Lecture – 9 Date: 01.02.2016

- Contours (Cartesian, Cylindrical, and Spherical)
- Surfaces (Cartesian, Cylindrical, and Spherical)
- Volume
- Gradient, Divergence, and Curl

The Contour *C*

Mathematically, a **contour** is described by:

2 equalities (e.g., **$x = 2$, $y = -4$; $r = 3$, $\phi = \pi/4$**) **AND**

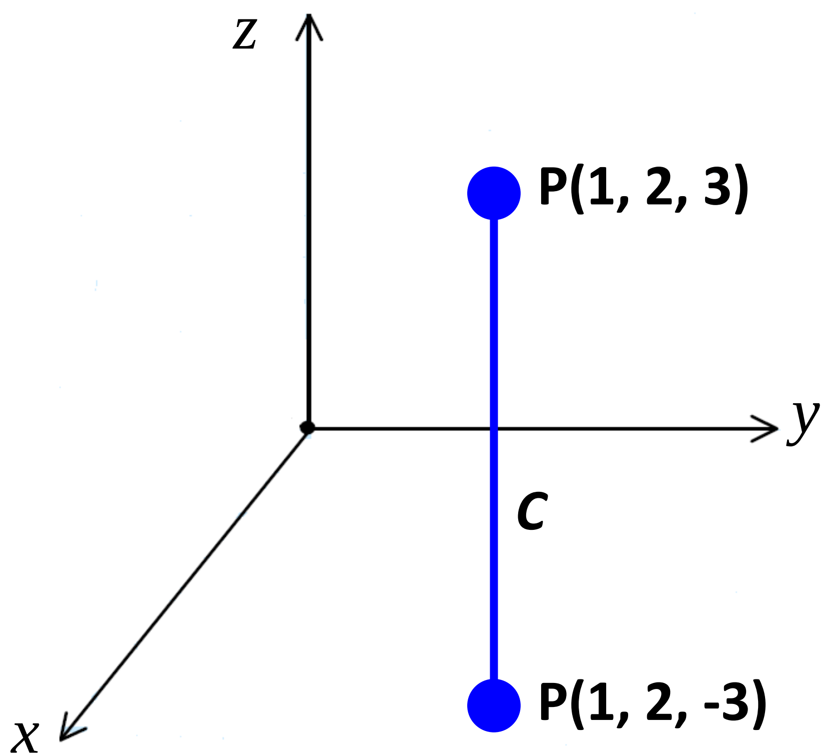
1 inequality (e.g., **$-1 < z < 5$; $0 < \theta < \pi/2$**)

- **Likewise,** we need to explicitly determine the **differential displacement vector** $\overline{dl}$ for each contour.

Recall we have studied **seven** coordinate parameters (x, y, z, ρ, ϕ, r, θ). As a result, we can form **seven** different contours C!

Cartesian Contours

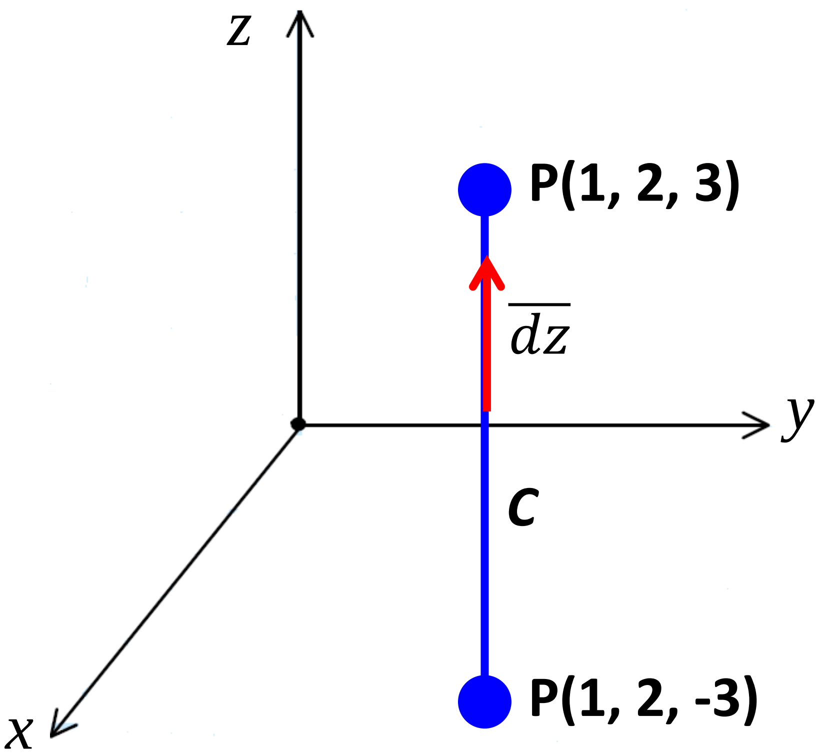
- Say we move a point from P(x =1, y =2, z =-3) to P(x =1, y =2, z=3) by changing **only the coordinate variable z from z =-3 to z=3**. In other words, the coordinate values x and y remain **constant** at x = 1 and y = 2.
- We form a contour that is a **line segment**, **parallel** to the z-axis!



Note that **every** point along this segment has coordinate values x =1 and y =2. As we move along the contour, **the only coordinate value that changes is *z*.**

Cartesian Contours (contd.)

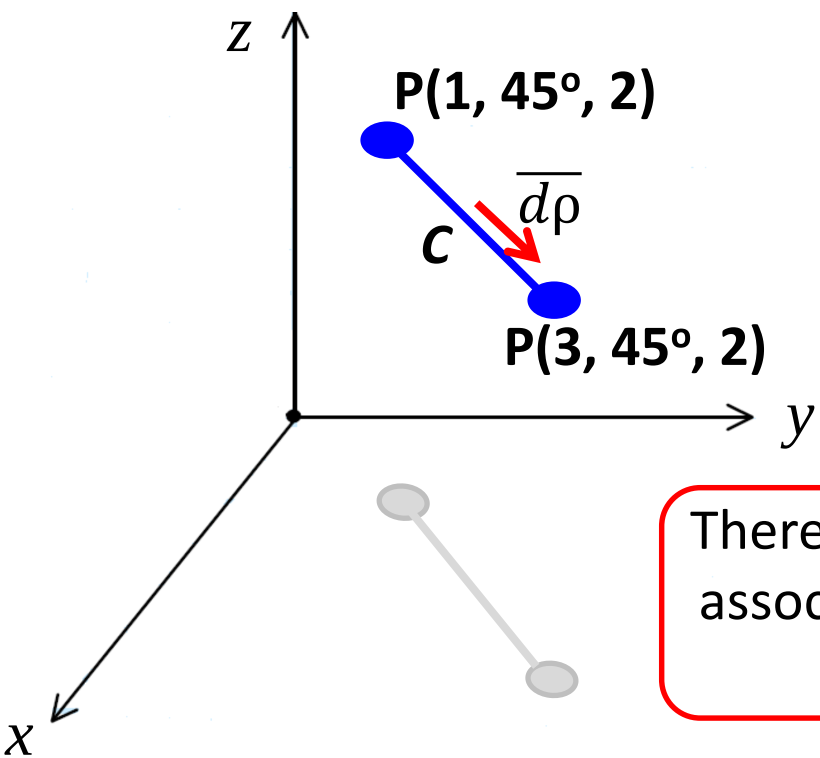
- Therefore, the **differential** directed distance associated with a change in position from z to z +dz, is $\overline{dl} = \overline{dz} = \hat{a}_z$dz



Similarly, a line segment parallel to the x-axis (or y-axis) can be formed by changing coordinate parameter x (or y), with a resulting differential displacement vector of $\overline{dl} = \overline{dx} = \hat{a}_x$dx (or $\overline{dl} = \overline{dy} = \hat{a}_y$dy).

Cylindrical Contours

- Say we move a point from P(ρ =1, ϕ = 45°, z =2) to P(ρ =3, ϕ = 45°, z =2) by changing only the coordinate variable ***ρ*** from ρ =1 to ρ =3. In other words, the coordinate values ϕ and z remain **constant** at ϕ = 45° and z =2.
- We form a contour that is a **line segment**, **parallel** to the x-y plane (i.e., perpendicular to the z-axis).

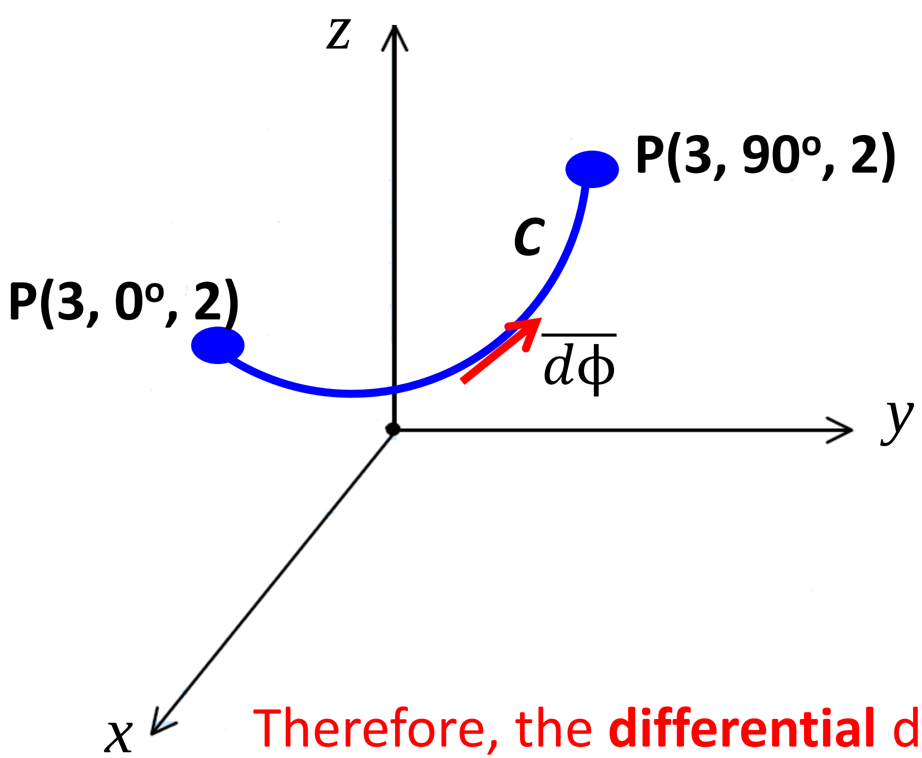


Note that **every** point along this segment has coordinate values ϕ = 45° and z =2. As we move along the contour, the **only** coordinate value that changes is ρ.

Therefore, the **differential** directed distance associated with a change in position from ρ to ρ+dρ, is $\overline{dl} = \overline{d\rho} = \hat{a}_\rho d\rho$

Cylindrical Contours (contd.)

- Alternatively, say we move a point from P(ρ =3, ϕ = 0°, z =2) to P(ρ =3, ϕ = 90°, z =2) by changing **only** the coordinate variable ϕ from ϕ = 0° to ϕ = 90°. In other words, the coordinate values ρ and z remain **constant** at ρ =3 and z =2. We form a contour that is a **circular arc**, parallel to the x-y plane.



Note: if we move from ϕ = 0° to ϕ = 360°, a complete **circle** is formed around the z-axis.

Every point along the arc has coordinate values ρ = 3 and z =2. As we move along the contour, the **only** coordinate value that changes is ϕ.

Therefore, the **differential** directed distance associated with a change in position from ϕ to ϕ+dϕ is $\overline{dl} = \overline{d\varphi} = \hat{a}_{\varphi}\rho d\phi$

Cylindrical Contours (contd.)

The three cylindrical contours are therefore described as:

1. Line segment parallel to the z-axis

$$\rho = c_\rho \qquad \phi = c_\phi \qquad c_{z1} \le z \le c_{z2} \quad \Longrightarrow \quad \overline{dl} = \hat{a}_z dz$$

2. Circular arc parallel to the xy-plane

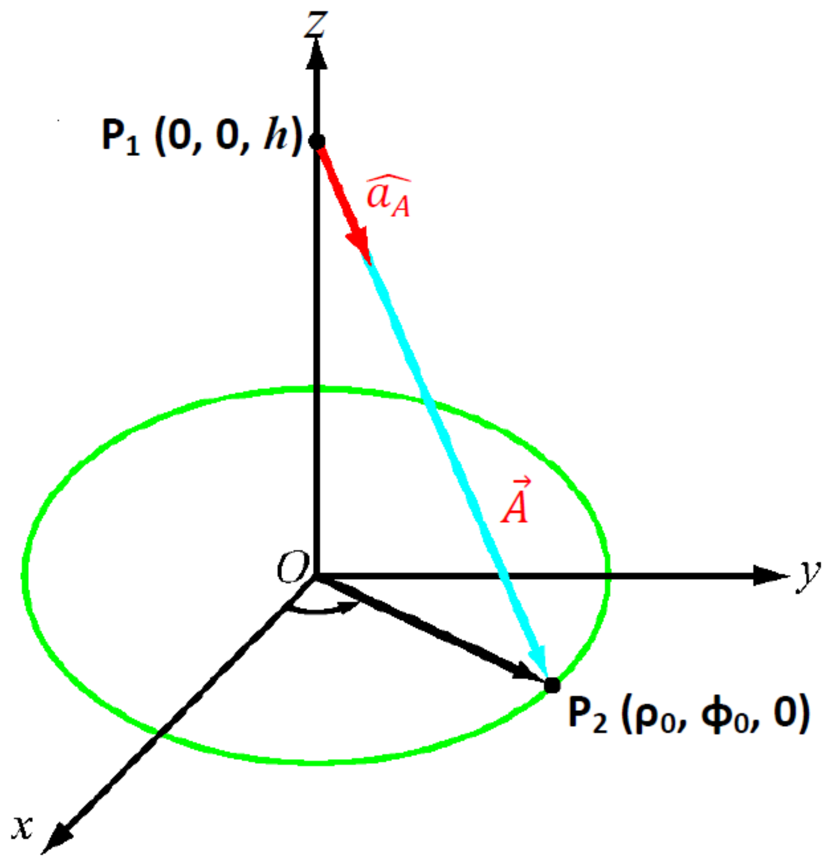
$$\rho = c_\rho \qquad z = c_z \qquad c_{\phi 1} \le \phi \le c_{\phi 2} \quad \Longrightarrow \quad \overline{dl} = \hat{a}_\phi \rho d\phi$$

3. Line segment parallel to the xy plane

$$\phi = c_\phi \qquad z = c_z \qquad c_{\rho 1} \le \rho \le c_{\rho 2} \quad \Longrightarrow \quad \overline{dl} = \hat{a}_\rho d\rho$$

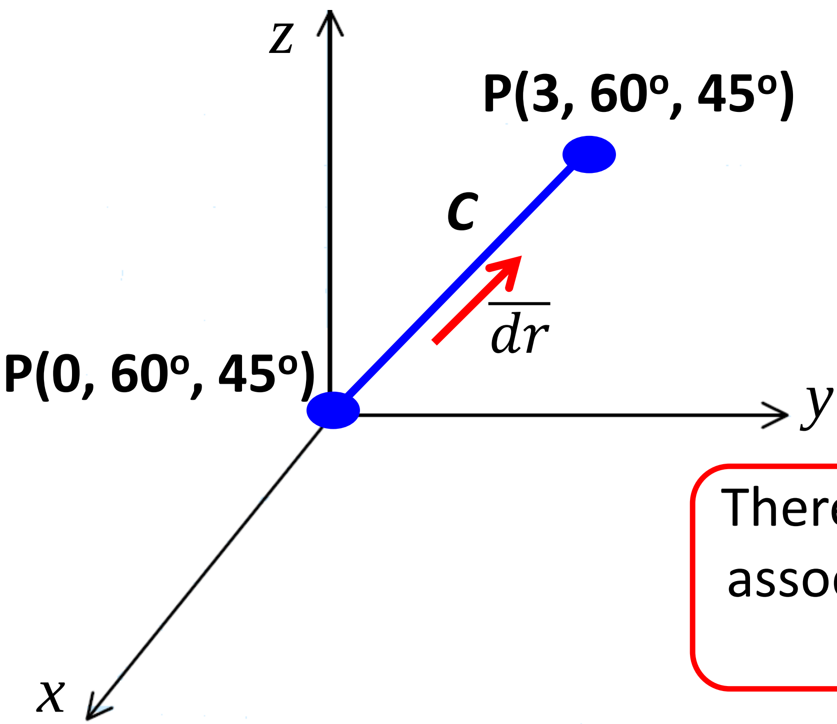
Example

Find an expression for the unit vector of $\vec{A}$ shown in the following Figure in cylindrical coordinates.



Spherical Contours

- Say we move a point from P(r =0, θ = 60°, ϕ = 45°) to P(r =3, θ = 60°, ϕ = 45°) by changing **only** the coordinate variable r from r=0 to r =3. In other words, the coordinate values θ and ϕ remain **constant** at θ = 60° and ϕ = 45°.
- We form a contour that is a **line segment**, emerging from the origin.

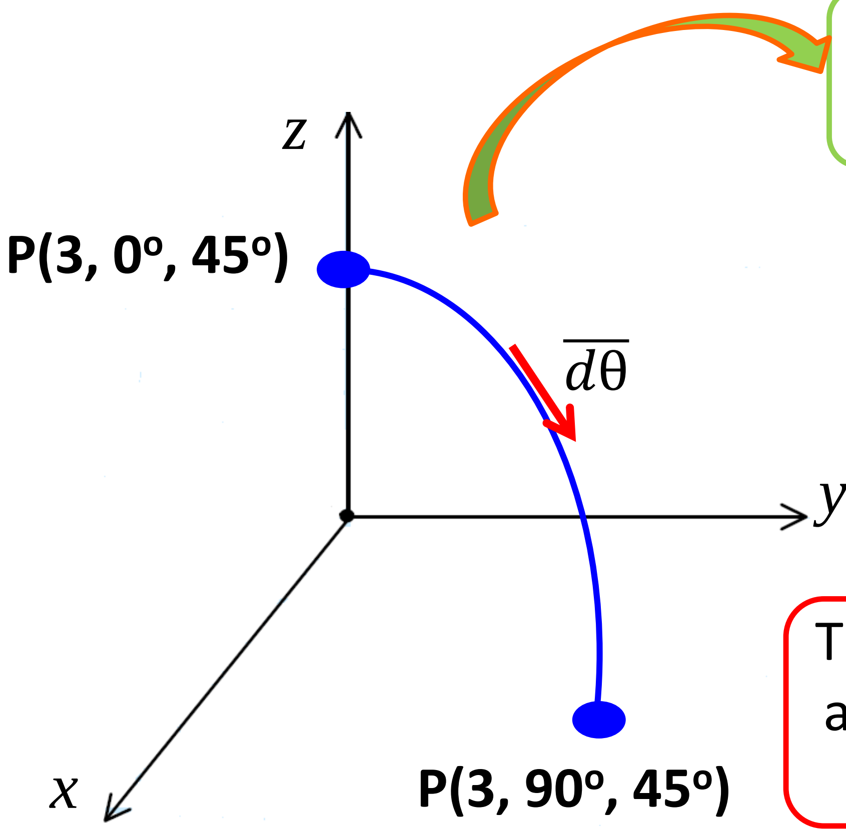


Every point along the line segment has coordinate values θ = 60° and ϕ = 45°. As we move along the contour, the **only** coordinate value that changes is r.

Therefore, the **differential** directed distance associated with a change in position from r to $r+dr$, is $\overline{dl} = \overline{dr} = \hat{a}_r$dr

Spherical Contours (contd.)

- Alternatively, say we move a point from P(r =3, θ = 0 º, ϕ = 45º) to P(r =3, θ = 90º, ϕ = 45º) by changing **only** the coordinate variable θ from θ = 0 º to θ =90º. In other words, the coordinate values r and ϕ remain **constant** at r = 3 and ϕ = 45º



We form a **circular arc**, whose plane includes the z-axis.

Every point along the arc has coordinate values r = 3 and ϕ = 45º. As we move along the contour, the **only** coordinate value that changes is θ.

Therefore, the **differential** directed distance associated with a change in position from θ to θ+dθ, is $\overline{dl} = \overline{d\theta} = \hat{a}_\theta r d\theta$

Spherical Contours (contd.)

- Finally, we could fix coordinates r and θ and vary coordinate ϕ only—but we **already** did this in cylindrical coordinates! We **again** find that a **circular arc** is generated, an arc that is parallel to the x-y plane.

The three spherical contours are therefore:

1. Circular arc parallel to the xy-plane

$$r = c_r \quad \theta = c_\theta \quad c_{\phi 1} \le \phi \le c_{\phi 2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_\phi r \sin\theta \, d\phi$$

2. Circular arc in a plane that includes z-axis

$$r = c_r \quad \phi = c_\phi \quad c_{\theta 1} \le \theta \le c_{\theta 2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_\theta r \, d\theta$$

3. Line segment directed towards the origin

$$\theta = c_\theta \quad \phi = c_\phi \quad c_{r1} \le r \le c_{r2} \quad \Rightarrow \quad \overline{dl} = \hat{a}_r \, dr$$

The Surface S

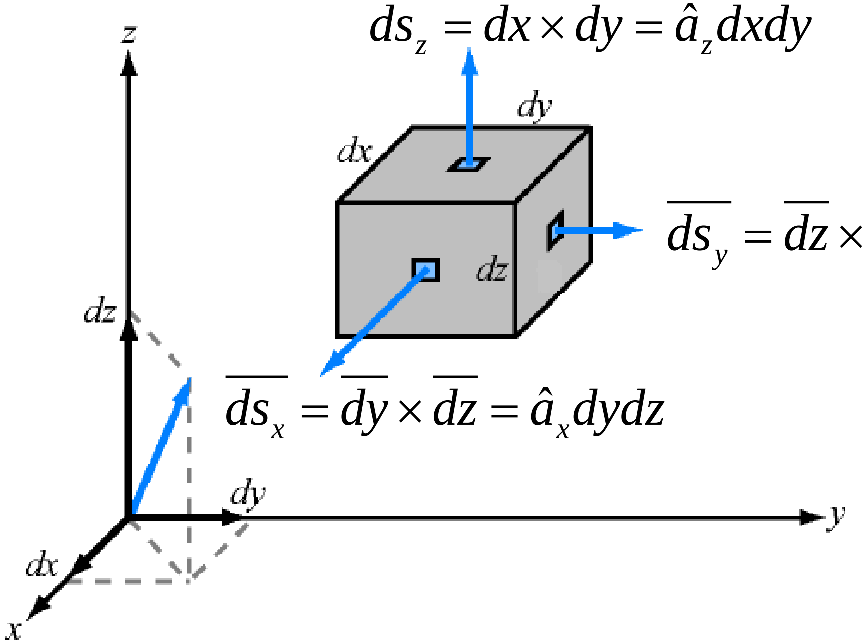
- Although **S** represents **any** surface, no matter how **complex** or **convoluted**, we will study only **basic** surfaces. In other words, $\overline{ds}$ will correspond to one of the differential surface vectors from Cartesian, cylindrical, or spherical coordinate systems.
- In this class, we will limit ourselves to studying only those surfaces that are formed when we change the location of a point by varying **two** coordinate parameters. In other words, the other coordinate parameters will remain **fixed**.

Mathematically, therefore, a surface is described by:

1 equality (e.g., **x=5** **OR** **r = 3**) **AND** **2 inequalities** (e.g., **-1 < y < 5** **and** **-2 < z < 7** **OR** **0 < θ < π/2** **and** **0 < ϕ < π**)

- Therefore, we will need to **explicitly** determine the **differential surface vector** $\overline{ds}$ for each contour.

Cartesian Coordinate Surfaces



$$\overline{ds_z} = \overline{dx} \times \overline{dy} = \hat{a}_z dxdy$$

$$\overline{ds_y} = \overline{dz} \times \overline{dx} = \hat{a}_y dxdz$$

$$\overline{ds_x} = \overline{dy} \times \overline{dz} = \hat{a}_x dydz$$

1. Flat plane parallel to y-z plane.

$$x = c_x \qquad c_{y1} \le y \le c_{y2} \qquad c_{z1} \le z \le c_{z2}$$

$$\overline{ds} = \pm \overline{ds_x} = \pm \hat{a}_x dydz$$

2. Flat plane parallel to x-z plane.

$$y = c_y \qquad c_{x1} \le x \le c_{x2} \qquad c_{z1} \le z \le c_{z2}$$

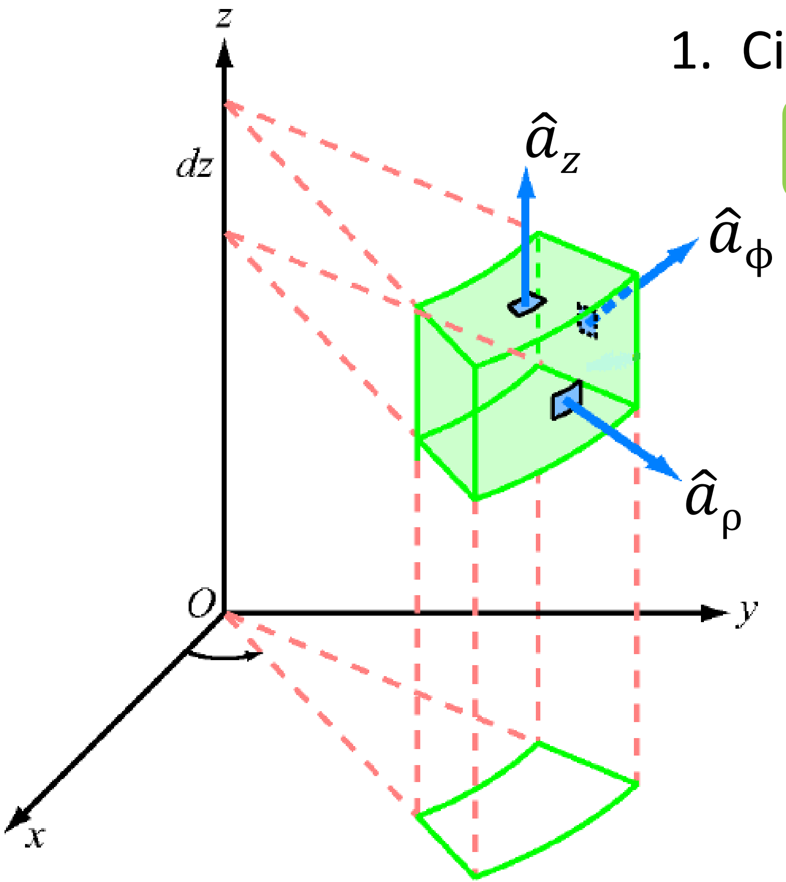
$$\overline{ds} = \pm \overline{ds_y} = \pm \hat{a}_y dxdz$$

3. Flat plane parallel to x-y plane.

$$z = c_z \qquad c_{x1} \le x \le c_{x2} \qquad c_{y1} \le y \le c_{y2}$$

$$\overline{ds} = \pm \overline{ds_z} = \pm \hat{a}_z dxdz$$

Cylindrical Coordinate Surfaces



1. Circular cylinder centered around the z-axis.

$$\rho = c_\rho \qquad c_{\phi 1} \le \phi \le c_{\phi 2} \qquad c_{z1} \le z \le c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\rho} = \pm \hat{a}_\rho \rho d\phi dz$$

2. Vertical plane extending from the z-axis

$$\phi = c_\phi \qquad c_{\rho 1} \le \rho \le c_{\rho 2} \qquad c_{z1} \le z \le c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi d\rho dz$$

3. Flat plane parallel to x-y plane.

$$z = c_z \qquad c_{\rho 1} \le \rho \le c_{\rho 2} \qquad c_{\phi 1} \le \phi \le c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_z} = \hat{a}_z \rho d\phi d\rho$$

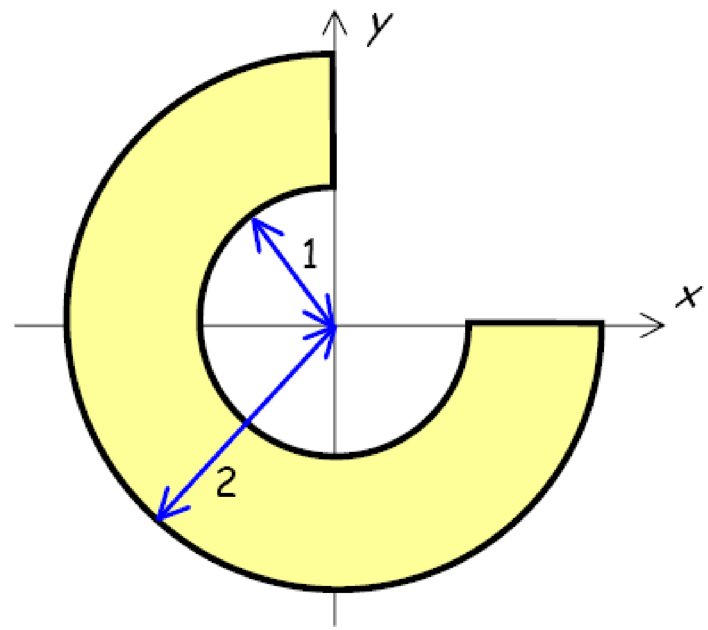
Cylindrical Coordinate Surfaces

Now let's see if **you've** been paying attention! Determine the two **inequalities** that define this flat surface.

$$z = 0$$

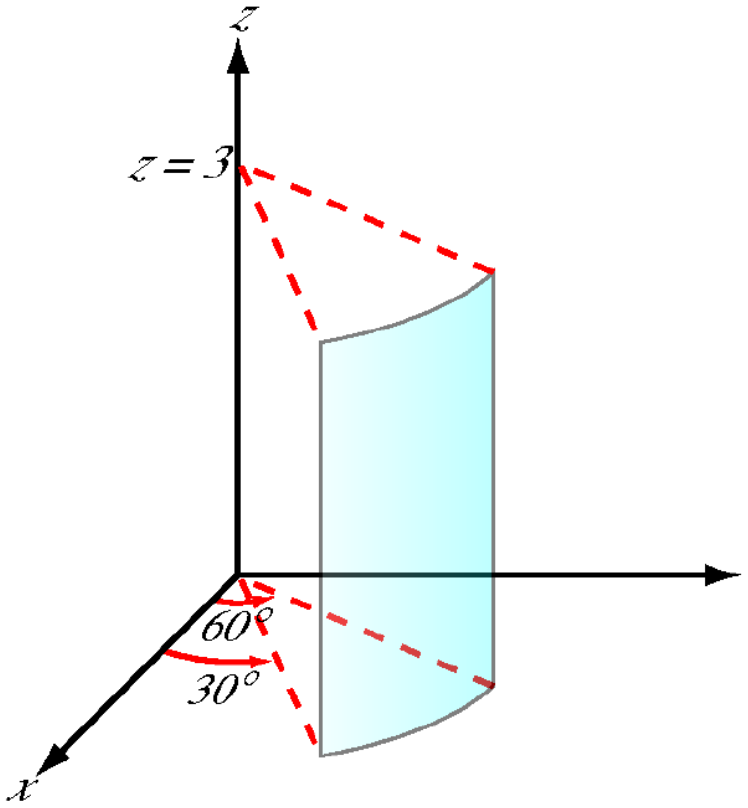
$$\leq \rho \leq$$

$$\leq \phi \leq$$

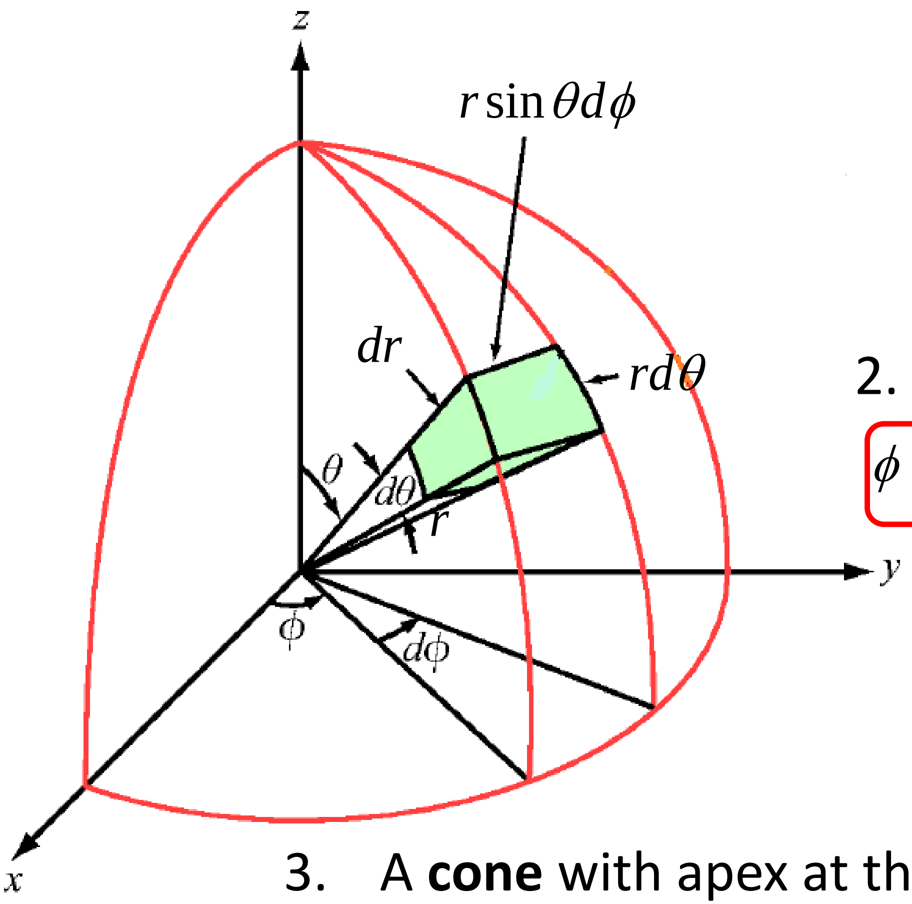


Example

Find the area of a cylindrical surface described by ρ = 5, $30° \leq \varphi \leq 60°$ in the following figure.



Spherical Coordinate Surfaces



1. **Sphere** centered at the origin.

$$r = c_r \qquad c_{\theta 1} \le \theta \le c_{\theta 2} \qquad c_{\phi 1} \le \phi \le c_{\phi 2}$$

$$\overline{ds} = \pm\overline{ds_r} = \pm\hat{a}_r r^2 \sin\theta d\theta d\phi$$

2. Vertical **plane** extending from the z-axis

$$\phi = c_\phi \qquad c_{\theta 1} \le \theta \le c_{\theta 2} \qquad c_{r1} \le r \le c_{r2}$$

$$\overline{ds} = \pm\overline{ds_z} = \pm\hat{a}_\phi r dr d\theta$$

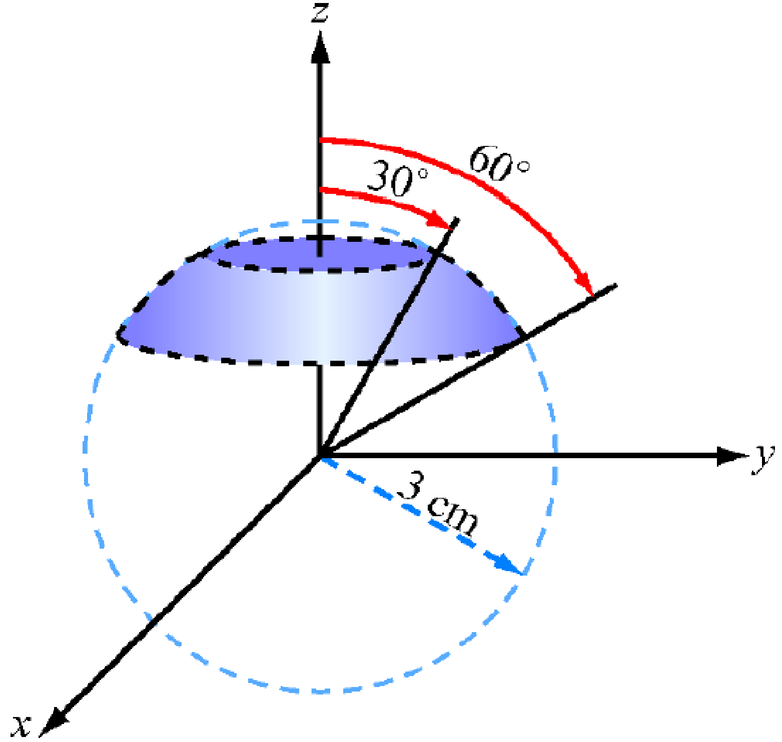
3. A **cone** with apex at the origin and aligned with the z-axis

$$\theta = c_\theta \qquad c_{r1} \le r \le c_{r2} \qquad c_{\phi 1} \le \phi \le c_{\phi 2}$$

$$\overline{ds} = \pm\overline{ds_\theta} = \pm\hat{a}_\theta r \sin\theta d\phi dr$$

Example

The spherical strip shown in following figure is a section of a sphere of radius 3 cm. Find the area of the strip.



The Volume V

- As we might expect from our knowledge about how to specify a **point** P (3 equalities), a **contour** C (2 equalities and 1 inequality), and a **surface** S (1 equality and 2 inequalities), a **volume** v is defined by **3 inequalities**.

Cartesian

The inequalities: $c_{x1} \leq x \leq c_{x2}$ $c_{y1} \leq y \leq c_{y2}$ $c_{z1} \leq z \leq c_{z2}$

define a **rectangular volume**, whose sides are parallel to the x-y, y-z, and x-z planes.

- The differential volume **dv** used for constructing this Cartesian volume is:

$$dv = dxdydz$$

$$\therefore v = \int_{c_{x1}}^{c_{x2}} \int_{c_{y1}}^{c_{y2}} \int_{c_{z1}}^{c_{z2}} dxdydz$$

The Volume V

Cylindrical

The inequalities: $c_{\rho 1} \leq \rho \leq c_{\rho 2}$ $c_{\phi 1} \leq \phi \leq c_{\phi 2}$ $c_{z1} \leq z \leq c_{z2}$

defines a **cylinder,** or some **subsection** thereof (e.g. a **tube**!).

- The differential volume **dv** is used for constructing this cylindrical volume is:

$$dv = \rho d\rho d\phi dz$$

$$\therefore v = \int_{c_{\rho 1}}^{c_{\rho 2}} \int_{c_{\phi 1}}^{c_{\phi 2}} \int_{c_{z1}}^{c_{z2}} \rho d\rho d\phi dz$$

Spherical

The inequalities: $c_{r1} \leq r \leq c_{r2}$ $c_{\theta 1} \leq \theta \leq c_{\theta 2}$ $c_{\phi 1} \leq \phi \leq c_{\phi 2}$

defines a **sphere**, or some subsection thereof (e.g., an "**orange slice**" !).

- The differential volume **dv** used for constructing this spherical volume is:

$$dv = r^2 \sin\theta \, dr d\theta d\phi$$

$$\therefore v = \int_{c_{r1}}^{c_{r2}} \int_{c_{\theta 1}}^{c_{\theta 2}} \int_{c_{\phi 1}}^{c_{\phi 2}} \rho d\rho d\phi dz$$

Example: The Volume Integral

Let's evaluate the **volume** integral: $\iiint_v g(\bar{r})dv$

where g($\bar{r}$) =1 and the volume *v* is a **sphere** with radius R.

In other words, the volume *v* is described for:

$$\iiint_v g(\bar{r})dv$$
$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$

- Therefore we use for the **differential** volume **dv**:

$$dv = \overline{dr}.\overline{d\theta} \times \overline{d\phi} = r^2 \sin\theta dr d\theta d\phi$$

- Therefore: $\iiint_v g(\bar{r})dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta dr d\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{R} r^2 dr = (2\pi)(2)\left(\frac{R^3}{3}\right)$

$$\therefore \iiint_v g(\bar{r})dv = \frac{4\pi R^3}{3}$$

Example: The Volume Integral

Q: So what's the volume integral even good for?

A: Generally speaking, the scalar function $g(\bar{r})$ will be a density function, with units of **things/unit volume**. Integrating $g(\bar{r})$ with the volume integral provides us the **number of things** within the space v!

For example, let's say $g(\bar{r})$ describes the **density** of a big **swarm of insects**, using units of **insects/m³** (i.e., insects are the **things**).

Note that $g(\bar{r})$ must indeed be a **function** of position, as the density of insects changes at different locations throughout the swarm.

Example: The Volume Integral

- Now, say we want to know the total number of insects within the swarm, which occupies some space v. We can determine this by simply applying the volume integral!

$$\text{number of insects in swarm} = \iiint_{v} g(\bar{r})\,dv$$

where space v completely encloses the insect swarm.

The Gradient Operator in Coordinate Systems

- For the **Cartesian** coordinate system, the Gradient of a scalar field T is expressed as:

$$\nabla T = \frac{\partial T}{\partial x}\hat{a}_x + \frac{\partial T}{\partial y}\hat{a}_y + \frac{\partial T}{\partial z}\hat{a}_z$$

Gradient Operator: $\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$

- Now let's consider the gradient operator in the **other** coordinate systems.

- Pfft! This is easy! The gradient operator in the spherical coordinate system is:

$$\nabla T = \frac{\partial T}{\partial r}\hat{a}_r + \frac{\partial T}{\partial \theta}\hat{a}_\theta + \frac{\partial T}{\partial \phi}\hat{a}_\phi$$

Right ??

NO!! The above equation is **not** correct!

- Instead, for **spherical** coordinates, the gradient is expressed as:

$$\nabla T = \frac{\partial T}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{a}_\phi$$

- And for the **cylindrical** coordinate system:

$$\nabla T = \frac{\partial T}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\hat{a}_\phi + \frac{\partial T}{\partial z}\hat{a}_z$$

Example

Find the directional derivative of $T = x^2 + y^2 z$ along direction $2\hat{a}_x + 3\hat{a}_y - 2\hat{a}_z$ and evaluate it at $(1, -1, 2)$.

Example

Find the gradient of $\mathrm{V} = V_0 e^{-2\rho} sin3\varphi$ at $(1, \pi/2, 3)$ in cylindrical coordinates.

Example

Find the gradient of $\mathrm{U} = U_0 \left(\frac{a}{r}\right) cos2\theta$ at $(2a, 0, \pi)$ in spherical coordinates.

The Conservative Vector Field

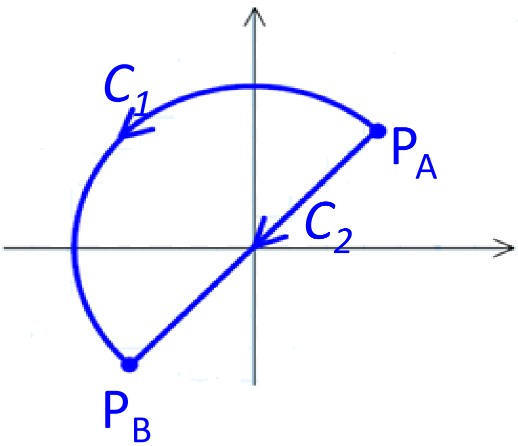
- Of all possible vector fields $\vec{A}(\bar{r})$, there is a subset of vector fields called **conservative** fields. A conservative vector field is a vector field that can be expressed as the **gradient** of some scalar field $g(\bar{r})$:

$$\vec{C}(\bar{r}) = \Delta g(\bar{r})$$

In other words, the gradient of **any** scalar field **always** results in a conservative field!

- A conservative field has the interesting property that its line integral is dependent on the **beginning** and **ending** points of the contour **only**! In other words, for the two contours:

$$\int_{C_1} \vec{C}(\bar{r}).\overline{dl} = \int_{C_2} \vec{C}(\bar{r}).\overline{dl}$$



- We therefore say that the line integral of a conservative field **is path independent**.

The Conservative Vector Field (contd.)

- This path independence is evident when considering the **integral identity**:

$$\int_C \nabla g(\bar{r}).\overline{dl} = g\left(\bar{r} = \bar{r}_B\right) - g\left(\bar{r} = \bar{r}_A\right)$$

position vector $\bar{r}_B$ denotes the **ending** point (P_B) of contour C, and $\bar{r}_A$ denotes the **beginning** point (P_A). $g(\bar{r} = \bar{r}_B)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_B$, and $g(\bar{r} = \bar{r}_A)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_A$.

- For **one** dimension, the above identity simply reduces to the familiar expression:

$$\int_{x_a}^{x_b} \frac{\partial g(x)}{\partial x} dx = g\left(x = x_b\right) - g\left(x = x_a\right)$$

- Since **every** conservative field can be written in terms of the **gradient** of a scalar field, we can use this identity to conclude:

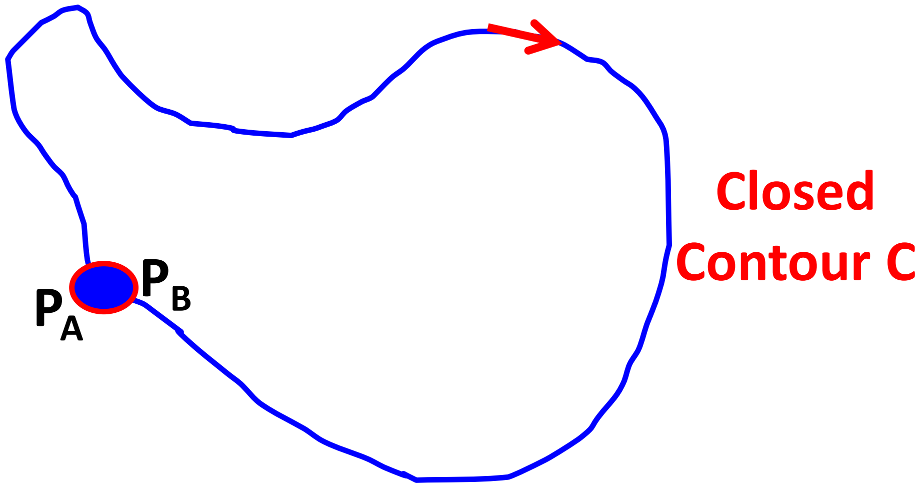
$$\int_C \vec{C}(\bar{r}).\overline{dl} = \int_C \nabla g(\bar{r}).\overline{dl} \quad \Rightarrow \quad \therefore \int_C \vec{C}(\bar{r}).\overline{dl} = g\left(\bar{r} = \bar{r}_B\right) - g\left(\bar{r} = \bar{r}_A\right)$$

Consider then what happens then if we integrate over a **closed** contour.

The Conservative Vector Field (contd.)

Q: What the heck is a closed contour ??

A: A closed contour's beginning and ending is the same point! e.g.,



A contour that is **not** closed is referred to as an **open** contour.

- Integration over a closed contour is **denoted** as:

$$\oint_C \vec{A}(\bar{r}).\overline{dl}$$

- The integration of a **conservative** field over a **closed** contour is therefore:

$$\oint_C \vec{C}(\bar{r}).\overline{dl} = \oint_C \nabla g(\bar{r}).\overline{dl} \Rightarrow = g(\bar{r}=\bar{r}_B) - g(\bar{r}=\bar{r}_A) \Rightarrow = 0$$

This result is due to the fact that $\bar{r}_A = \bar{r}_B \Rightarrow g(\bar{r}=\bar{r}_B) = g(\bar{r}=\bar{r}_A)$

The Conservative Vector Field (contd.)

- Let's **summarize** what we know about a **conservative** vector field:

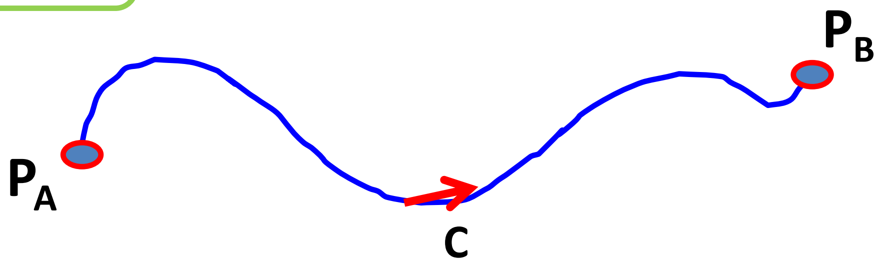
1. A conservative vector field can always be expressed as the **gradient** of a **scalar** field.
2. The gradient of **any** scalar field is therefore a conservative vector field.
3. Integration over an **open** contour is dependent **only** on the value of scalar field $g(\bar{r})$ at the beginning and ending points of the contour (i.e., integration is **path independent**).
4. Integration of a conservative vector field over any **closed** contour is always equal to **zero**.

Example

- Consider the conservative vector field: $\vec{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$
- Evaluate the contour integral: $\int_C \vec{A}(\bar{r}).\overline{dl}$

where $\vec{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$

and contour C is:



- The **beginning** of contour C is the point denoted as: $\bar{r}_A = 3\hat{a}_x - \hat{a}_y + 4\hat{a}_z$
- while the **end** point is denoted with position vector: $\bar{r}_B = -3\hat{a}_x - 2\hat{a}_z$

Note that ordinarily, this would be an **impossible** problem for **us** to do!

Example (contd.)

- we note that vector field $\vec{A}(\bar{r})$ is **conservative,** therefore:

$$\int_C \vec{A}(\bar{r}).\overline{dl} = \int_C \nabla g(\bar{r}).\overline{dl} \Rightarrow = g(\bar{r} = \bar{r}_B) - g(\bar{r} = \bar{r}_A)$$

- For this problem, it is evident that: $g(\bar{r}) = (x^2 + y^2)z$

- Therefore, $g(\bar{r} = \overline{r_A})$ is the **scalar** field evaluated at $x = 3, y = -1, z = 4$; while $g(\bar{r} = \overline{r_B})$ is the **scalar** field evaluated at at $x = -3, y = 0, z = -2$.

$$g(\bar{r} = \bar{r}_A) = \left((3)^2 + (-1)^2\right)4 = 40 \qquad g(\bar{r} = \bar{r}_B) = \left((-3)^2 + (0)^2\right)(-2) = -18$$

Therefore:

$$\int_C \vec{A}(\bar{r}).\overline{dl} = \int_C \nabla g(\bar{r}).\overline{dl} \Rightarrow = -18 - 40 = -58$$

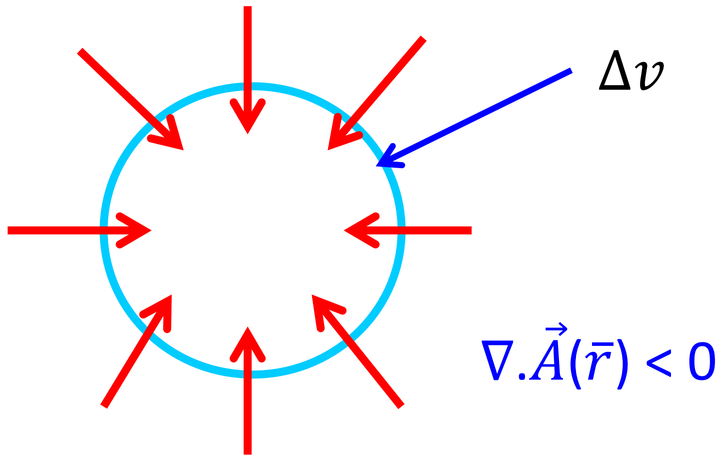
The Divergence of a Vector Field

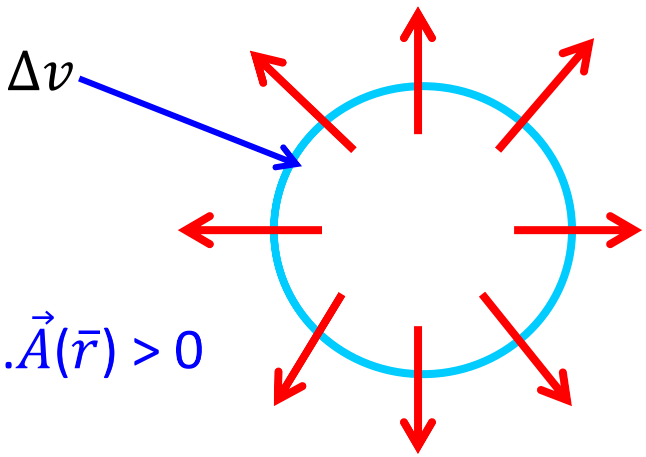
- The **mathematical** definition of divergence is:

$$\nabla.\vec{A}(\bar{r}) = \lim_{\Delta v \to 0} \frac{\oiint_S \vec{A}(\bar{r}).\overline{ds}}{\Delta v}$$

where the surface S is a **closed** surface that **completely** surrounds a **very small** volume Δv at point $\bar{r}$, and $\overline{ds}$ points **outward** from the closed surface.

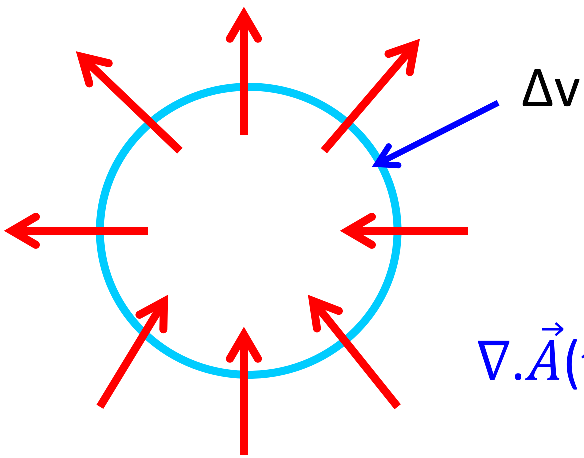
- The divergence indicates the amount of vector field $\vec{A}(\bar{r})$ that is **converging to**, or **diverging from**, a given point.
- For example, consider the vector fields in the region of a **specific point**:



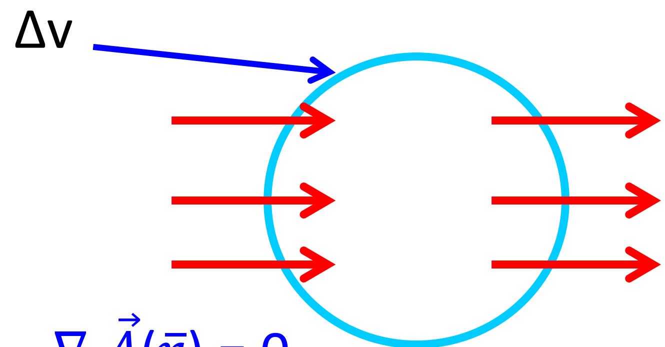


The Divergence of a Vector Field (contd.)

- Lets consider some **other** vector fields in the region of a specific point:



$\nabla . \vec{A}(\bar{r}) = 0$



$\nabla . \vec{A}(\bar{r}) = 0$

Cartesian

$$\nabla . \vec{A}(\bar{r}) = \frac{\partial A_x(\bar{r})}{\partial x} + \frac{\partial A_y(\bar{r})}{\partial y} + \frac{\partial A_z(\bar{r})}{\partial z}$$

Cylindrical

$$\nabla . \vec{A}(\bar{r}) = \frac{1}{\rho}\left[\frac{\partial\left(\rho A_\rho(\bar{r})\right)}{\partial \rho}\right] + \frac{1}{\rho}\frac{\partial A_\phi(\bar{r})}{\partial \phi} + \frac{\partial A_z(\bar{r})}{\partial z}$$

Spherical

$$\nabla . \vec{A}(\bar{r}) = \frac{1}{r^2}\left[\frac{\partial\left(r^2 A_r(\bar{r})\right)}{\partial r}\right] + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta(\bar{r})\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi(\bar{r})}{\partial \phi}$$

The Divergence Theorem

- Recall we studied volume integrals of the form: $\iiint_v g(\bar{r})dv$

- It turns out that **any** and **every** scalar field can be written as the divergence of some **vector** field, i.e.: $g(\bar{r}) = \nabla.\vec{A}(\bar{r})$

- Therefore we can equivalently write any volume integral as: $\iiint_v \nabla.\vec{A}(\bar{r})dv$

- The **divergence theorem** states that this integral is equal to: $\iiint_v \nabla.\vec{A}(\bar{r})dv = \oiint_S \vec{A}(\bar{r}).\overline{ds}$

where S is the **closed** surface that completely surrounds volume v, and vector $\overline{ds}$ points **outward** from the closed surface. For example, if volume v is a **sphere**, then S is the **surface** of that sphere.

The divergence theorem states that the **volume** integral of a scalar field can be likewise evaluated as a **surface** integral of a vector field!

Example

Determine the divergence of $\vec{E} = 3x^2\hat{a}_x + 2z\hat{a}_y + x^2z\hat{a}_z$ and evaluate it at $(2, -2, 0)$.

Example

Determine the divergence of $\vec{E} = \hat{a}_r\left({a^3 cos\theta}/{r^2}\right) - \hat{a}_\theta\left({a^3 sin\theta}/{r^2}\right)$ and evaluate it at $\left(\frac{a}{2}, 0, \pi\right)$.

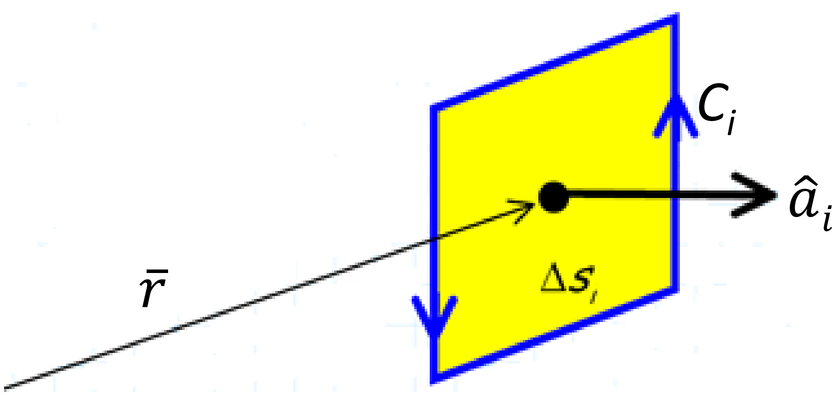
The Curl of a Vector Field

Say $\nabla \times \vec{A}(\bar{r}) = \vec{B}(\bar{r})$. The **mathematical** definition of Curl is given as:

$$B_i(\bar{r}) = \lim_{\Delta s \to 0} \frac{\oint_{C_i} \vec{A}(\bar{r}).\overline{dl}}{\Delta s_i}$$

This rather complex equation requires some **explanation** !

- $B_i(\bar{r})$ is the scalar component of vector $\vec{B}(\bar{r})$ in the direction defined by unit vector $\hat{a}_i$ (e.g., $\hat{a}_x$, $\hat{a}_\rho$, $\hat{a}_\theta$).
- The small surface Δs_i is centered at point $\bar{r}$, and oriented such that it is normal to unit vector $\hat{a}_i$.
- The contour C_i is the closed contour that surrounds surface Δs_i.



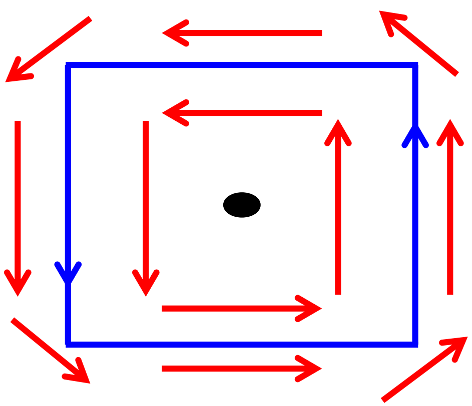
Note that this derivation must be completed for **each** of the **three** orthonormal base vectors in order to completely define $\nabla \times \vec{A}(\bar{r}) = \vec{B}(\bar{r})$.

The Curl of a Vector Field (contd.)

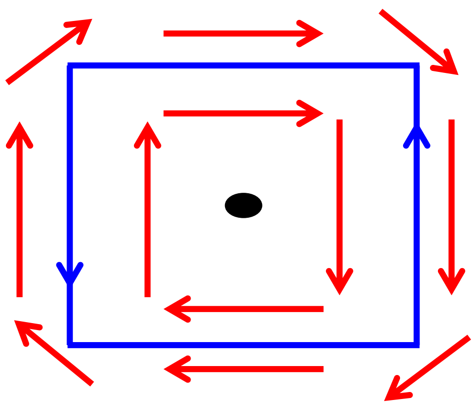
Q: What does curl tell us ?

A: Curl is a measurement of the **circulation** of vector field $\vec{A}(\bar{r})$ around point $\bar{r}$.

- If a component of vector field $\vec{A}(\bar{r})$ is pointing in the direction $\overline{dl}$ at every point on contour C_i (i.e., **tangential** to the contour). Then the line integral, and thus the curl, will be **positive**.
- If, however, a component of vector field $\vec{A}(\bar{r})$ points in the opposite direction $(-\overline{dl})$ at every point on the contour, the curl at point $\bar{r}$ will be **negative**.



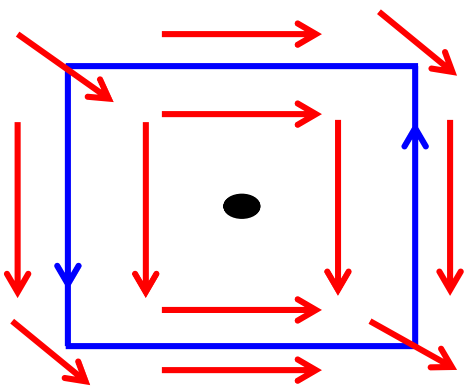
$B_i > 0$



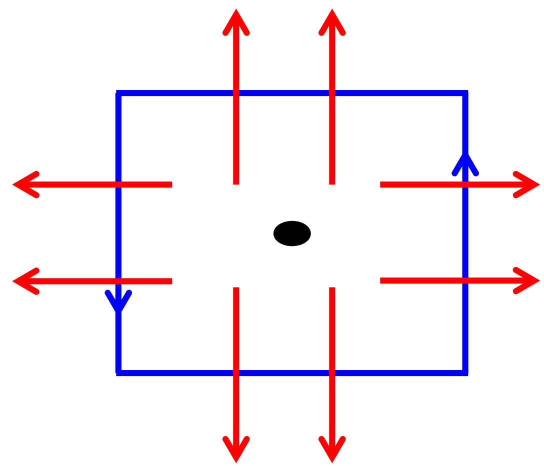
$B_i < 0$

The Curl of a Vector Field (contd.)

- **following** vector fields will result in a curl with **zero** value at point $\bar{r}$:



$B_i = 0$



$B_i = 0$

- **Generally**, the curl of a vector field result in another vector field whose magnitude is positive in some regions of space, negative in other regions, and zero elsewhere.
- For most **physical** problems, the curl of a vector field provides another vector field that indicates **rotational sources** (i.e., “paddle wheels”) of the original vector field.

Curl in Coordinate Systems

- Consider now the curl of vector fields expressed using our coordinate systems.

$$\nabla\times\vec{A}(\bar{r})=\left[\frac{\partial A_y(\bar{r})}{\partial z}-\frac{\partial A_z(\bar{r})}{\partial y}\right]\hat{a}_x+\left[\frac{\partial A_z(\bar{r})}{\partial x}-\frac{\partial A_x(\bar{r})}{\partial z}\right]\hat{a}_y+\left[\frac{\partial A_x(\bar{r})}{\partial y}-\frac{\partial A_y(\bar{r})}{\partial x}\right]\hat{a}_z$$

$$\nabla\times\vec{A}(\bar{r})=\left[\frac{1}{\rho}\frac{\partial A_z(\bar{r})}{\partial\phi}-\frac{\partial A_\phi(\bar{r})}{\partial z}\right]\hat{a}_\rho+\left[\frac{\partial A_\rho(\bar{r})}{\partial z}-\frac{\partial A_z(\bar{r})}{\partial\rho}\right]\hat{a}_\phi+\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\phi(\bar{r})\right)-\frac{1}{\rho}\frac{\partial A_\rho(\bar{r})}{\partial\phi}\right]\hat{a}_z$$

$$\nabla\times\vec{A}(\bar{r})=\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi(\bar{r})\right)-\frac{1}{r\sin\theta}\frac{\partial A_\theta(\bar{r})}{\partial\phi}\right]\hat{a}_r+\left[\frac{1}{r\sin\theta}\frac{\partial A_r(\bar{r})}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi(\bar{r})\right)\right]\hat{a}_\theta$$
$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta(\bar{r})\right)-\frac{1}{r}\frac{\partial A_r(\bar{r})}{\partial\theta}\right]\hat{a}_\phi$$

Yikes! These expressions are **very** complex. Precision, organization, and patience are required to **correctly** evaluate the **curl** of a vector field !

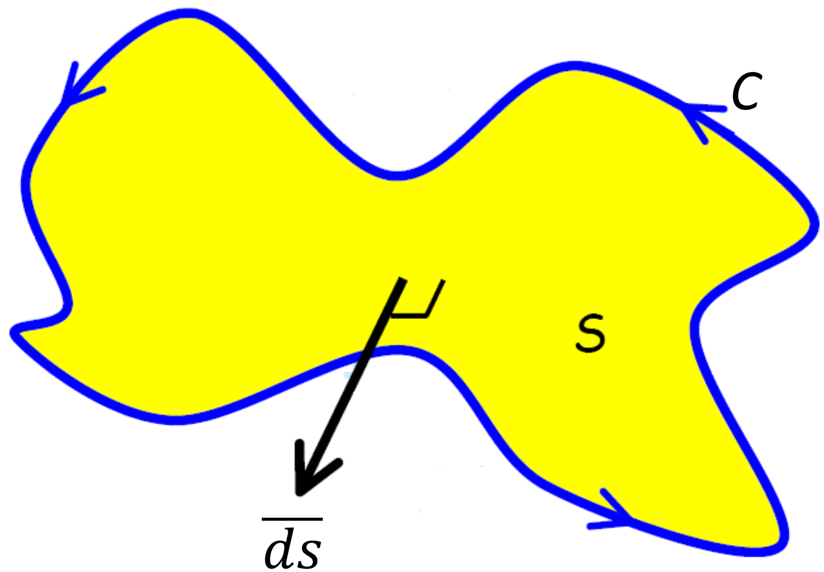
Stokes' Theorem

- Consider a vector field $\vec{B}(\bar{r})$ where: $\vec{B}(\bar{r}) = \nabla \times \vec{A}(\bar{r})$

- Say we wish to integrate this vector field over an **open** surface **S**: $\iint_S \vec{B}(\bar{r}).\overline{dS} = \iint_S \nabla \times \vec{A}(\bar{r}).\overline{dS}$

- We can likewise evaluate this integral using **Stokes' Theorem**: $\iint_S \nabla \times \vec{A}(\bar{r}).\overline{dS} = \oint_C \vec{A}(\bar{r}).\overline{dl}$

- In this case, the contour C is a **closed** contour that **surrounds** surface **S**. The direction of C is defined by $\overline{ds}$ and the **right - hand rule**. In other words C rotates **counter clockwise** around $\overline{ds}$. e.g.,



Example

Determine the curl of $\vec{A} = 10e^{-2\rho}\hat{a}_\rho cos\varphi + 10sin\varphi\hat{a}_z$ and evaluate it at $(2, 0, 3)$ in cylindrical coordinates.

Example

Determine the curl of $\vec{B} = 12sin\theta\hat{a}_\theta$ and evaluate it at $(3, \pi/6, 0)$ in spherical coordinates.

The Curl of Conservative Fields

- Recall that every **conservative** field can be written as the gradient of some scalar field:

$$\vec{C}(\bar{r}) = \nabla g(\bar{r})$$

Therefore:

$$C_x(\bar{r}) = \frac{\partial g(\bar{r})}{\partial x} \qquad C_y(\bar{r}) = \frac{\partial g(\bar{r})}{\partial y} \qquad C_z(\bar{r}) = \frac{\partial g(\bar{r})}{\partial z}$$

- Consider now the **curl of a conservative field**:

$$\nabla \times \vec{C}(\bar{r}) = \nabla \times \nabla g(\bar{r})$$

- Recall that if $\vec{C}(\bar{r})$ is expressed using the **Cartesian** coordinate system, the curl of $\vec{C}(\bar{r})$ is:

$$\nabla \times \vec{C}(\bar{r}) = \left[\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right]\hat{a}_z$$

- Likewise, the **gradient** of $g(\bar{r})$ is:

$$\nabla \times \vec{C}(\bar{r}) = \left[\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right]\hat{a}_z$$

- Combining the two results:

$$\nabla \times \nabla g(\bar{r}) = \nabla \times \vec{C}(\bar{r}) = \left[\frac{\partial^2 g(\bar{r})}{\partial y \partial z} - \frac{\partial^2 g(\bar{r})}{\partial z \partial y}\right]\hat{a}_x + \left[\frac{\partial^2 g(\bar{r})}{\partial z \partial x} - \frac{\partial^2 g(\bar{r})}{\partial x \partial z}\right]\hat{a}_y + \left[\frac{\partial^2 g(\bar{r})}{\partial x \partial y} - \frac{\partial^2 g(\bar{r})}{\partial y \partial x}\right]\hat{a}_z$$

The Curl of Conservative Fields (contd.)

- We know: $$\frac{\partial^2 g(\bar{r})}{\partial y \partial z} = \frac{\partial^2 g(\bar{r})}{\partial z \partial y}$$

- each component of $\nabla \times \nabla g(\bar{r})$ is then equal to **zero**, and we can say: $$\nabla \times \nabla g(\bar{r}) = \nabla \times \vec{C}(\bar{r}) = 0$$

→ The **curl** of every **conservative** field is **equal to zero** !

Q: Are there some **non-**conservative fields whose curl is also equal to zero?
A: NO! The curl of a conservative field, and **only** a conservative field, is equal to **zero**.

- Thus, we have way to **test** whether some vector field $\vec{A}(\bar{r})$ is conservative: **evaluate its curl**!

  1. If the result **equals zero**—the vector field **is** conservative.
  2. If the result is **non-zero**—the vector field **is not** conservative.

The Curl of Conservative Fields (contd.)

- Let's again **recap** what we've learnt about **conservative** fields:

1. The line integral of a conservative field is **path independent**.
2. Every conservative field can be expressed as the **gradient** of some scalar field.
3. The gradient of **any** and **all** scalar fields is a conservative field.
4. The line integral of a conservative field around any **closed** contour is equal to zero.
5. The **curl** of every conservative field is equal to **zero**.
6. The **curl** of a vector field is zero **only** if it is conservative.

The Solenoidal Vector Field

1. We know that a **conservative** vector field $\vec{C}(\bar{r})$ can be identified from its curl, which is always equal to zero:

$$\nabla \times \vec{C}(\bar{r}) = 0$$

- Similarly, there is **another** type of vector field $\vec{S}(\bar{r})$, called a **solenoidal** field, whose **divergence** always equals zero:

$$\nabla . \vec{S}(\bar{r}) = 0$$

Moreover, it should be noted that **only** solenoidal vector have zero divergence! Thus, zero divergence is a **test** for determining if a given vector field is solenoidal.

We sometimes refer to a solenoidal field as a **divergenceless** field.

The Solenoidal Vector Field (contd.)

2. Recall that **another** characteristic of a **conservative** vector field is that it can be expressed as the **gradient** of some **scalar** field (i.e., $\vec{C}(\bar{r})=\nabla g(\bar{r})$).

- Solenoidal vector fields have a **similar** characteristic! Every solenoidal vector field can be expressed as the **curl** of some other vector field (say $\vec{A}(\bar{r})$).

$$\vec{S}(\bar{r})=\nabla\times\vec{A}(\bar{r})$$

- Additionally, it is important to note that **only** solenoidal vector fields can be expressed as the curl of some other vector field.

The curl of **any** vector field **always** results in a solenoidal field!

- Note if we **combine** these two previous equations, we get a **vector identity**:

$$\nabla.\nabla\times\vec{A}(\bar{r})=0$$

a result that is always true for **any** and **every** vector field $\vec{A}(\bar{r})$.

The Solenoidal Vector Field (contd.)

3. Now, let's recall the **divergence theorem**:

$$\iiint_v \nabla.\vec{A}(\bar{r})dv = \oiint_S \vec{A}(\bar{r}).\overline{ds}$$

- **If** the vector field $\vec{A}(\bar{r})$ is **solenoidal**, we can write this theorem as:

$$\iiint_v \nabla.\vec{S}(\bar{r})dv = \oiint_S \vec{S}(\bar{r}).\overline{ds}$$

But the divergence of a solenoidal field is **zero:**

$$\nabla.\vec{S}(\bar{r}) = 0$$

As a result, the **left** side of the divergence theorem is zero, and we can conclude that:

$$\oiint_S \vec{S}(\bar{r}).\overline{ds} = 0$$

In other words the **surface** integral of **any** and **every** solenoidal vector field across a **closed** surface is equal to zero.

- Note this result is **analogous** to evaluating a line integral of a conservative field over a closed contour:

$$\oint_C \vec{C}(\bar{r}).\overline{dl} = 0$$

The Solenoidal Vector Field (contd.)

- Lets **summarize** what we know about **solenoidal** vector fields:

1. **Every** solenoidal field can be expressed as the **curl** of some **other** vector field.
2. The curl of **any** and **all** vector fields always results in a solenoidal vector field.
3. The **surface integral** of a solenoidal field across any **closed** surface is equal to **zero.**
4. The **divergence** of every solenoidal vector field is equal to **zero**.
5. The divergence of a vector field is zero **only** if it is **solenoidal**.