Lecture – 19

Date: 18.02.2014

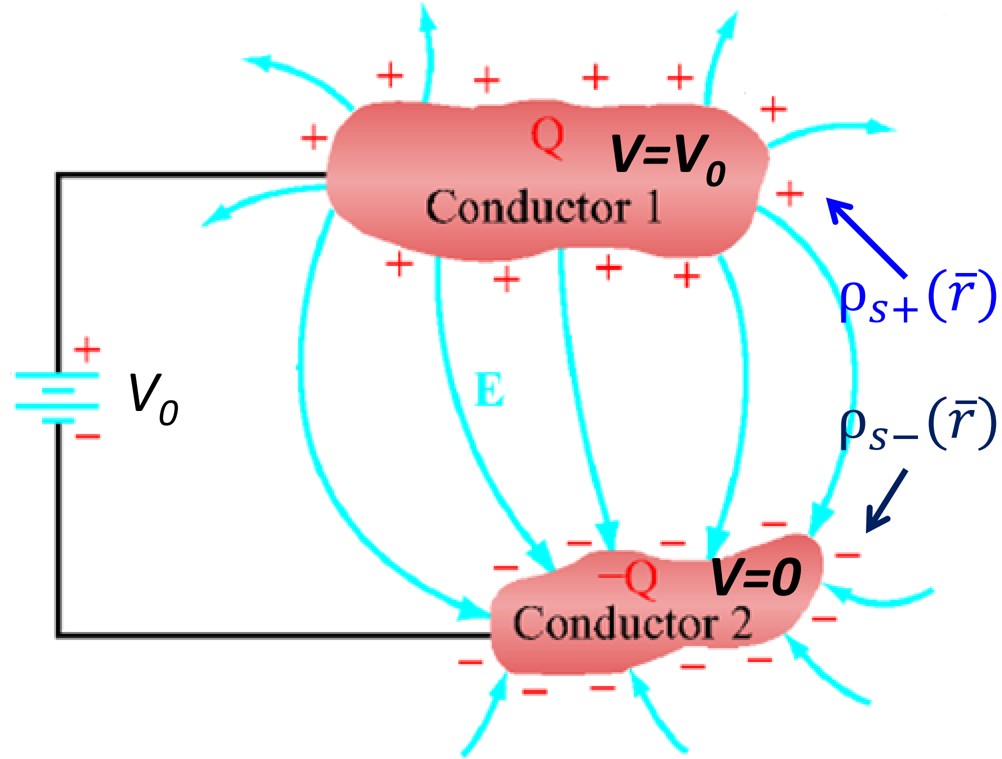
- Capacitances
- Energy Storage in a Capacitor

Capacitance

- Any two conducting bodies, when separated by an insulating (dielectric) medium, regardless of their shapes and sizes form a capacitor.
- If a dc voltage is connected across them, the surfaces of conductors connected to the positive and negative source terminals will accumulate charges +Q and –Q respectively.
- If a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor → to ensure that electric potential is same at every point in the conductor.

Capacitance (contd.)

- Consider two **conductors**, with a **potential difference** of V volts.



- Since there is a potential difference between the conductors, there must be an **electric potential field** $V(\bar{r})$, and therefore an **electric field** $\vec{E}(\bar{r})$ in the region between the conductors.

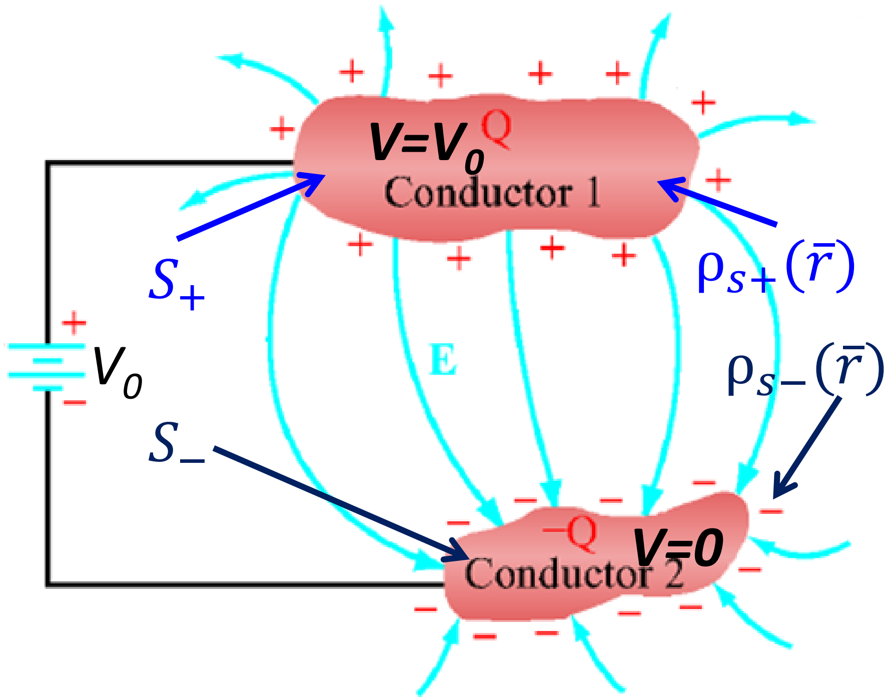
- Likewise, if there is an electric field, then we can specify an **electric flux density** $\vec{D}(\bar{r})$, which we can use to determine the **surface charge density** $\rho_s(\bar{r})$ on each of the conductors.

Capacitance (contd.)

- We find that if the total net charge on **one** conductor is Q then the charge on the **other** will be equal to $-Q$.
- In other words, the total net charge on each conductor will be **equal** but **opposite**!
- Note that this does **not** mean that the surface charge densities on each conductor are equal (i.e., $\rho_{s+}(\bar{r}) \neq \rho_{s-}(\bar{r})$). Rather, it means that:

$$\oiint_{S_+} \rho_{s+}(\bar{r})dS = -\oiint_{S_-} \rho_{s-}(\bar{r})dS = Q$$

where surface S_+ is the surface surrounding the conductor with the positive charge (and the higher electric potential), while the surface S_- surrounds the conductor with the negative charge.



Capacitance (contd.)

Q: How much free **charge** Q is there on each conductor, and how does this charge relate to the **voltage** V_0?

A: We can determine this from the mutual **capacitance** C of these conductors!

- The mutual **capacitance** between two conductors is **defined** as:

$$C = \frac{Q}{V} \qquad \left[\frac{Coulombs}{Volts} \equiv Farad\right]$$

where Q is the **total charge** on **each conductor**, and V is the **potential difference** between each conductor (for our example, $V = V_0$).

- Recall that the total charge on a conductor can be determined by **integrating** the surface charge density $\rho_s(\bar{r})$ across the **entire surface** S of a conductor:

$$Q = \oiint_{S_+} \rho_{s+}(\bar{r})dS = -\oiint_{S_-} \rho_{s-}(\bar{r})dS$$

Capacitance (contd.)

- But recall also that the surface charge density on the surface of a conductor can be determined from the **electric flux density** $\vec{D}(\bar{r})$:

$$\rho_s(\bar{r}) = \vec{D}(\bar{r}).\hat{a}_n$$

where $\hat{a}_n$ is a unit vector **normal** to the conductor.

- **Combining** the two equations, we get:

$$Q = \oint\!\!\!\oint_{S_+} \vec{D}(\bar{r}).\hat{a}_n dS = -\oint\!\!\!\oint_{S_-} \vec{D}(\bar{r}).\hat{a}_n dS \Rightarrow Q = \oint\!\!\!\oint_{S_+} \vec{D}(\bar{r}).\overline{dS} = -\oint\!\!\!\oint_{S_-} \vec{D}(\bar{r}).\overline{dS}$$

where we remember that $\overline{dS} = \hat{a}_n dS$.

- Hey! This is **no surprise**! We **already** knew that:

$$Q = \oint\!\!\!\oint_{S} \vec{D}(\bar{r}).\overline{dS}$$

This expression is also known as___________________!!

Capacitance (contd.)

- Note since $\vec{D}(\bar{r}) = \varepsilon\vec{E}(\bar{r})$ we can also say:

$$Q = \oiint_S \varepsilon \vec{E}(\bar{r}).\overline{dS}$$

- The **potential difference** V between two conductors can likewise be determined as:

$$V = \int_C \vec{E}(\bar{r}).\overline{dl}$$

where C is **any contour** that leads from one conductor to the other.

Q: Why **any** contour?
A:

- We can therefore determine the **capacitance** between two conductors as:

$$C = \frac{Q = \oiint_S \varepsilon \vec{E}(\bar{r}).\overline{dS}}{V = \int_C \vec{E}(\bar{r}).\overline{dl}} \quad [Farad]$$

- Where the contour C must start at **some** point on surface S_+ and end at **some** point on surface S_-.
- $S_+ = S_-$

Capacitance (contd.)

- Note this expression can be written as: $Q = CV$

In other words, the charge **stored** by two conductors is equal to the product of their mutual capacitance and the potential difference between them.

Therefore, the **greater** capacitance, the **greater** the amount of **charge** that is stored.

- Furthermore, try taking the **time derivative** of the above equation:

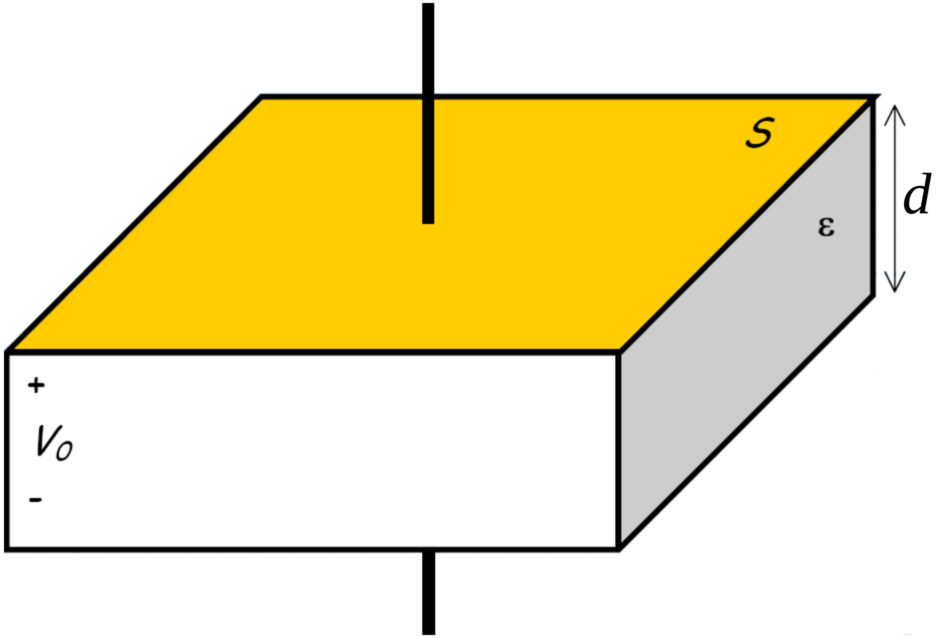
$$\frac{dQ}{dt} = C\frac{dV}{dt} \quad \Rightarrow \quad I = C\frac{dV}{dt}$$

Look familiar ?

By the way, the current I in this equation is **displacement current**.

The Parallel Plate Capacitor

- Consider the geometry of a **parallel plate capacitor**:



Recall that we determined the fields and surface charge density of an **infinite** pair of parallel plates. We can use those results to approximate the fields and charge densities of this **finite** structure, where the **area** of each plate is S.

Where:

V_0 = the **potential difference** between the plates

S = **surface area** of each conducting plate

d = **distance** between plates

ε = **permittivity** of the dielectric between the plates

The Parallel Plate Capacitor (contd.)

- For example, we determined that the **surface charge density** on the upper plate is:

$$\rho_{s+}(\bar{r}) = \frac{\varepsilon V_0}{d}$$

- The **total charge** on the upper plate is therefore:

$$Q = \iint_{S_+} \rho_{s+}(\bar{r})\,ds \;\Rightarrow\; Q = \iint_{S_+} \frac{\varepsilon V_0}{d}\,ds \;\Rightarrow\; Q = \frac{\varepsilon V_0}{d}\iint_{S_+} ds \;\Rightarrow\; Q = \frac{\varepsilon V_0 S}{d}$$

- The **capacitance** of this structure is therefore:

$$C = \frac{Q}{V} \;\Rightarrow\; C = \left(\frac{\varepsilon V_0 S}{d}\right)\left(\frac{1}{V_0}\right) \;\Rightarrow\; \therefore C = \frac{\varepsilon S}{d}$$

The Parallel Plate Capacitor (contd.)

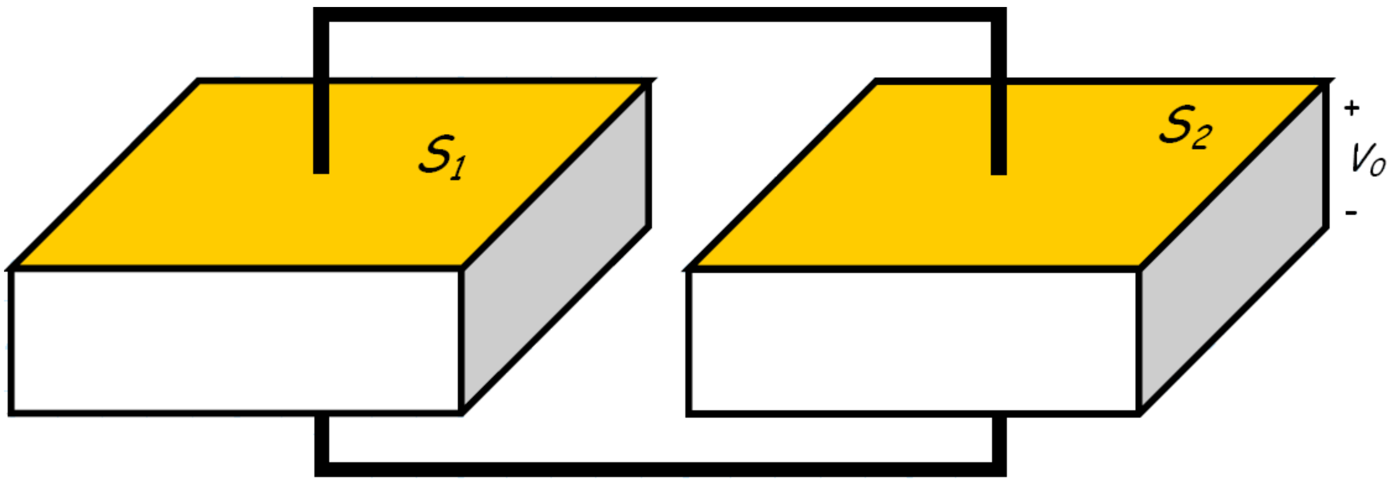
$$\therefore C = \frac{\varepsilon S}{d}$$

Therefore, we can **increase** the capacitance of a parallel plate capacitor by:

1) Increasing surface area S.
2) Decreasing separation distance d.
3) Increasing the dielectric permittivity ε.

The Parallel Plate Capacitor (contd.)

- Consider now the structure:



Note the **two** upper plates form **one** conducting structure, and the **two** bottom plates form **another**.

Q: What is the **capacitance** between these two conducting structures?

A: The potential difference between them is V_0. The **total charge** on one conducting structure is simply the **sum** of the charges on **each plate**:

$$Q = Q_1 + Q_2 = \frac{\varepsilon V_0 S_1}{d} + \frac{\varepsilon V_0 S_2}{d}$$

The Parallel Plate Capacitor (contd.)

- Therefore, the **capacitance** of this structure is:

$$C = \frac{Q}{V} = \left(\frac{\varepsilon V_0 (S_1 + S_2)}{d}\right)\left(\frac{1}{V_0}\right)$$

$$\Rightarrow C = \frac{\varepsilon (S_1 + S_2)}{d}$$

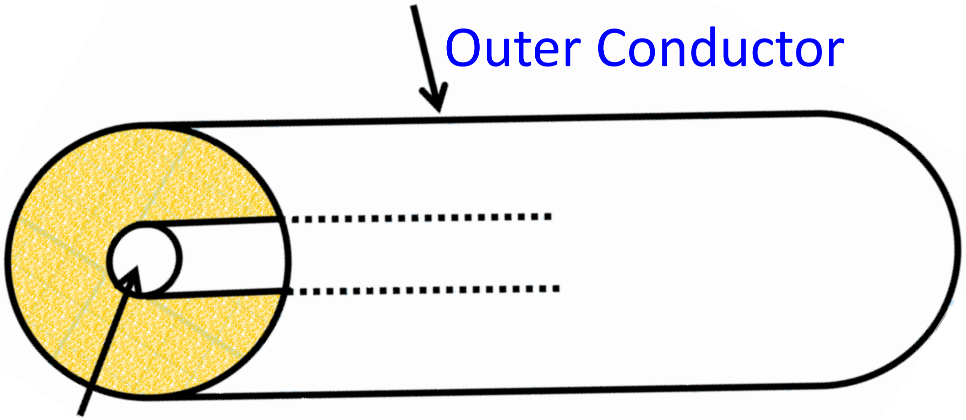
$$\Rightarrow C = \frac{\varepsilon S_1}{d} + \frac{\varepsilon S_2}{d}$$

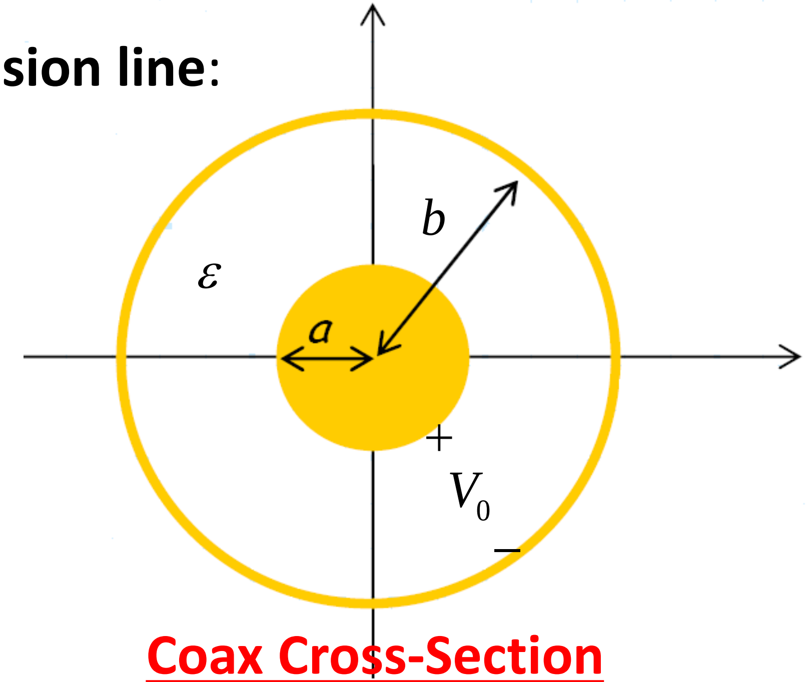
$$\Rightarrow C = C_1 + C_2$$

But **you** knew this! The total capacitance of two capacitors in **parallel** is equal to the **sum** of **each** capacitance.

Capacitance of a Coaxial Transmission Line

- Recall the geometry of a **coaxial transmission line**:





Coax Cross-Section

- We earlier determined that if a **potential difference** of V_0 volts is placed across the conductors, the **surface charge density** on the **inner** conductor is:

$$\rho_{sa}(\bar{r}) = \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \qquad \rho = a$$

Capacitance of a Coaxial Transmission Line (contd.)

- The **total charge** Q on the **inner** conductor of a coax of length l is determined by **integrating** the surface charge density across the **conductor surface**:

$$Q = \iint_{S_+} \rho_{s+}(\bar{r})ds$$

$$Q = \int_0^l \int_0^{2\pi} \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \rho d\phi dz$$

$$Q = \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \rho \int_0^l \int_0^{2\pi} d\phi dz$$

$$Q = \left[\frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \rho \right]_{\rho=a} \int_0^l \int_0^{2\pi} d\phi dz$$

$$\therefore Q = \frac{\varepsilon V_0}{\ln[b/a]} 2\pi l$$

Capacitance of a Coaxial Transmission Line (contd.)

- We can now determine the **capacitance** of this coaxial line!
- Since $C = Q/V$, and since the **potential difference** between the conductors is $V = V_0$, we find:

$$C = \frac{Q}{V} = \left(\frac{\varepsilon V_0}{\ln[b/a]} 2\pi l \right) \left(\frac{1}{V_0} \right) \Rightarrow C = \frac{2\pi\varepsilon}{\ln[b/a]} l$$

- This value represents the capacitance of a coaxial line of length l. A more useful expression is the capacitance of a coaxial line **per unit length** (e.g. farads/meter). We find this simply by **dividing** it by length l:

$$\frac{C}{l} = \frac{2\pi\varepsilon}{\ln[b/a]} \qquad \left[\frac{Farads}{metre}\right]$$

Capacitance of a Coaxial Transmission Line (contd.)

$$C = \frac{2\pi\varepsilon}{\ln\left[b/a\right]} l$$

Note the **longer** the transmission line, the **greater** the capacitance!

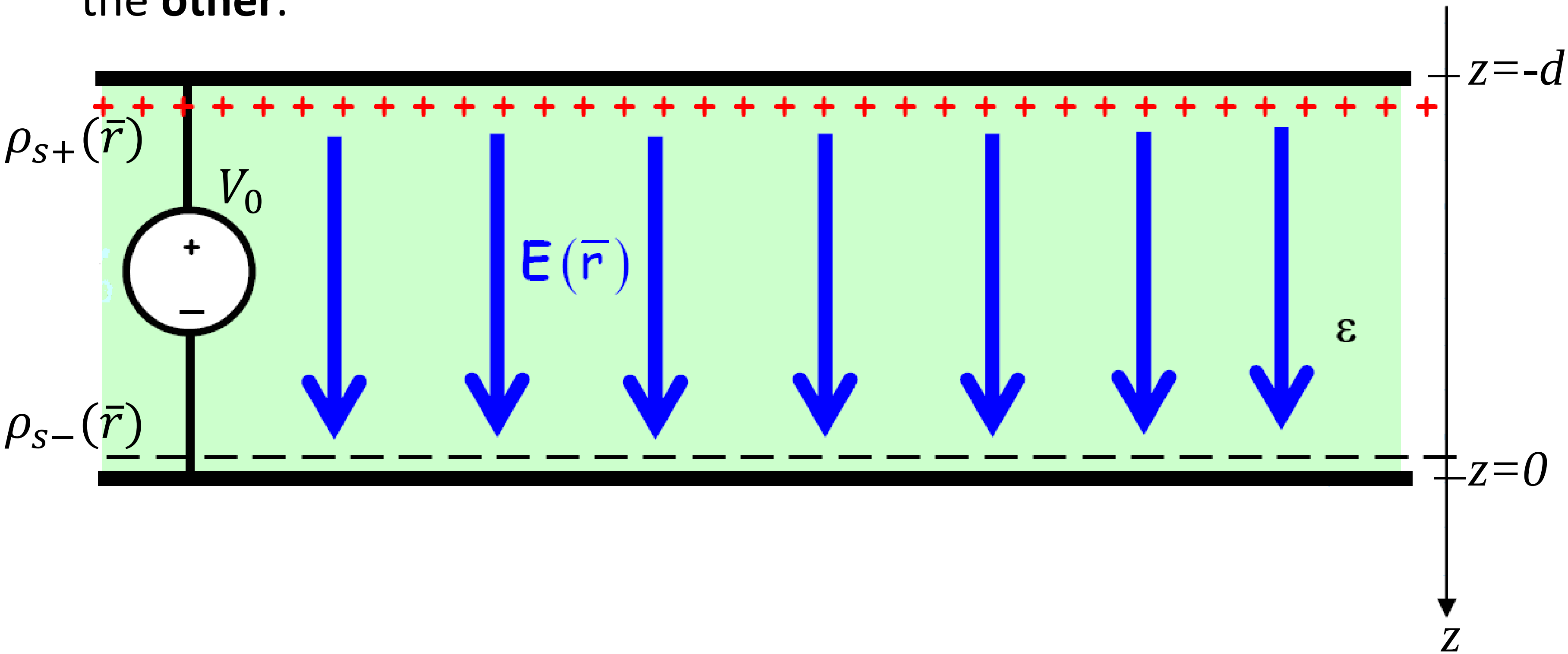
This can cause **great difficulty** if the voltage across the transmission line conductors is **time varying** (as it almost certainly will be!).

For **long** transmission lines, engineers cannot consider a transmission line simply as a "**wire**" conductor that connects circuit elements together. Instead, capacitance (and inductance) make the transmission line **itself** a **circuit element**!

In this case, engineers must use **transmission line theory** to analyze circuits!

Energy Storage in Capacitors

- Recall in a **parallel plate capacitor**, a surface charge distribution $\rho_{s+}(\bar{r})$ is created on **one** conductor, while charge distribution $\rho_{s-}(\bar{r})$ is created on the **other**.



Q: How much **energy** is stored by these charges?

Energy Storage in Capacitors (contd.)

- We learned that the energy **stored** by a **charge distribution** is:

$$W_e = \frac{1}{2}\iiint_v \rho_v(\bar{r})V(\bar{r})dv$$

- The **equivalent** equation for **surface** charge distributions is:

$$W_e = \frac{1}{2}\iint_S \rho_s(\bar{r})V(\bar{r})dS$$

- For the parallel plate capacitor, we must integrate over **both** plates:

$$W_e = \frac{1}{2}\iint_{S_+} \rho_{s+}(\bar{r})V(\bar{r})dS + \frac{1}{2}\iint_{S_-} \rho_{s-}(\bar{r})V(\bar{r})dS$$

- But on the **top** plate (i.e., S_+), we know that: $V(z=-d) = V_0$

- While on the **bottom** plate (i.e., S_-): $V(z=0) = 0$

Energy Storage in Capacitors (contd.)

- Therefore:

$$W_e = \frac{V_0}{2}\iint_{S_+} \rho_{s+}(\bar{r})dS + \frac{0}{2}\iint_{S_-} \rho_{s-}(\bar{r})dS$$

$$W_e = \frac{V_0}{2}\iint_{S_+} \rho_{s+}(\bar{r})dS$$

here

$$\iint_{S_+} \rho_{s+}(\bar{r})dS = Q$$

$$\therefore W_e = \frac{1}{2}QV$$

here

- $Q = CV$
- and V is the **potential difference** between the two conductors

- Combining these **two** equations, we find:

$$W_e = \frac{1}{2}CV^2$$

Energy Storage in Capacitors (contd.)

$$W_e = \frac{1}{2} C V^2$$

It shows that the **energy stored** within a capacitor is proportional to the product of its capacitance and the squared value of the voltage across the capacitor.

- Recall that we also can determine the stored energy from the **fields** within the dielectric:

$$W_e = \frac{1}{2} \iiint_v \vec{D}(\bar{r}).\vec{E}(\bar{r}) dv \quad \Rightarrow \quad W_e = \frac{1}{2} \frac{\varepsilon V^2}{d^2} (volume)$$

- Here $volume = Sd$, therefore:

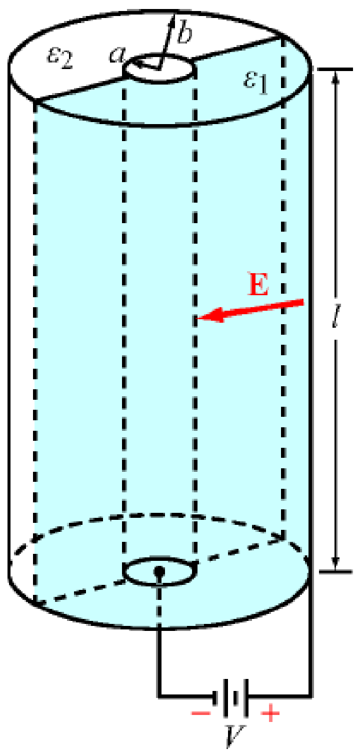
$$W_e = \frac{1}{2} \frac{\varepsilon S}{d} V^2 \quad \Rightarrow \quad W_e = \frac{1}{2} C V^2$$

Example

- A coaxial capacitor consists of two concentric, conducting, cylindrical surfaces, one of radius a and another of radius b. The insulating layer separating the two conducting surfaces is divided equally into two semi-cylindrical sections, one filled with dielectric ε_1 and the other filled with dielectric ε_2.

(a) Develop an expression for C in terms of the length l and the given quantities.

(b) Evaluate the value of C for a = 2 mm, b = 6 mm, ε_{r1} = 2, ε_{r2} = 4, and l = 4 cm.



Example (contd.)

(a) For the indicated voltage polarity, the **electric field** inside the capacitor exists in only the dielectric materials and **points radially inward**. Let $\vec{E}_1$ be the field in dielectric ε_1 and $\vec{E}_2$ be the field in dielectric ε_2.

(b) At the interface between the two dielectric sections, $\vec{E}_1$ is parallel to $\vec{E}_2$ and both are tangential to the interface.

(c) Since boundary conditions require that the tangential components of $\vec{E}_1$ and $\vec{E}_2$ be the same, it follows that:

$$\vec{E}_1 = \vec{E}_2 = -E\hat{a}_\rho$$

- At r = a (surface of inner conductor), in medium 1, the boundary condition on $\vec{D}$, leads to:

$$\vec{D}_1 = \varepsilon_1\vec{E}_1 = \rho_{s1}\hat{a}_n \quad \Rightarrow \quad -\varepsilon_1 E\hat{a}_\rho = \rho_{s1}\hat{a}_\rho \quad \Rightarrow \quad \rho_{s1} = -\varepsilon_1 E$$

Example (contd.)

- Similarly, in medium 2: $\rho_{s2} = -\varepsilon_2 E$
- Thus, the electric fields will be the same in the two dielectrics, but the charge densities will be different along the two sides of the inner conducting cylinder.
- Since the same voltage applies for the two sections of the capacitor, we can treat them as two capacitors in parallel. For first half of the cylinder that includes dielectric ε_1, we can express:

$$C_1 = \frac{\pi \varepsilon_1}{\ln[b/a]} l$$

Only half cylinder

- Similarly:

$$C_2 = \frac{\pi \varepsilon_2}{\ln[b/a]} l$$

Therefore:

$$C = C_1 + C_2 = \frac{\pi l (\varepsilon_1 + \varepsilon_2)}{\ln[b/a]}$$