Lecture – 2

Date: 07.01.2016

- Review of Complex Numbers
- Review of Phasors
- Examples

Project Themes

1. Microwave/THz Imaging
2. Automotive Radar
3. Mobile Phone Technology
4. RF Circuits and Systems
5. RF Sensors for Measurement Applications

Minimal requirement: thorough reading and understanding the concept by going through various papers, book chapters, white papers etc.. You are expected to explore (such as simulation or test setup development) as much as possible if you want to get the optimum marks and bonus marks.

Example – 1

- An acoustic wave traveling in x-direction in a fluid is characterized by a differential pressure $p(x,t)$. The unit for pressure is $^{N}/_{m^2}$. Find an expression for $p(x,t)$ for a sinusoidal sound wave traveling in the positive x-direction in water, given that the wave frequency is $1\,kHz$, the velocity of sound in water is $1.5\,Km/s$, the wave amplitude is $10\,^{N}/_{m^2}$, and $p(x,t)$ was observed to be at its maximum at $t = 0$ and $x = 0.25m$. Treat water as a lossless medium.

- Wave in positive $x - direction$: $p(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \varphi_0\right)$

- Given: $\mathrm{A} = 10\,^{N}/_{m^2}$ $\quad \mathrm{T} = \frac{1}{\mathrm{f}} = 10^{-3}\,s$ $\quad \lambda = \frac{v_p}{\mathrm{f}} = 1.5\mathrm{m}$

- Hence: $p(x,t) = 10\cos\left(2\pi \times 10^3 t - \frac{4\pi}{3}x + \varphi_0\right)$

Example – 1 (contd.)

- at $t = 0$ and $x = 0.25m$, $\quad p(0.25, 0) = 10\,\frac{Np}{m^2}$:

$$10 = 10\cos\left(-\frac{4\pi}{3}(0.25) + \varphi_0\right) \Rightarrow 10 = 10\cos\left(-\frac{\pi}{3} + \varphi_0\right) \Rightarrow \therefore \varphi_0 = \frac{\pi}{3}$$

$$\boxed{p(x,t) = 10\cos\left(2\pi \times 10^3 t - \frac{4\pi}{3}x + \frac{\pi}{3}\right)}$$

Example – 2

- A traveling wave along a string is given by: $y(x,t) = 2\sin(4\pi t + 10\pi x)$ cm

  Where x is the distance along the string in metres and y is the vertical displacement. Determine: (a) the direction of wave travel, (b) the reference phase φ_0, (c) the frequency, (d) the wavelength, and (e) the phase velocity.

(a) We convert the given expression into cosine form:

$$y(x,t) = 2\sin(4\pi t + 10\pi x) \Rightarrow y(x,t) = 2\cos\left(4\pi t + 10\pi x - \frac{\pi}{2}\right)$$

Since the coefficients of t and x both have the same sign, the wave is traveling in the negative x-direction.

(b) Phase reference, $\varphi_0 = -\frac{\pi}{2}$

(c) $\omega = 2\pi f = 4\pi$

$\therefore f = 2Hz$

Example – 2 (contd.)

(d) $2\pi/\lambda = 10\pi$ $\therefore \lambda = 0.2m$

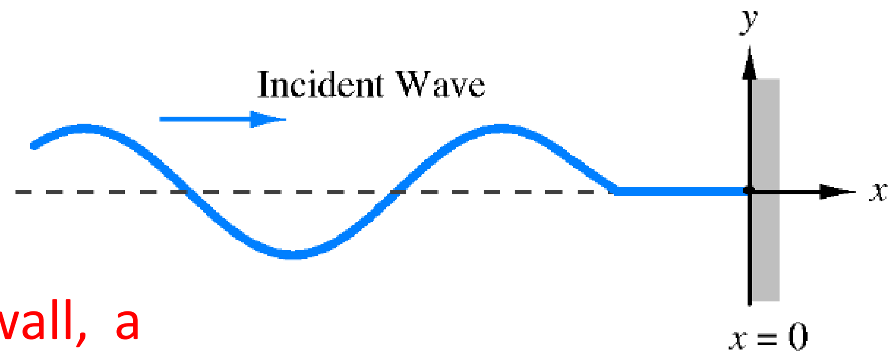
(e) $v_p = f\lambda = 2 \times 0.2 = 0.4\ m/s$

Example – 3

- A wave traveling along a string in the positive x-direction is given by:

$$y_1(x,t) = A\cos(\omega t - \beta x)$$

Where x=0 is the end of the string, which is tied rigidly to a wall as shown.



When wave $y_1(x,t)$ arrives at the wall, a reflected wave $y_2(x,t)$ is generated. Hence, at any location on the string, the vertical displacement y_s is the sum of the incident and reflected waves:

$$y_s(x,t) = y_1(x,t) + y_2(x,t)$$

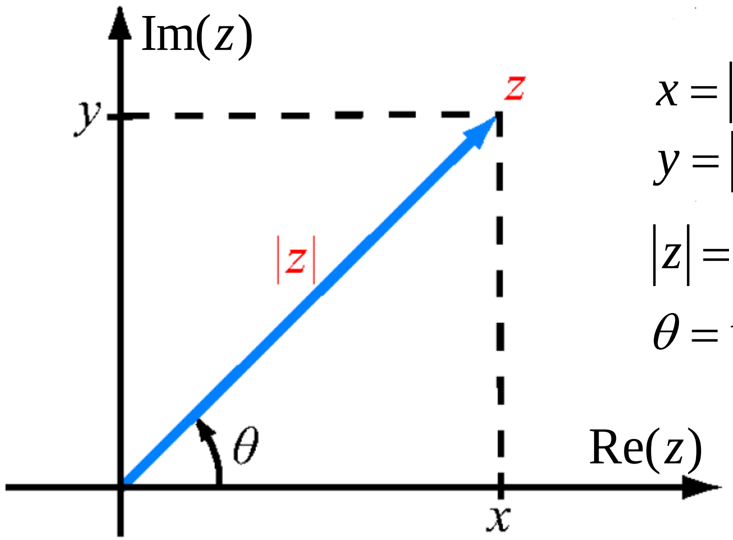
(a) Write an expression for $y_2(x,t)$, keeping in mind its direction of travel and the fact that the end of the string can't move

(b) Generate plots for $y_1(x,t)$, $y_2(x,t)$, and $y_s(x,t)$ versus x over the range $-2\lambda \leq x \leq 0$ at $\omega t = \frac{\pi}{4}$ and at $\omega t = \frac{\pi}{2}$.

Review of Complex Numbers

- Any complex number ***z*** can be expressed in rectangular form as: $z = x + jy$

$$x = \mathrm{Re}(z) \qquad y = \mathrm{Im}(z)$$



$$x = |z|\cos\theta$$
$$y = |z|\sin\theta$$
$$|z| = \sqrt[+]{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$

- Alternatively:

$$z = |z|e^{j\theta} = |z|\angle\theta$$

- The complex conjugate is denoted as:

$$z^* = x - jy = |z|e^{-j\theta} = |z|\angle -\theta$$

- The magnitude is given by:

$$|z| = \sqrt[+]{zz^*}$$

Operation on Complex Numbers

- If two complex numbers are given by:

$$z_1 = x_1 + jy_1 = |z_1| e^{j\theta_1}$$

$$z_2 = x_2 + jy_2 = |z_1| e^{j\theta_2}$$

Then these two are equal if and only if: $x_1 = x_2$ **and** $y_1 = y_2$

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$

Division: $z_2 \neq 0$ $\dfrac{z_1}{z_2} = \dfrac{x_1 + jy_1}{x_2 + jy_2} = \dfrac{(x_1 x_2 + y_1 y_2) + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

$$\frac{z_1}{z_2} = \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)} \Rightarrow \frac{z_1}{z_2} = \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} \left[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)\right]$$

Power: $z_n = \left(|z| e^{j\theta}\right)^n = |z|^n e^{jn\theta} = |z|^n (\cos n\theta + j\sin n\theta)$

Example – 4

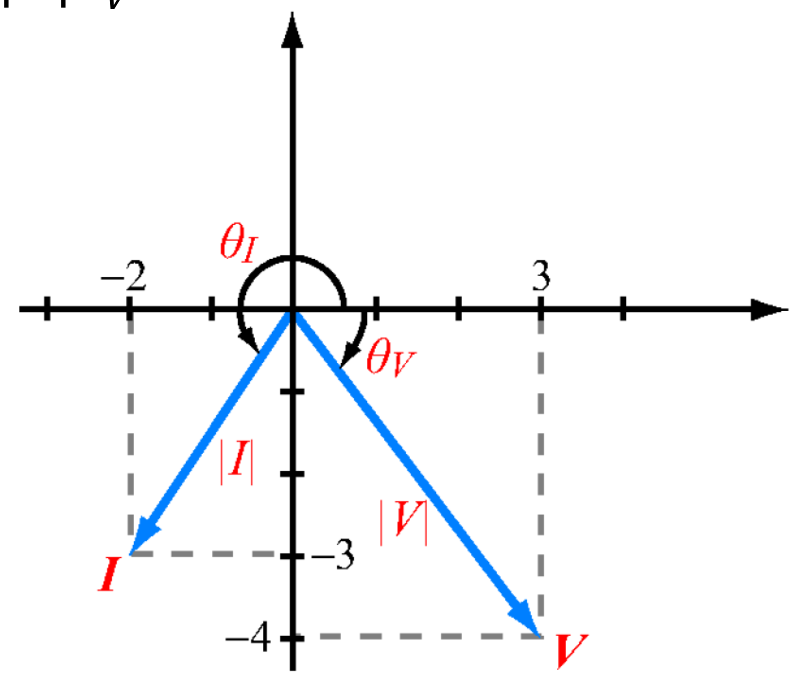
- Given complex numbers: $V = 3 - j4$ **and** $I = -(2 + j3)$

(a) Express V and I in polar form, and find (b) VI, (c)VI*, (d) V/I, and (e) $\sqrt{I}$

$$|V| = \sqrt[+]{VV^*} \quad |V| = \sqrt[+]{(3 - j4)(2 + j3)} = \sqrt[+]{9 + 16} \quad \therefore |V| = 5$$

$$\theta_V = tan^{-1}(-4/3) = -53.1° \quad \therefore V = |V|\theta_V = 5e^{-j53.1°}$$

Similarly: $I = |I|\theta_I = 3.61e^{j236.3°}$



Example – 5

- Express the following complex functions in polar form:

$$z_1 = (4 - j3)^2 \qquad z_2 = (4 - j3)^{1/2}$$

Answer: $z_1 = 25\angle -73.7^o \qquad z_2 = \pm 5\angle -18.4^o$

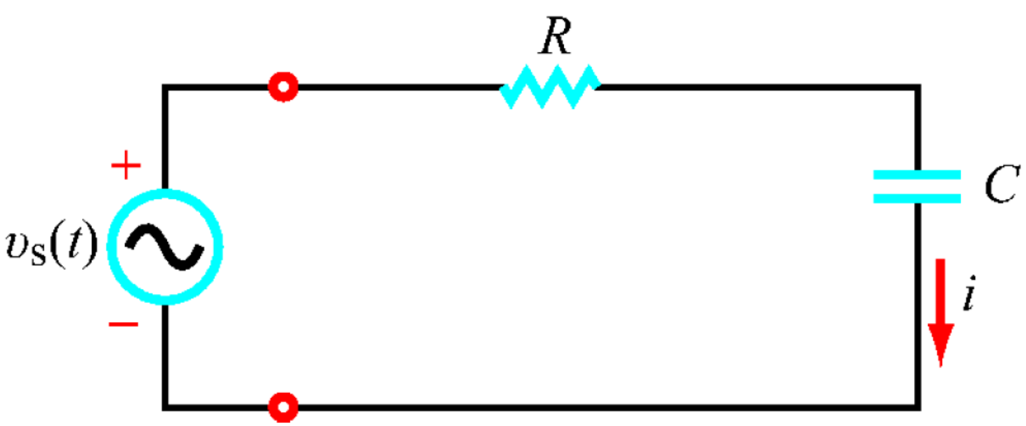
Example – 6

- Show that $\sqrt{2j} = \pm(1 + j)$

Review of Phasors

- Phasor analysis is a useful mathematical tool for solving problems involving linear systems in which the excitation, *also called forcing function*, is a periodic function.
- Phasor notation allows representation of linear integro-differential equation into a linear equation with no sinusoidal functions.
- The phasor technique can also be used for analyzing linear systems when the excitation is any arbitrary (nonsinusoidal) periodic time function such as a square wave or a sequence of pulses.

Lets consider simple RC circuit:



Assume: $v_s(t) = V_0 \sin\left(\omega t + \varphi_0\right)$

Application of KVL results:

$$Ri(t) + \frac{1}{C}\int i(t)dt = v_s(t)$$

Time Domain

Solution Procedure

Step-1: *Adopt a cosine reference*

$$v_s(t) = V_0 \sin(\omega t + \varphi_0) \quad = V_0 \cos\left(\frac{\pi}{2} - \omega t - \varphi_0\right) \quad = V_0 \cos\left(\omega t + \varphi_0 - \frac{\pi}{2}\right)$$

Step-2: *Express time dependent variables as phasors*

- Any sinusoidally time varying function *z(t)* can be expressed as:

$$z(t) = \tilde{Z} e^{j\omega t}$$

Where $\tilde{Z}$ is a time-independent function called the phasor of the instantaneous function $z(t)$.

- The voltage $v_s(t)$ is then:

$$v_s(t) = Re\left[V_0 e^{j(\omega t + \varphi_0 - \pi/2)}\right] = Re\left[V_0 e^{j(\varphi_0 - \pi/2)} e^{j\omega t}\right] \quad = Re\left[\tilde{V}_s e^{j\omega t}\right]$$

Where: $\tilde{V}_s = V_0 e^{j(\varphi_0 - \pi/2)}$ ← Phasor

Solution Procedure (contd.)

- Similarly, the unknown quantity $i(t)$ is: $i(t) = Re\left[\tilde{I}e^{j\omega t}\right]$
- For derivative and integral one can express:

$$\frac{d}{dt}i(t) = \frac{d}{dt}Re\left[\tilde{I}e^{j\omega t}\right] = Re\left[\frac{d}{dt}\left(\tilde{I}e^{j\omega t}\right)\right] = Re\left[j\omega\tilde{I}e^{j\omega t}\right]$$

$$\int i dt = \int Re\left[\tilde{I}e^{j\omega t}\right]dt = Re\left(\int \tilde{I}e^{j\omega t}dt\right) = Re\left(\frac{\tilde{I}}{j\omega}e^{j\omega t}\right)$$

Step-3: *Recast the differential/integral equation in phasor form*

$$R \times Re\left[\tilde{I}e^{j\omega t}\right] + \frac{1}{C}Re\left(\frac{\tilde{I}}{j\omega}e^{j\omega t}\right) = Re\left[\tilde{V}_s e^{j\omega t}\right]$$

$$Re\left\{\left[\left(R + \frac{1}{j\omega C}\right)\tilde{I} - \tilde{V}_s\right]e^{j\omega t}\right\} = 0$$

Solution Procedure (contd.)

- Similarly, if sine reference is adopted then:

$$Im\left\{\left[\left(R+\frac{1}{j\omega C}\right)\tilde{I}-\widetilde{V_s}\right]e^{j\omega t}\right\}=0$$

- As $e^{j\omega t}\neq 0$ $\Rightarrow$ $\left(R+\frac{1}{j\omega C}\right)\tilde{I}=\widetilde{V_s}$ $\Leftarrow$ Phasor Domain Equivalent Equation

Step-4: *Solve the phasor-domain equation*

$$\tilde{I}=\frac{\widetilde{V_s}}{\left(R+\frac{1}{j\omega C}\right)} \Rightarrow \tilde{I}=V_0e^{j(\varphi_0-\pi/2)}\frac{j\omega C}{(j\omega CR+1)}$$

$$\Rightarrow \tilde{I}=V_0e^{j(\varphi_0-\pi/2)}\left[\frac{\omega C\,e^{j\pi/2}}{\overset{+}{}\sqrt{1+\omega^2R^2C^2}\;e^{j\varphi_1}}\right]$$

Solution Procedure (contd.)

$$\therefore \tilde{I} = \frac{\omega C V_0}{\overset{+}{\sqrt{1+\omega^2 R^2 C^2}}} e^{j(\varphi_0-\varphi_1)}$$

Step-5: *Find the instantaneous value*

$$i(t) = Re\left[\tilde{I} e^{j\omega t}\right]$$

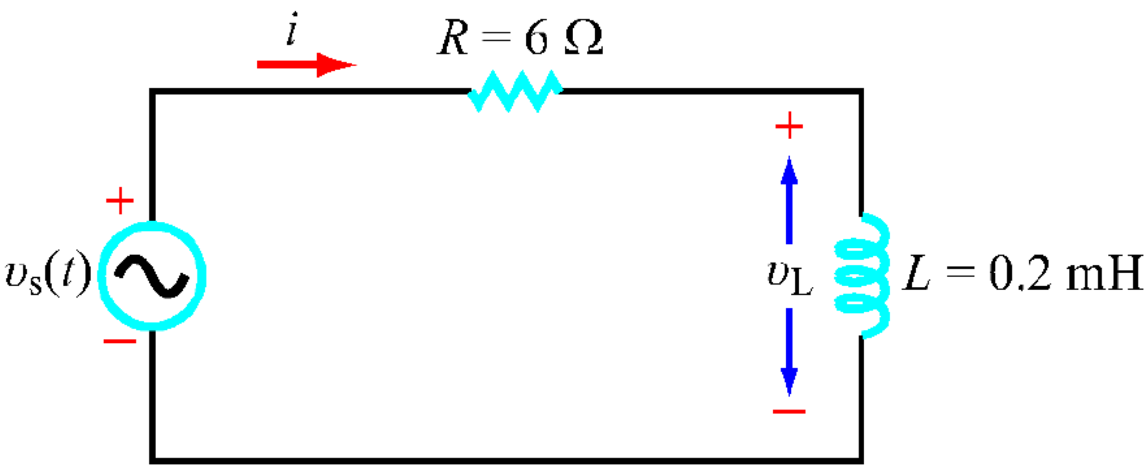
$$i(t) = Re\left[\frac{\omega C V_0}{\overset{+}{\sqrt{1+\omega^2 R^2 C^2}}} e^{j(\varphi_0-\varphi_1)} e^{j\omega t}\right]$$

$$\therefore i(t) = \frac{\omega C V_0}{\overset{+}{\sqrt{1+\omega^2 R^2 C^2}}} cos(\omega t + \varphi_0 - \varphi_1)$$

Example – 7

- The voltage source of the circuit shown in following figure is given by:

$$v_s(t) = 5\sin(4 \times 10^4 t - 30°) \qquad V$$



Obtain an expression for the voltage across the inductor.

Solution:

The voltage loop equation is: $v_s(t) = Ri + L\dfrac{di}{dt}$

Example – 7 (contd.)

- Let us convert the input voltage source to cosine reference:

$$v_s(t) = 5\sin(4 \times 10^4 t - 30°) \quad V \qquad = 5\cos(4 \times 10^4 t - 120°) \quad V$$

- The phasor form of voltage source: $\widetilde{V_s} = 5e^{-j120°} \qquad V$

- The phasor equation of the circuit: $R\tilde{I} + j\omega L\tilde{I} = \widetilde{V_s}$

- Solving for the current phasor: $\tilde{I} = \dfrac{\widetilde{V_s}}{R + j\omega L}$

$$\tilde{I} = \frac{5e^{-j120°}}{6 + j4 \times 10^4 \times 2 \times 10^{-4}} \Rightarrow \tilde{I} = \frac{5e^{-j120°}}{6 + j8} \Rightarrow \tilde{I} = \frac{5e^{-j120°}}{10e^{j53.1°}}$$

$$\therefore \tilde{I} = 0.5e^{-j173.1°}$$

Example – 7 (contd.)

- The voltage phasor across the inductor: $\widetilde{V_L} = j\omega L\tilde{I}$

$$\widetilde{V_L} = j4 \times 10^4 \times 2 \times 10^{-4} \times 0.5e^{-j173.1°}$$

$$\widetilde{V_L} = 4e^{j(90°-173.1°)} \qquad \therefore \widetilde{V_L} = 4e^{-j83.1°}$$

- Therefore the corresponding instantaneous voltage is:

$$v_L(t) = Re\left[\widetilde{V_L}e^{j\omega t}\right] \Rightarrow v_L(t) = Re\left[4e^{-j83.1°}e^{j\omega t}\right]$$

$$\therefore v_L\,(t) = 4\cos(4 \times 10^4 t - 83.1°) \qquad V$$

Traveling Waves in Phasor

$$y(x,t) = A\cos(\omega t - \beta x + \varphi_0) \xrightarrow{\varphi_0} y(x,t) = A\cos(\omega t - \beta x)$$

$$A\cos(\omega t - \beta x) \Longleftrightarrow Ae^{-j\beta x}$$

In the phasor domain, a wave of amplitude A traveling in the positive x-direction in a lossless medium with phase constant β is given by negative exponent $Ae^{-j\beta x}$, and conversely, a wave traveling in the negative x-direction is given by $Ae^{j\beta x}$. Thus, sign of x in the exponent is opposite to the direction of travel.