 0)\hat{a}_x + (0 + 4yz)\hat{a}_y + (y^2 - x^2)\hat{a}_z$

$$\boxed{\therefore \vec{B}(-1, 2, 5) = 20\hat{a}_x + 40\hat{a}_y + 3\hat{a}_z}$$

Example – 3 (contd.)

(b) The flux through the given surface:

$$\psi = \int_S \vec{B}.\overline{ds} \Rightarrow \psi = \int_{y=-1}^{4}\int_{x=0}^{1}\left(y^2 - x^2\right)\partial x\partial y \Rightarrow \psi = 20Wb$$

Alternatively:

$$\psi = \int_C \vec{A}.\overline{dl} \Rightarrow \psi = \int_0^1 x^2(-1)\partial x + \int_{-1}^{4} y^2(1)\partial y + \int_1^0 x^2(4)\partial x + 0 \Rightarrow \psi = 20Wb$$