Lecture – 7

Date: 21.01.2016

- Vector Arithmetic (Review)
- Coordinate System and Transformations
- Examples

Vector Addition

Q: Say we **add** two vectors $\vec{A}$ and $\vec{B}$ together; what is the **result**?

A: The addition of two vectors results in **another vector**, which we will denote as $\vec{C}$. Therefore, we can say: $\vec{A}+\vec{B}=\vec{C}$

The **magnitude** and **direction** of $\vec{C}$ is determined by the **head-to-tail rule**.

This is not a **provable** result, rather the head-to-tail rule is the **definition** of vector addition. This definition is used because it has many **applications** in physics.

Some important properties of vector addition:

1. Vector addition is **commutative:** $\vec{A}+\vec{B}=\vec{B}+\vec{A}$
2. Vector addition is **associative:** $\left(\vec{X}+\vec{Y}\right)+\vec{Z}=\vec{X}+(\vec{Y}+\vec{Z})=\vec{K}$

From these two properties, we can conclude that the addition of **several** vectors can be executed in **any order**

- We consider the addition of a negative vector as a **subtraction**.

Vector Multiplication

- Consider a scalar quantity a and a vector quantity $\vec{B}$. We express the multiplication of these two values as:

$$a\vec{B} = \vec{C}$$

In other words, the product of a scalar and a vector is a **vector**!

Q: OK, but what **is** vector $\vec{C}$**?** What is the **meaning** of $\vec{C}$**?**

A: The resulting vector $\vec{C}$ has a **magnitude** that is equal to a times the **magnitude** of $\vec{B}$. In other words:

$$\left|\vec{C}\right| = a\left|\vec{B}\right|$$

The **direction** of vector $\vec{C}$ is **exactly** that of $\vec{B}$**.**

→ Jut to reiterate, multiplying a vector by a scalar changes the **magnitude** of the vector, but **not** its direction.

Multiplication (contd.)

Some important properties of vector multiplication:

1. The scalar-vector multiplication is **distributive**: $a\vec{B} + b\vec{B} = (a+b)\vec{B}$
2. also **distributive** as: $a\vec{B} + a\vec{C} = a(\vec{B} + \vec{C})$
3. Scalar-Vector multiplication is also **commutative**: $a\vec{B} = \vec{B}a$
4. Multiplication of a vector by a **negative** scalar is interpreted as: $-a\vec{B} = a(-\vec{B})$
5. **Division** of a vector by a scalar is the same as multiplying the vector by the **inverse** of the scalar: $\frac{\vec{B}}{a} = \left(\frac{1}{a}\right)\vec{B}$

Unit Vector

- Lets begin with vector $\vec{A}$. Say we **divide** this vector by its **magnitude** (a scalar value). We create a new vector, which we will denote as $\hat{a}_A$:

$$\hat{a}_A = \frac{\vec{A}}{\left|\vec{A}\right|}$$

Q: How is vector $\hat{a}_A$ related to vector $\vec{A}$**?**

A: Since we divided $\vec{A}$ by a scalar value, the vector $\hat{a}_A$ has the **same direction** as vector $\vec{A}$.

- But, the **magnitude** of $\hat{a}_A$ is:

$$\left|\hat{a}_A\right| = \frac{\left|\vec{A}\right|}{\left|\vec{A}\right|} = 1$$

The vector $\hat{a}_A$ has a magnitude equal to **one**! We call such a vector a **unit vector**.

- A unit vector is essentially a **description of direction** only, as its magnitude is always **unit valued** (i.e., equal to one). Therefore:

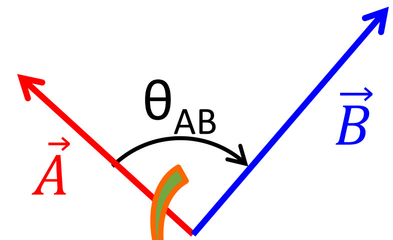
- $|\vec{A}|$ is a scalar value that describes the **magnitude** of vector $\vec{A}$.
- $\hat{a}_A$ is a vector that describes the **direction** of $\vec{A}$**.**

The Dot Product

- The **dot product** of two vectors, $\vec{A}$ and $\vec{B}$, is **denoted** as $\vec{A}\,.\,\vec{B}$

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB}$$

angle θ_{AB} is the angle formed **between** the vectors $\vec{A}$ and $\vec{B}$.



$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB}$$

- Note also that the dot product is **commutative**:

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB}$$

- The dot product of a vector **with itself** is equal to the **magnitude** of the vector **squared.**

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

$$0 \le \theta_{AB} \le \pi$$

- If $\vec{A}\,.\,\vec{B} = 0$ (and $\vec{A} \neq 0$, $\vec{B} \neq 0$), then it must be true that:

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

- If $\vec{A}\,.\,\vec{B} = |\vec{A}||\vec{B}|$, then it must be true that:

$$\vec{A}.\vec{A} = |\vec{A}|.|\vec{A}|\cos 0^\circ = |\vec{A}|^2$$

- The dot product is **distributive** with addition:

$$|\vec{A}| = \sqrt{\vec{A}.\vec{A}}$$

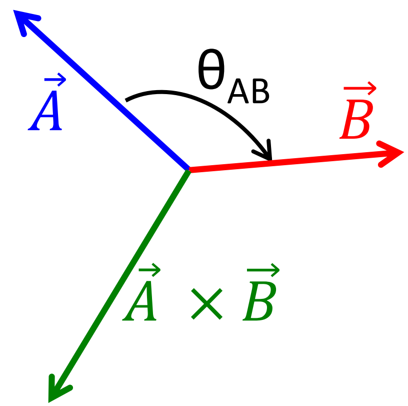
The Cross Product

- The **cross product** of two vectors, $\vec{A}$ and $\vec{B}$, is **denoted** as $\vec{A} \times \vec{B}$.

$$\vec{A} \times \vec{B} = \hat{a}_n |\vec{A}||\vec{B}| \sin\theta_{AB}$$

$$0 \le \theta_{AB} \le \pi$$

Just as with the dot product, the angle θ_{AB} is the angle between the vectors $\vec{A}$ and $\vec{B}$. The unit vector $\hat{a}_n$ is **orthogonal** to both $\vec{A}$ and $\vec{B}$ (i.e., $\hat{a}_n.\vec{A}$ =0 and $\hat{a}_n.\vec{B}$ =0.)



IMPORTANT NOTE: The cross product is an operation involving **two vectors**, and the result is also a **vector**. e.g.,:

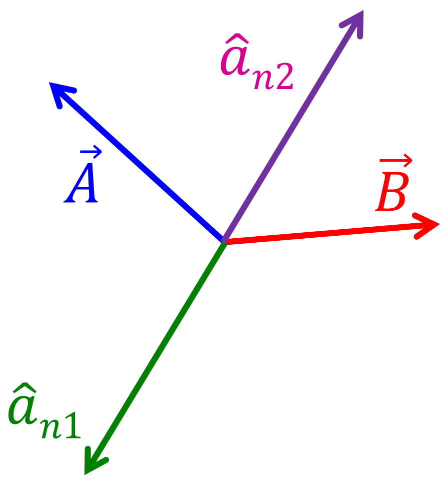
$$\vec{A} \times \vec{B} = \vec{C}$$

- The **magnitude** of vector $\vec{A} \times \vec{B}$ is therefore: $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta_{AB}$

While the **direction** of vector $\vec{A} \times \vec{B}$ is described by unit vector $\hat{a}_n$.

The Cross Product (contd.)

Problem! There are **two** unit vectors that satisfy the equations $\hat{a}_n . \vec{A}$ =0 and $\hat{a}_n . \vec{B}$ =0!! These two vectors are **antiparallel**.



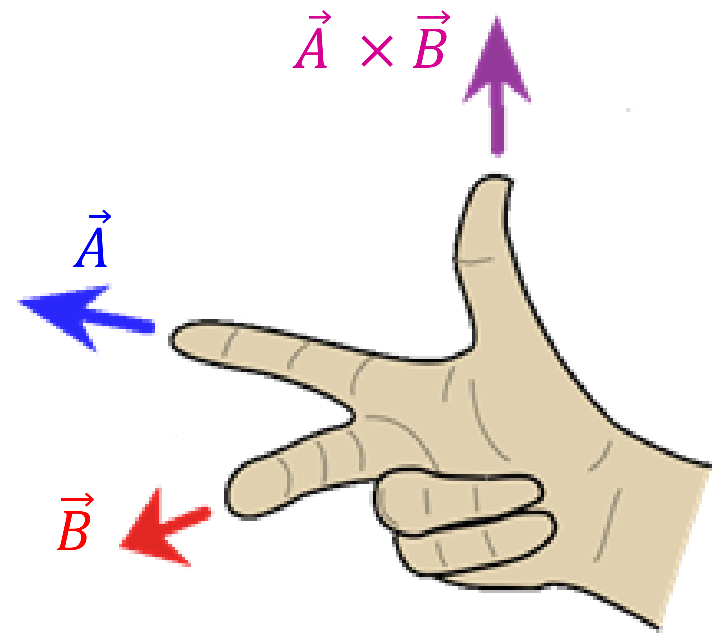
$\vec{A}.\hat{a}_{n1} = \vec{A}.\hat{a}_{n2} = 0$

$\vec{A}.\hat{a}_{n1} = \vec{A}.\hat{a}_{n2} = 0$

$\vec{B}.\hat{a}_{n1} = \vec{B}.\hat{a}_{n2} = 0$

Q: Which unit vector is correct?

A: Use the **right-hand rule**



The Cross Product (contd.)

1. If θ_{AB} = 90º (i.e., **orthogonal**), then:

$$\vec{A}\times\vec{B}=\hat{a}_n\left|\vec{A}\right|\left|\vec{B}\right|\sin 90^\circ=\hat{a}_n\left|\vec{A}\right|\left|\vec{B}\right|$$

2. If θ_{AB} = 0º (i.e., **parallel**), then:

$$\vec{A}\times\vec{B}=\hat{a}_n\left|\vec{A}\right|\left|\vec{B}\right|\sin 0^\circ=0$$

Note that $\vec{A}\times\vec{B}=\mathbf{0}$ also if θ_{AB} = 180º.

3. The cross product is **not** commutative! In other words, $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$.

While evaluating the cross product of two vectors, the **order** is important !

$$\vec{A}\times\vec{B}\neq-(\vec{B}\times\vec{A})$$

4. The **negative** of the cross product is:

$$-(\vec{A}\times\vec{B})=\vec{A}\times(-\vec{B})$$

5. The cross product is also **not** associative:

$$\vec{A}\times\vec{B}\times\vec{C}\neq\vec{A}\times\left(\vec{B}\times\vec{C}\right)$$

6. But, the cross product is **distributive**, in that:

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\left(\vec{A}\times\vec{B}\right)+\left(\vec{A}\times\vec{C}\right)$$

The Triple Product

- The **triple product** is not a "new" operation, as it is simply a combination of the **dot** and **cross** products.
- For example, the triple product of vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ is **denoted** as:

$$\vec{A}.\vec{B}\times\vec{C}$$

Q: Yikes! Does this mean: $\left(\vec{A}.\vec{B}\right)\times\vec{C}$ **OR** $\vec{A}.\left(\vec{B}\times\vec{C}\right)$

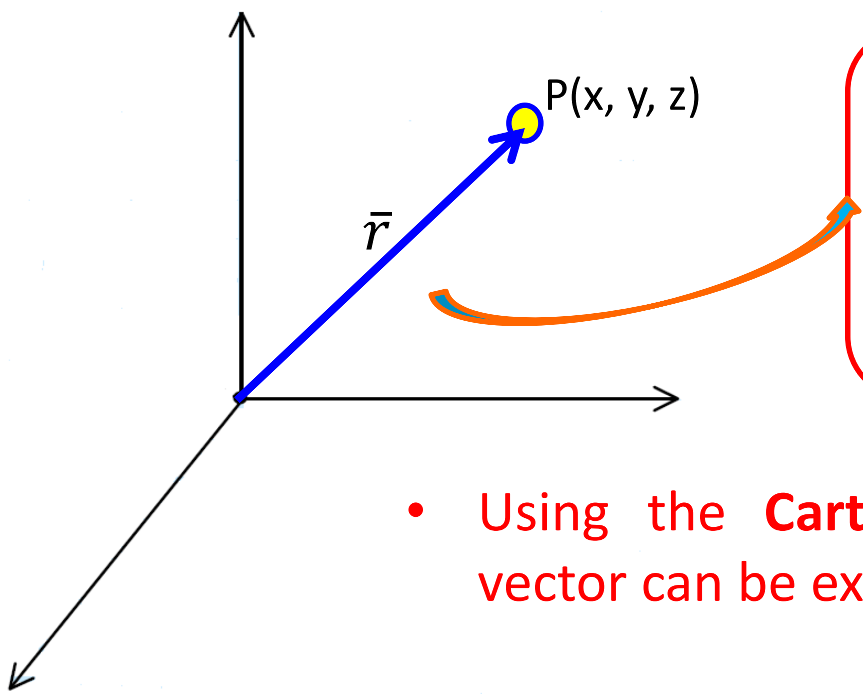
A: The answer is **easy**! Only one of these two interpretations makes sense:

$\left(\vec{A}.\vec{B}\right)\times\vec{C} =$ **Scalar X Vector** ← **makes no sense**

$\vec{A}.\left(\vec{B}\times\vec{C}\right) =$ **Vector . Vector** ← **dot product**

The Position Vector

- Consider a point whose location in space is specified with Cartesian coordinates (e.g., P(x, y, z)). Now consider the **directed distance** (a vector quantity!) extending from the origin to this point.



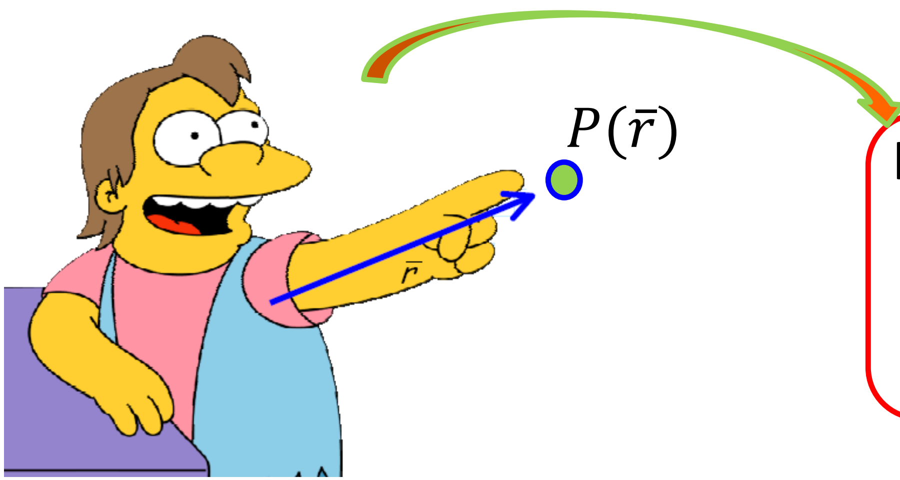
This **particular** directed distance—a vector beginning at the **origin** and extending outward to a point—is a **very important** and fundamental directed distance known as the **position vector** $\bar{r}$

- Using the **Cartesian** coordinate system, the position vector can be explicitly written as:

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

The Position Vector (contd.)

- Note that given the **coordinates** of some point (e.g., x =1, y =2, z =-3), we can easily determine the **corresponding position vector** (e.g., $\bar{r} = \hat{a}_x + 2\hat{a}_y - 3\hat{a}_z$).
- Moreover, given some **specific position vector** (e.g., $\bar{r} = 4\hat{a}_y - 2\hat{a}_z$), we can easily determine the **corresponding coordinates of that point** (e.g., x =0, y =4, z =-2).
- In other words, a position vector $\bar{r}$ is an alternative way to denote the location of a point in space! We can use **three coordinate values** to specify a point's location, **or** we can use a **single position vector** $\bar{r}$.



I see! The position vector is essentially a **pointer.** Look at the end of the vector, and you will find the **point specified**!

The magnitude of $\bar{r}$

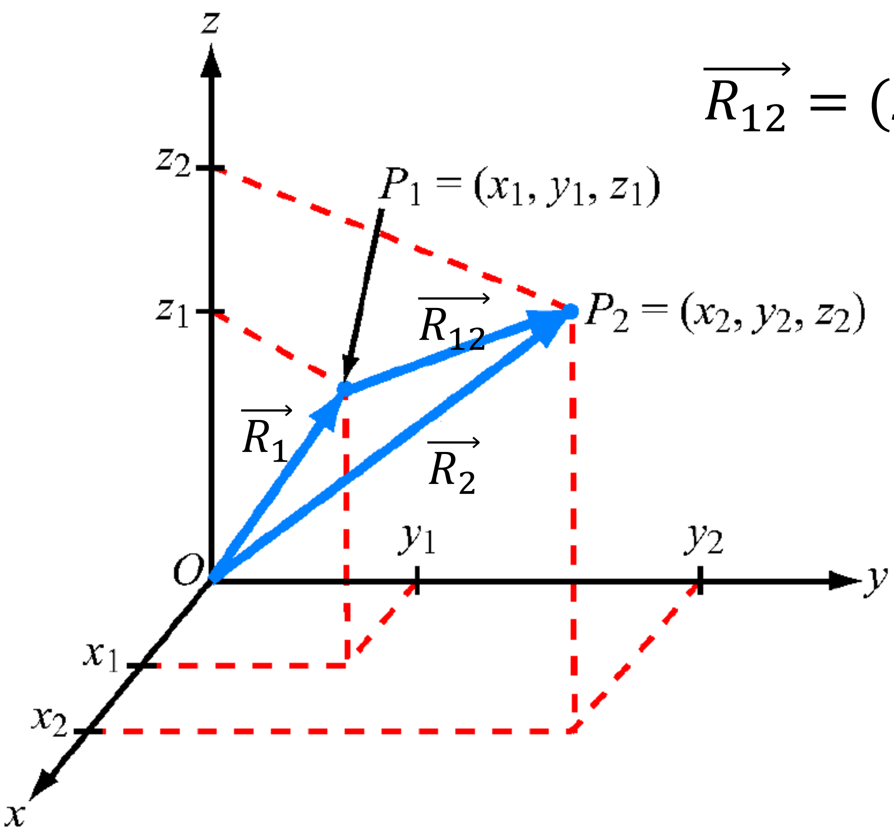
- Note the **magnitude** of any and all position vectors is:

$$\left|\bar{r}\right| = \sqrt{\bar{r}.\bar{r}} = \sqrt{x^2 + y^2 + z^2} = r$$

Q: Hey, this makes **perfect sense**! Doesn't the coordinate value r have a **physical** interpretation as the **distance** between the **point** and the **origin**?

A: That's right! The **magnitude** of a **directed distance** vector is equal to the **distance** between the two points—in this case the distance between the **specified point** and the **origin**!

The Distance Vector



$$\overrightarrow{R_{12}} = \overrightarrow{P_1P_2} = \overrightarrow{R_2} - \overrightarrow{R_1}$$

$$\overrightarrow{R_{12}} = (x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z$$

$$d = \left|\overrightarrow{R_{12}}\right|$$

Example – 1

In Cartesian coordinates, Vector $\vec{A}$ points from the origin to point $P_1 = (2,3,3)$, and Vector $\vec{B}$ is directed from P_1 to point $P_2 = (1,-2,2)$. Find:

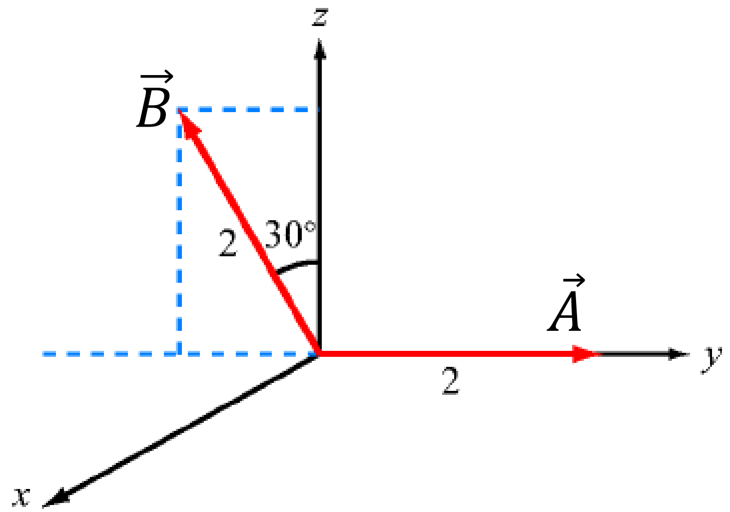
(a) Vector $\vec{A}$, its magnitude A, and unit vector $\hat{a}$.
(b) The angle between $\vec{A}$ and the y-axis.
(c) Vector $\vec{B}$
(d) The angle θ_{AB} between $\vec{A}$ and $\vec{B}$.
(e) Then find the angle θ_{AB} from the cross product between $\vec{A}$ and $\vec{B}$.
(f) The perpendicular distance from the origin to Vector $\vec{B}$
(g) Find the angle between Vector $\vec{B}$ and the z-axis.

Example – 2

- Find the distance vector between $P_1 = (1, 2, 3)$ and $P_2 = (-1, -2, 3)$

Example – 3

- Vectors $\vec{A}$ and $\vec{B}$ lie in the y-z plane and both have the same magnitude of 2. Determine (a) $\vec{A}.\vec{B}$ and (b) $\vec{A} \times \vec{B}$.



Example – 4

- If $\vec{A}.\vec{B} = \vec{A}.\vec{C}$ then does it mean that $\vec{B} = \vec{C}$??

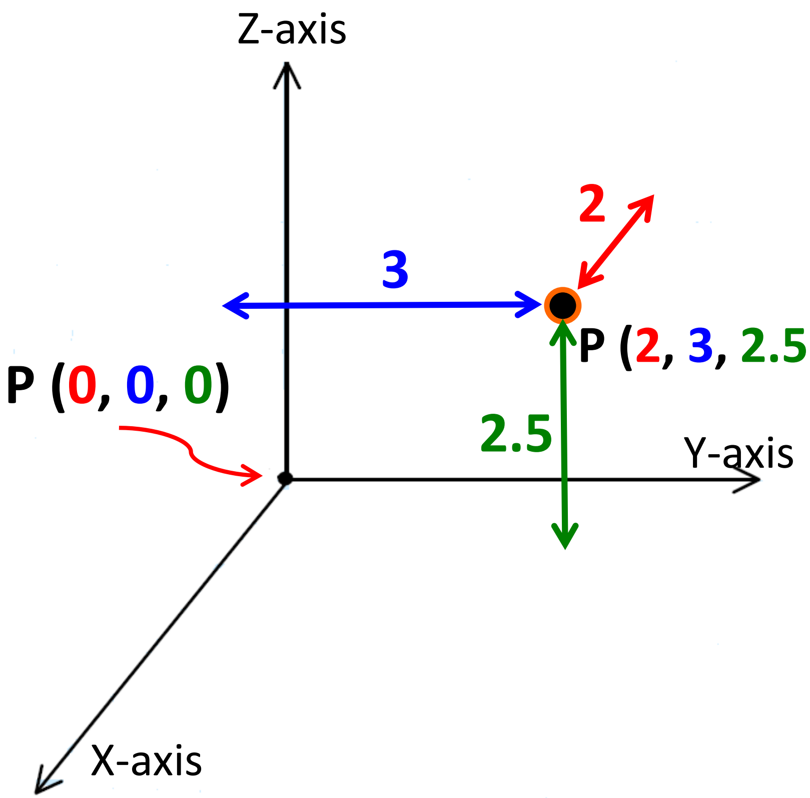
Example – 5

- Given $\vec{A} = \hat{a}_x - \hat{a}_y + 2\hat{a}_z$ $\quad \vec{B} = \hat{a}_y + \hat{a}_z$ $\quad \vec{C} = -2\hat{a}_x + 3\hat{a}_z$

  Find $(\vec{A} \times \vec{B}) \times \vec{C}$ and compare it with $\vec{A} \times (\vec{B} \times \vec{C})$

Cartesian Coordinates

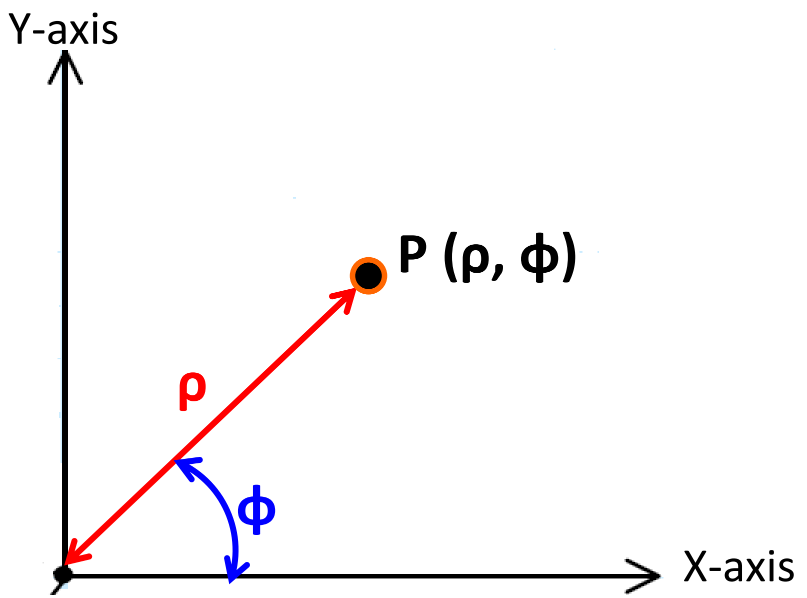
- Note the coordinate values in the Cartesian system effectively represent the **distance** from a **plane** intersecting the origin.
- For **example**, x =3 means that the point is **3 units** from the y-z **plane** (i.e., the x = 0 plane).
- Likewise, the y coordinate provides the **distance** from the x-z (y=0) plane, and the z coordinate provides the **distance** from the x-y (z =0) plane.
- Once **all three** distances are specified, the **position** of a point is **uniquely** identified.

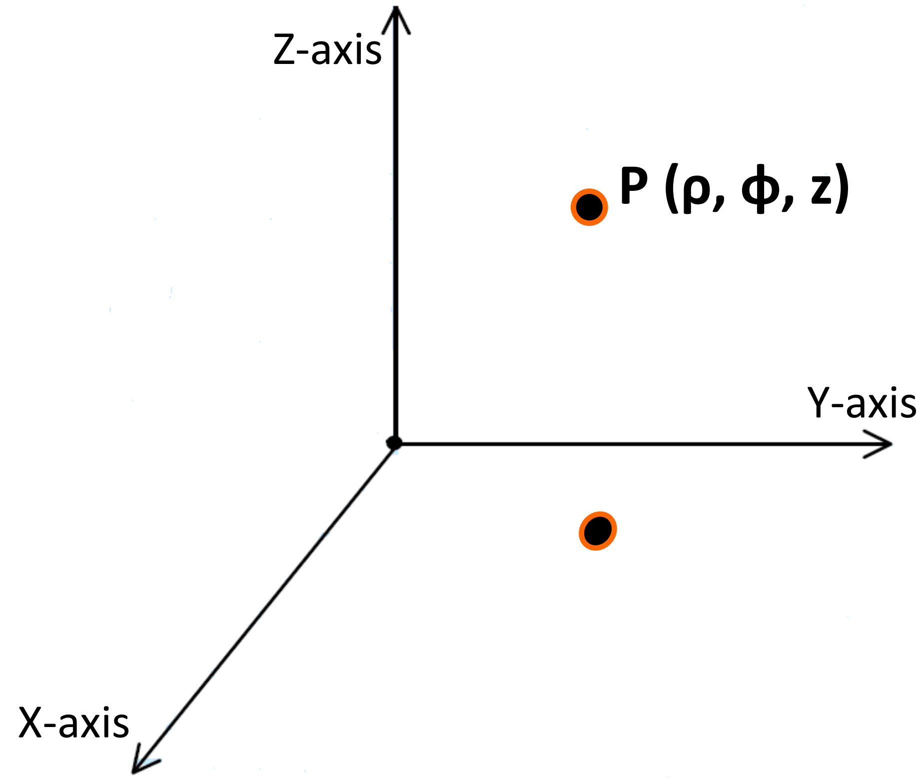


Cylindrical Coordinates

- You’re also familiar with **polar coordinates**. In **two** dimensions, we specify a point with **two** scalar values, generally called ρ and ϕ.

We can extend this to **3**-dimensions, by adding a **third** scalar value z. This method for identifying the position of a point is referred to as **cylindrical coordinates**.

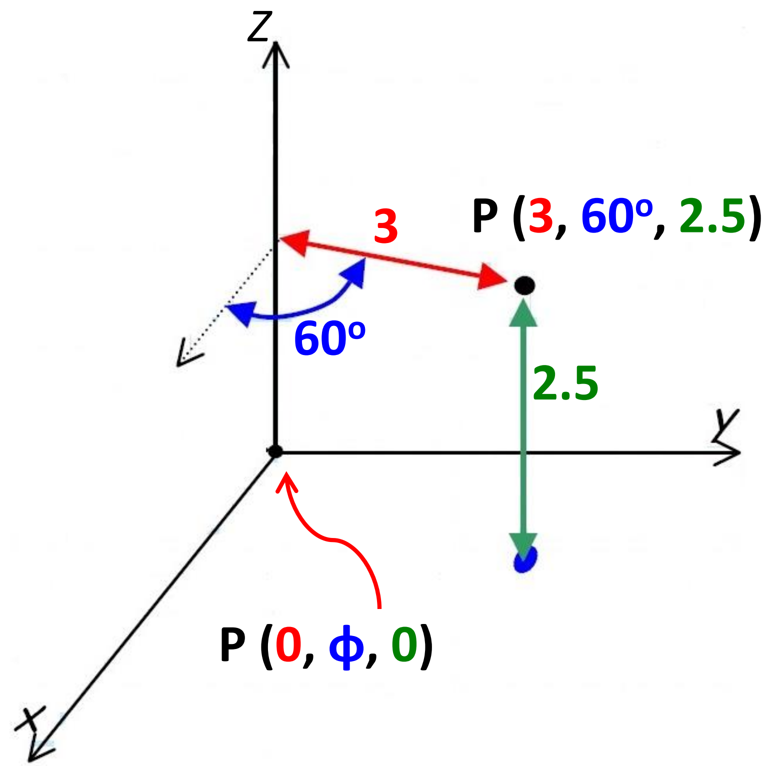




Cylindrical Coordinates

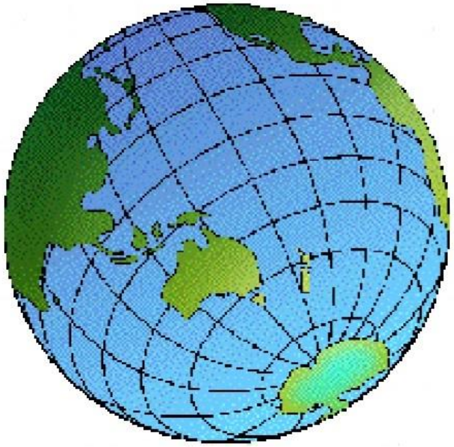
Note the **physical** significance of each parameter of **cylindrical** coordinates:

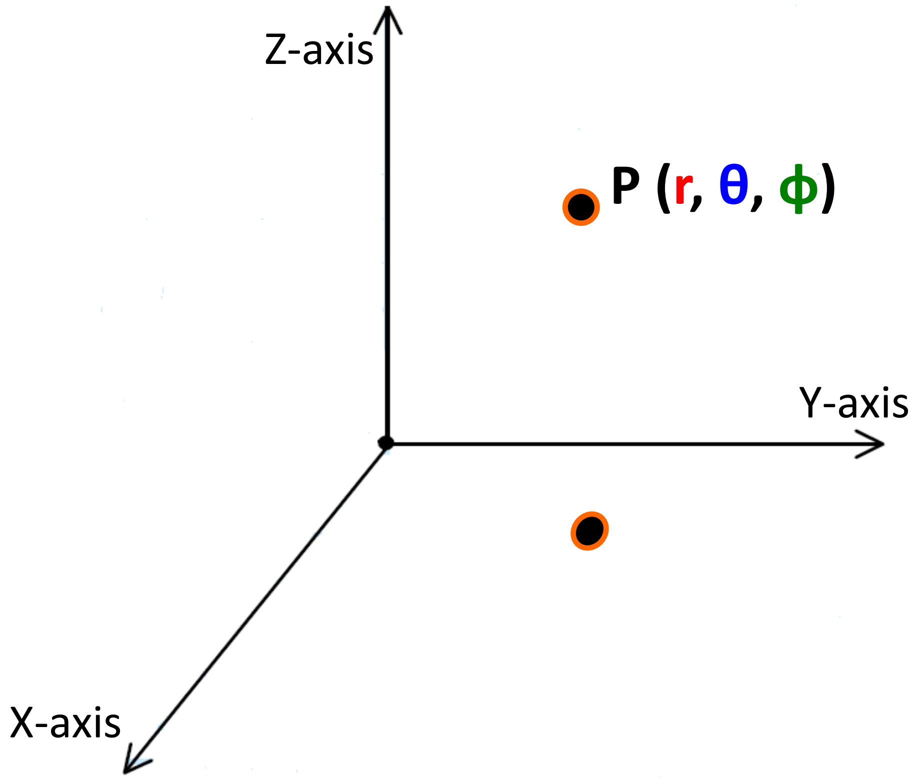
1. The value **ρ** indicates the **distance** of the point from the **z-axis** ($0 \le \rho < \infty$).
2. The value **ϕ** indicates the **rotation angle** around the **z-axis** ($0 \le \phi < 2\pi$), **precisely** the same as the angle **ϕ** used in **spherical** coordinates.
3. The value **z** indicates the **distance** of the point from the x-y (z = 0) plane ($-\infty < z < \infty$), **precisely** the same as the coordinate **z** used in **Cartesian** coordinates.
4. Once **all three** values are specified, the **position** of a point is **uniquely** identified.



Spherical Coordinates

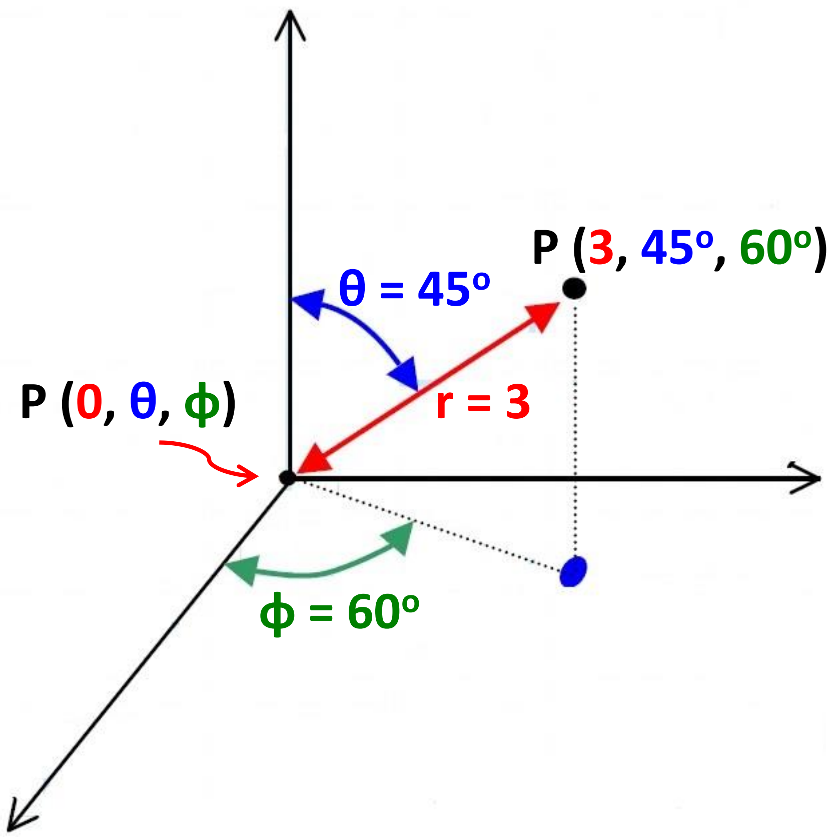
- **Geographers** specify a location on the Earth's surface using **three** scalar values: **longitude**, **latitude**, and **altitude**.
- Both longitude and latitude are **angular** measures, while altitude is a measure of **distance.**
- Latitude, longitude, and altitude are similar to **spherical coordinates.**
- Spherical coordinates consist of one scalar value (r), with units of **distance**, while the other two scalar values (θ, ϕ) have **angular** units (degrees or radians).





Spherical Coordinates

- For spherical coordinates, **r** ($0 \le r < \infty$) expresses the **distance** of the point from the **origin** (i.e., similar to **altitude**).
- Angle **θ** ($0 \le \theta \le \pi$) represents the angle formed **with the z-axis** (i.e., similar to **latitude**).
- Angle **ϕ** ($0 \le \phi < 2\pi$) represents the rotation angle around the z-axis, **precisely** the same as the **cylindrical** coordinate ϕ (i.e., similar to **longitude**).



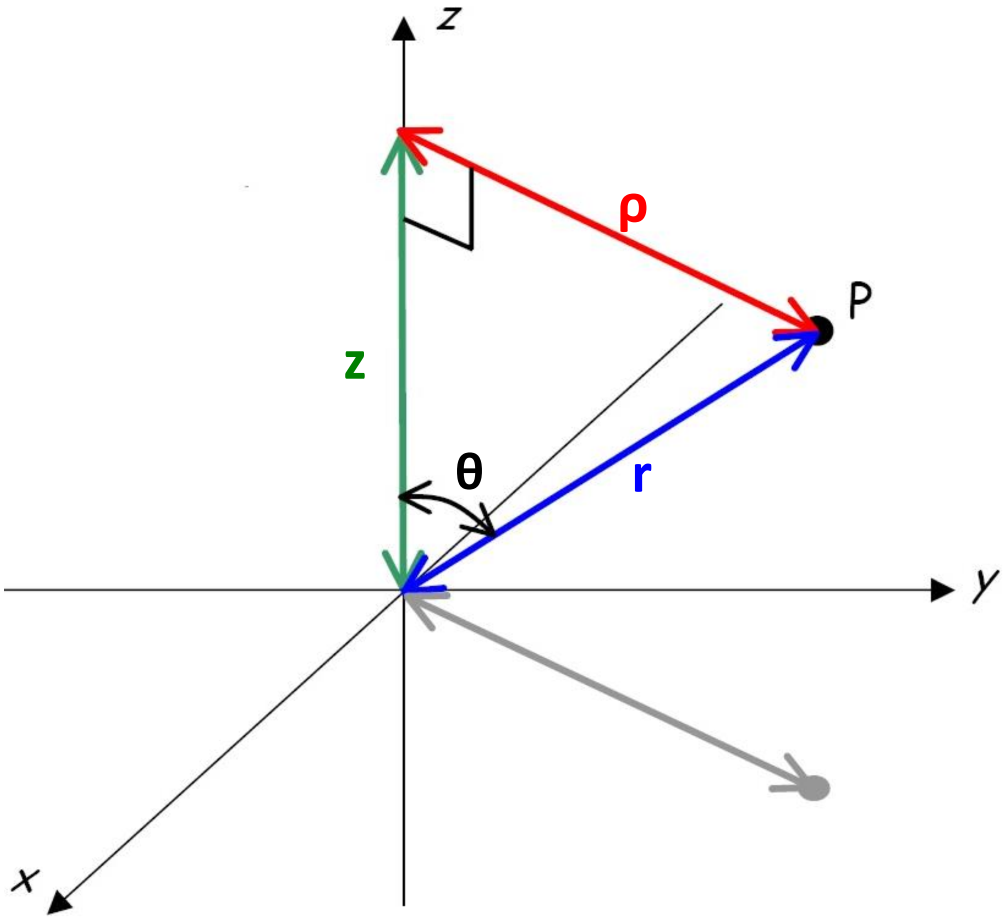
Thus, using **spherical** coordinates, a point in space can be unambiguously defined by **one distance** and **two angles**.

Coordinate Transformations

- Say we **know** the location of a point, or the description of some scalar field in terms of **Cartesian** coordinates (e.g., T (x, y, z)).
- What if we decide to express this point or this scalar field in terms of **cylindrical** or **spherical** coordinates **instead**?
- We see that the coordinate values ***z, ρ, r,*** and θ are all variables of a **right triangle**! We can use our knowledge of trigonometry to relate them to each other.
- In fact, we can **completely derive** the relationship between **all six** independent coordinate values by considering just **two very important right triangles**!
  - Hint: **Memorize these 2 triangles!!!**

Coordinate Transformations (contd.)

Right Triangle #1



$$z = r \times \cos\theta = \rho \times \cot\theta = \sqrt{r^2 - \rho^2}$$

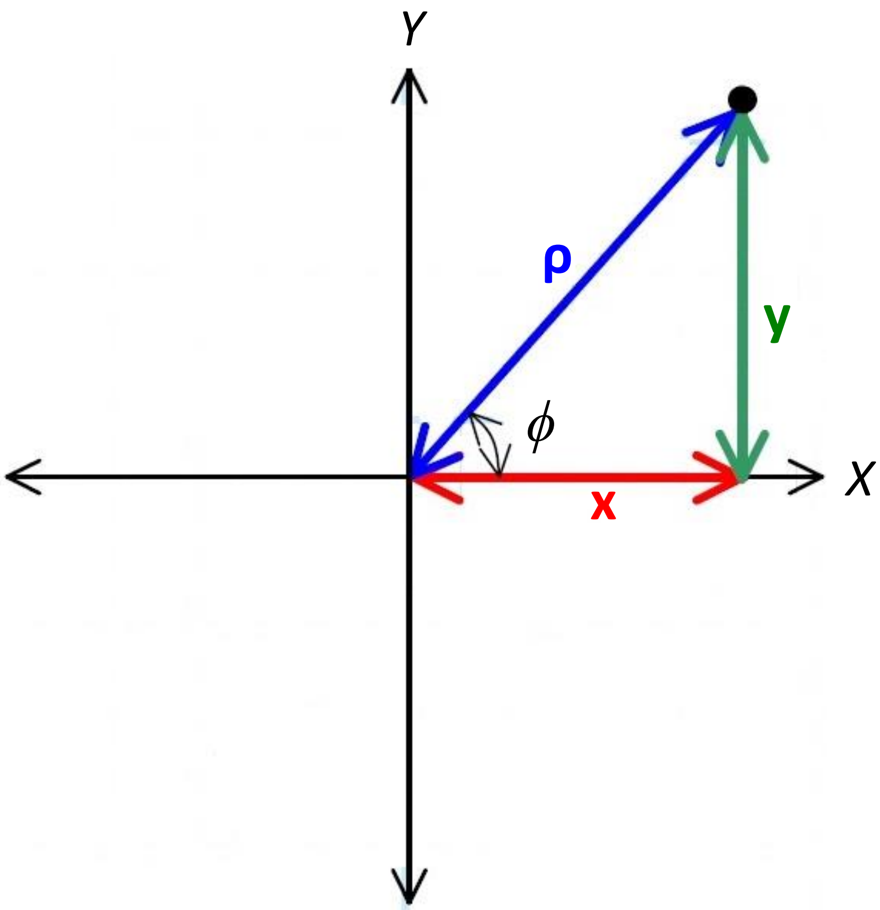
$$\rho = r \times \sin\theta = z \times \tan\theta = \sqrt{r^2 - z^2}$$

$$r = \sqrt{\rho^2 + z^2} = \rho \times \operatorname{cosec}\theta = z \times \sec\theta$$

$$\theta = \tan^{-1}\left[\frac{\rho}{z}\right] = \sin^{-1}\left[\frac{\rho}{r}\right] = \cos^{-1}\left[\frac{z}{r}\right]$$

Coordinate Transformations (contd.)

Right Triangle #2



$$x = \rho \times \cos\phi = y \times \cot\phi = \sqrt{\rho^2 - y^2}$$

$$y = \rho \times \sin\phi = x \times \tan\phi = \sqrt{\rho^2 - x^2}$$

$$\rho = \sqrt{x^2 + y^2} = x \times \sec\phi = y \times \operatorname{cosec}\phi$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right] = \sin^{-1}\left[\frac{y}{\rho}\right] = \cos^{-1}\left[\frac{x}{\rho}\right]$$

Coordinate Transformations (contd.)

Combining the results of the two triangles allows us to write each coordinate set in terms of each other

- **Cartesian and Cylindrical**

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right]$$

$$z = z$$

⇕

$$x = \rho \times \cos\phi$$

$$y = \rho \times \sin\phi$$

$$z = z$$

- **Cartesian and Spherical**

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right]$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right]$$

⇕

$$x = r \times \sin\theta \times \cos\phi$$

$$y = r \times \sin\theta \times \sin\phi$$

$$z = r \times \cos\theta$$

Coordinate Transformations

- **Cylindrical and Spherical**

$$\rho = r \times \sin\theta$$
$$\phi = \phi$$
$$z = r \times \cos\theta$$

⇔

$$r = \sqrt{\rho^2 + z^2}$$
$$\theta = \tan^{-1}\left[\frac{\rho}{z}\right]$$
$$\phi = \phi$$

Example – 1

- Say we have denoted a **point** in space (using **Cartesian** Coordinates) as $P(x = -3, y = -3, z = 2)$.
- Let's **instead** define this **same** point using **cylindrical** coordinates ρ, ϕ, z.

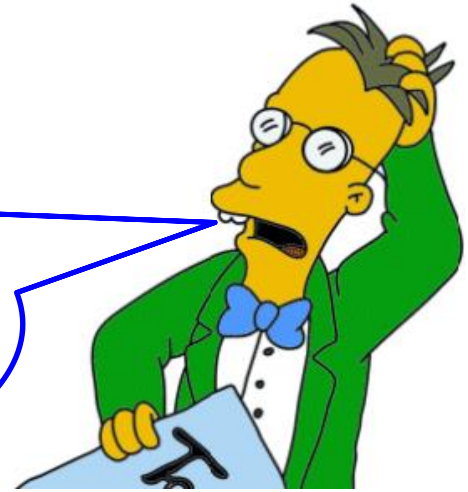
$$\rho = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$$

$$\phi = \tan^{-1}\left[\frac{-3}{-3}\right] = 45^o$$

$$z = 2$$

Therefore, the location of this point can **perhaps** be defined **also** as $P\left(\rho = 3\sqrt{2}, \phi = 45^o, z = 2\right)$.

Q: **Wait!** Something has gone **horribly wrong**. Coordinate ϕ = 45° indicates that point P is located in **quadrant-I**, whereas the coordinates x =-3, y =-3 tell us it is in fact in **quadrant-III**!



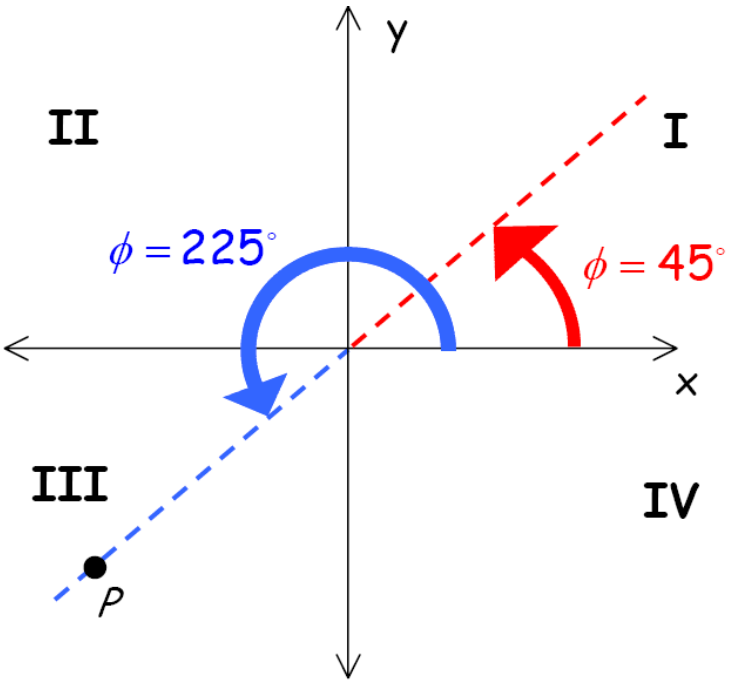
Example – 1 (contd.)

A: The problem is in the interpretation of the **inverse tangent!**

Remember that 0≤ϕ<360°, so that we must do a **four quadrant** inverse tangent. Your calculator likely only does a **two quadrant** inverse tangent (i.e., 90°≤ϕ≤−90°), so **be careful**!

Therefore, if we **correctly** find the coordinate ϕ:

$$\phi = \tan^{-1}\left[\frac{-3}{-3}\right] = 225^o$$



The location of point P can be expressed as **either** $P(x=-3, y=-3, z=2)$ or $P(\rho = 3\sqrt{2}, \phi = 225^o, z = 2)$.

Example – 2

Coordinate transformation on a Scalar field

- Consider the scalar field (i.e., scalar function): $g(\rho,\phi,z) = \rho^3 z \sin\phi$

rewrite this function in terms of **Cartesian** coordinates.

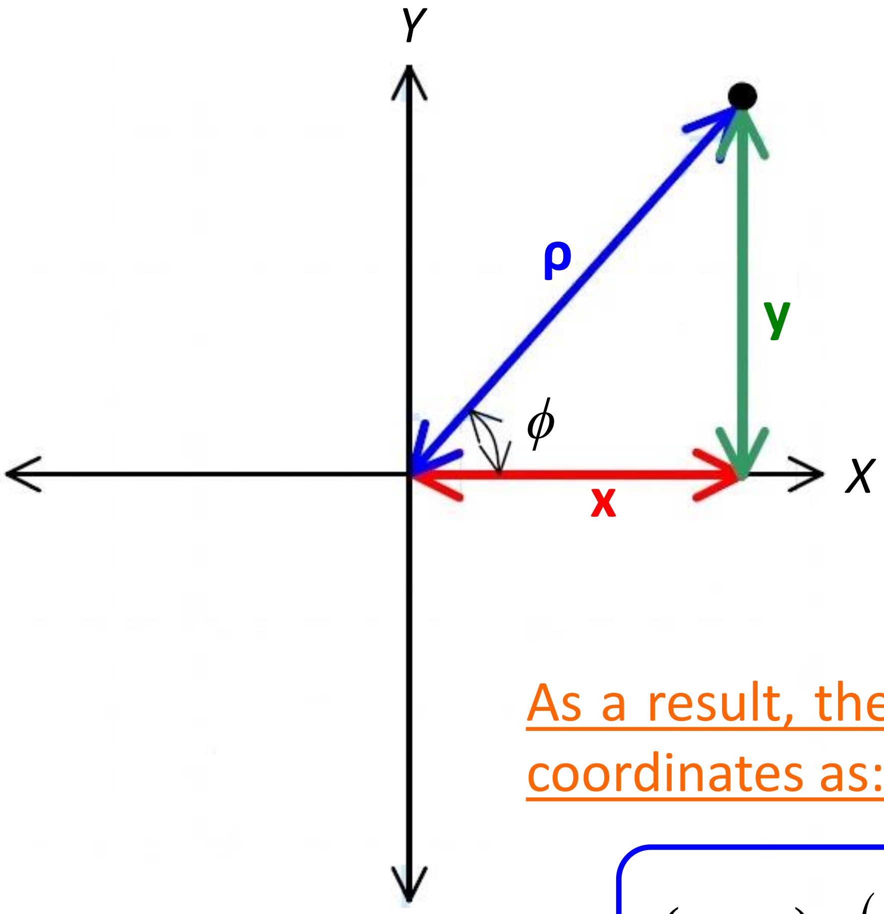
- Note that since $\rho = \sqrt{x^2 + y^2}$ → $\rho^3 = \left(x^2 + y^2\right)^{3/2}$
- Now, what about **sinϕ**?

We know that $\phi = \tan^{-1}\left[\frac{y}{x}\right]$, We might be tempted to write:

$$\sin\phi = \sin\left[\tan^{-1}\left[\frac{y}{x}\right]\right]$$

Technically correct, this is one **ugly** expression. We can instead turn to one of the **very important right triangles** that we discussed earlier

Example – 2 (contd.)



From **this** triangle, it is apparent that:

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}}$$

As a result, the scalar field can be written in **Cartesian** coordinates as:

$$g(x,y,z) = \left(x^2 + y^2\right)^{3/2} \frac{y}{\sqrt{x^2 + y^2}} z = \left(x^2 + y^2\right) yz$$

Example – 2 (contd.)

Although the scalar fields: $g(\rho,\phi,z)=\rho^3 z\sin\phi$ **and** $g(x,y,z)=\left(x^2+y^2\right)yz$

look very different, they are in fact **exactly** the same functions—only expressed using different **coordinate variables**.

- For **example**, if you **evaluate** each of the scalar fields at the **point** described earlier, you will get **exactly the same** result!

$$g(x=-3,y=-3,z=2)=-108$$

$$g(\rho=3\sqrt{2},\phi=225^o,z=2)=-108$$