Lecture – 24 and 25 Date: 11.04.2016

- Wave Polarization
- Incidence, Reflection, and Transmission of Plane Waves
- Reflection of Plane Wave at Oblique Incidence (Snells' Law, Brewster's Angle, Parallel Polarization, Perpendicular Polarization etc.)

General Relations Between $\vec{E}$ and $\vec{H}$

- We learnt earlier that if $\hat{a}_E$, $\hat{a}_H$ and $\hat{a}_k$ are unit vectors along $\vec{E}$, $\vec{H}$ and the direction of propagation, then:

$$\hat{a}_k \times \hat{a}_E = \hat{a}_H \qquad \hat{a}_k \times \hat{a}_H = -\hat{a}_E \qquad \hat{a}_E \times \hat{a}_H = \hat{a}_k$$

- In general it can be deduced that:

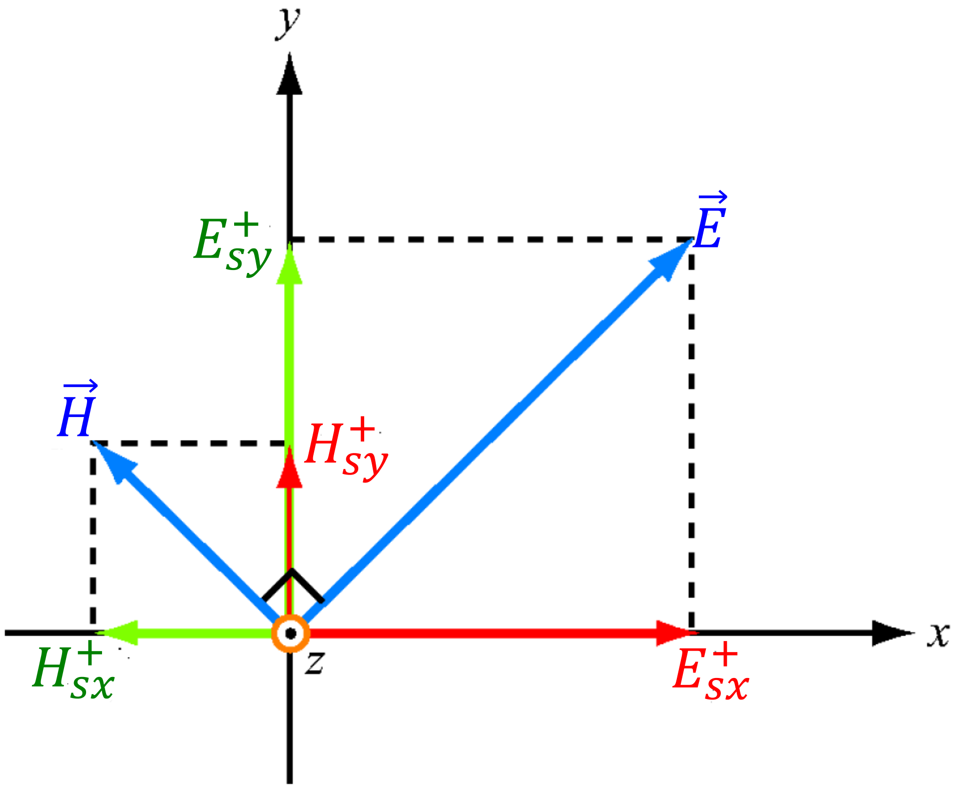
$$\vec{H}_s = \frac{1}{\eta}\hat{a}_k \times \vec{E}_s \qquad \vec{E}_s = -\eta \hat{a}_k \times \vec{H}_s$$

- Furthermore, a uniform plane wave travelling in the $+\hat{a}_z$ direction may have both $x -$ and $y - components$.
- In such a scenario: $\vec{E}_s = \hat{a}_x \vec{E}_{sx}^{+}(z) + \hat{a}_y \vec{E}_{sy}^{+}(z)$
- The associated magnetic field will be: $\vec{H}_s = \hat{a}_x \vec{H}_{sx}^{+}(z) + \hat{a}_y \vec{H}_{sy}^{+}(z)$
- The exact expression of magnetic field in terms of electric field will be:

$$\vec{H}_s = \frac{1}{\eta}\hat{a}_z \times \vec{E}_s = -\hat{a}_x \frac{\vec{E}_{sy}^{+}(z)}{\eta} + \hat{a}_y \frac{\vec{E}_{sx}^{+}(z)}{\eta}$$

- Thus: $\vec{H}_{xs}^{+} = -\dfrac{\vec{E}_{sy}^{+}(z)}{\eta}$ $\qquad \vec{H}_{ys}^{+}(z) = \dfrac{\vec{E}_{sx}^{+}(z)}{\eta}$

General Relations Between $\vec{E}$ and $\vec{H}$ (contd.)



In general, a TEM wave may have an electric field in any direction in the plane orthogonal to the direction of wave travel, and the associated magnetic field is also in the same plane with appropriate magnitude and direction.

Wave Polarization

- Very important concept considering its use in energy transmission of waves and its applications in the design of components such as Antenna.
- The $polarization$ of a uniform plane wave describes the locus traced by the tip of the $\vec{E}$ vector (in the phase orthogonal to the direction of propagation) at a given point in space as a function of time.
- In the most general case, the locus of tip of $\vec{E}$ is an ellipse, and wave is said to be $elliptically\ polarized$.
- Under certain conditions, the ellipse may degenerate into a circle or a straight line, in which case the $polarization\ state$ is called $circular$ or $linear$ respectively.
- We know that the $z - components$ of the electric and magnetic fields of a $z - propagating$ plane wave are both zero.
- Hence, in the most general case, the electric field phasor may consist of an $x - component$ and a $y - component$.

$$\boxed{\vec{E}_s = \hat{a}_x \vec{E}_{sx}^{+}(z) + \hat{a}_y \vec{E}_{sy}^{+}(z)}$$

Wave Polarization (contd.)

- With: $E_{sx}(z) = E_{x0}e^{-j\beta z}$ $E_{sy}(z) = E_{y0}e^{-j\beta z}$

Where, E_{x0} and E_{y0} are the amplitudes of $E_{sx}(z)$ and $E_{sy}(z)$ respectively.

- The amplitudes E_{x0} and E_{y0} are, in general, complex quantities → each characterized by phase and magnitude.
- The phase of a wave is defined relative to a reference state, such as $z = 0$, and $t = 0$ or any other combination of z and t.
- Essentially, the polarization of wave depends on phase of E_{y0} relative to that of E_{x0} and not the absolute phases of E_{x0} and E_{y0}.
- Therefore, for convenience, let us assign a phase of zero to E_{x0} and denote the phase of E_{y0}, relative to that of E_{x0}, as δ_p.
- Accordingly: $E_{x0} = A_x$ $E_{y0} = A_y e^{j\delta_p}$

Wave Polarization (contd.)

$$E_{x0} = A_x \qquad E_{y0} = A_y e^{j\delta_p}$$

Where, $A_x = |E_{x0}| \geq 0$ and $A_y = |E_{y0}| \geq 0$ are the magnitudes of E_{x0} and E_{y0} respectively.

Thus by definition, A_x and A_y may not assume negative values.

- Therefore, the electric field phasor is:

$$\vec{E}_s(z) = E_{sx}(z)\hat{a}_x + E_{sy}(z)\hat{a}_y \Rightarrow \vec{E}_s(z) = \left(\hat{a}_x A_x + \hat{a}_y A_y e^{j\delta_p}\right)e^{-j\beta z}$$

- The corresponding instantaneous field is:

$$\vec{E}(z,t) = \mathrm{Re}\left\{\vec{E}_s(z)e^{j\omega t}\right\} \Rightarrow \vec{E}(z,t) = \hat{a}_x A_x \cos\left(\omega t - \beta z\right) + \hat{a}_y A_y \cos\left(\omega t - \beta z + \delta_p\right)$$

- An electric field at a given point in space is characterized by its magnitude and direction.
- The magnitude of $\vec{E}(z,t)$ is: $\left|\vec{E}(z,t)\right| = \left[A_x^2 \cos^2(\omega t - \beta z) + A_y^2 \cos^2(\omega t - \beta z + \delta_p)\right]^{1/2}$

Wave Polarization (contd.)

- At a specific position z, the direction of $\vec{E}(z,t)$ is characterized by its $inclination\, angle$ ψ with respect to the x-axis:

$$\psi(z,t) = \tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

Linear Polarization

- Wave is said to be linearly polarized if for a fixed z, the tip of $\vec{E}(z,t)$ traces a straight line segment as a function of time $\leftrightarrow$ happens when $E_x(z,t)$ and $E_y(z,t)$ are $in-phase$ $(\delta_p = 0)$ or out $-$ of $-$ $phase$ $(\delta_p = \pi)$.

- Under these conditions:

$$\vec{E}(0,t) = \left(\hat{a}_x A_x + \hat{a}_y A_y\right)\cos\left(\omega t - \beta z\right)$$

In-phase

$$\vec{E}(0,t) = \left(\hat{a}_x A_x - \hat{a}_y A_y\right)\cos\left(\omega t - \beta z\right)$$

Out-of-phase

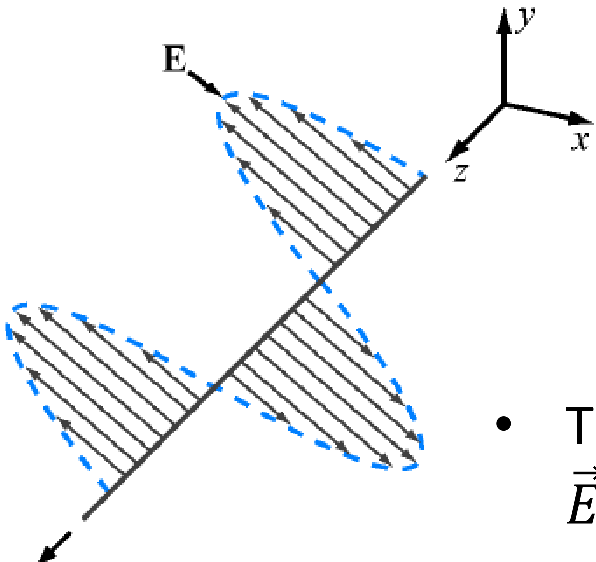
- Let us assume out $-$ of $-$ $phase$ case:

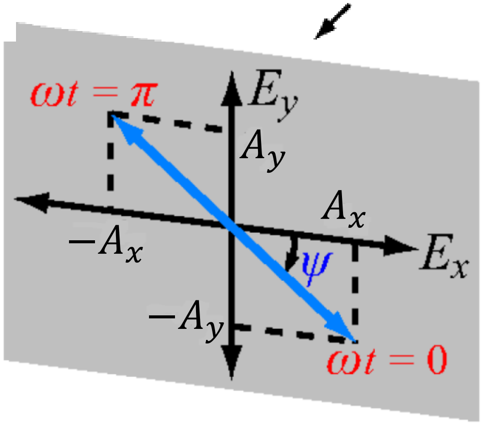
$$\left|\vec{E}(z,t)\right| = \left[A_x^2 + A_y^2\right]^{1/2} \left|\cos(\omega t - \beta z)\right|$$

$$\psi(z,t) = \tan^{-1}\left(\frac{-A_y}{A_x}\right)$$

Linear Polarization (contd.)

- We note that ψ is independent of both z and t.
- Following figure displays the line segment traced by the tip of $\vec{E}$ at $z = 0$ over half a cycle.





- The trace would be the same over any other value of z as well.
- At $z = 0, t = 0$:

$$\boxed{\left|\vec{E}(0,0)\right| = \left[A_x^2 + A_y^2\right]^{1/2}}$$

- The length of the vector representing $\vec{E}(0,t)$ decreases to zero at $\omega t = \pi/2$.
- The vector then reverses direction and increases in magnitude to:

$$\boxed{\left|\vec{E}(0,t)\right| = \left[A_x^2 + A_y^2\right]^{1/2}}$$

In the second quadrant of the $x - y$ *plane* at $\omega t = \pi$.

Linear Polarization (contd.)

- Since ψ is independent of both z and t, $\vec{E}(z,t)$ maintains a direction along the line making an angle ψ with the $x-axis$, while oscillating back and forth across the origin.
- If $A_y = 0$, then $\psi = 0$ or $180°$, and the wave is $x - polarized$.
- If $A_x = 0$, then $\psi = 90°$ or $-90°$, and the wave is y $- polarized$.

Circular Polarization

- Let us consider the special case when $A_x = A_y$ and $\delta_p = \pm\pi/2$.
- For reasons that will become evident shortly, the wave polarization is called $left - hand$ polarized when $\delta_p = \pi/2$, and $right - hand$ polarized when $\delta_p = -\pi/2$.

Left-Hand Circular Polarization (LHCP)

- For $A_x = A_y = A$ and $\delta_p = \pi/2$, the electric field phasor and instantaneous electric field become:

$$\boxed{\vec{E}_s(z) = \left(\hat{a}_x A + \hat{a}_y A e^{j\pi/2}\right)e^{-j\beta z} = A\left(\hat{a}_x + j\hat{a}_y\right)e^{-j\beta z}}$$

Left-Hand Circular Polarization (LHCP) (contd.)

$$\boxed{\vec{E}(z,t) = \hat{a}_x A\cos(\omega t - \beta z) + \hat{a}_y A\cos\left(\omega t - \beta z + \frac{\pi}{2}\right) = \hat{a}_x A\cos(\omega t - \beta z) - \hat{a}_y A\sin(\omega t - \beta z)}$$

- The corresponding magnitude and inclination angle are:

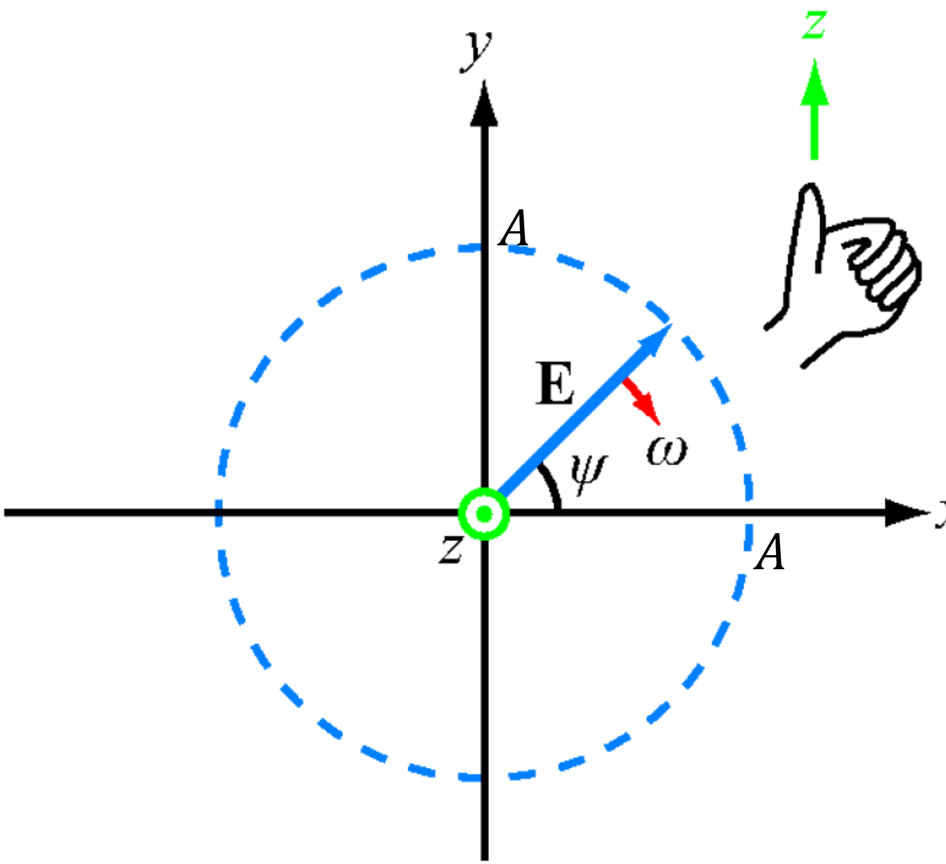
$$\boxed{\left|\vec{E}(z,t)\right| = A}$$

$$\boxed{\psi(z,t) = -(\omega t - \beta z)}$$

- Apparently the magnitude of $\vec{E}$ is independent of both z and t, whereas ψ depends on both variables → these functional dependencies are converse of those for the $linear\ polarization$ case.
- At $z = 0$, the inclination angle $\psi = -\omega t$.
- The negative sign implies that the inclination angle decreases with the increase in time.

Left-Hand Circular (LHC) Polarization (contd.)

- As seen in the figure, the tip of $\vec{E}(t)$ traces a circle in $x-y\,plane$ and rotates in clockwise direction as a function of time (when viewing the wave approaching).
- Such a wave is called $left-hand\ circularly$ polarized.

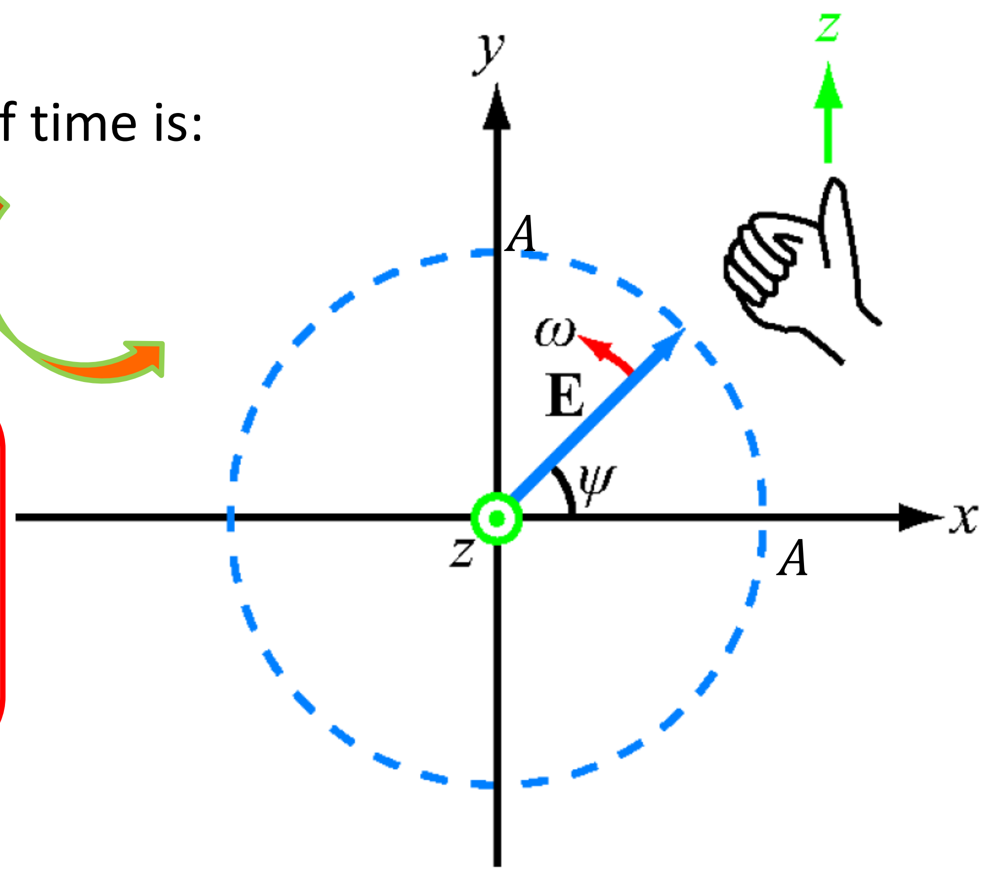


When the thumb of the left hand points along the direction of propagation (the $z-direction$ in this case), the other four fingers point in the direction of rotation of $\vec{E}$.

Right-Hand Circular Polarization (RHCP)

- For $A_x = A_y = A$ and $\delta_p = -\pi/2$, we get: $\boxed{\left|\vec{E}(z,t)\right| = A}$ $\boxed{\psi(z,t) = (\omega t - \beta z)}$

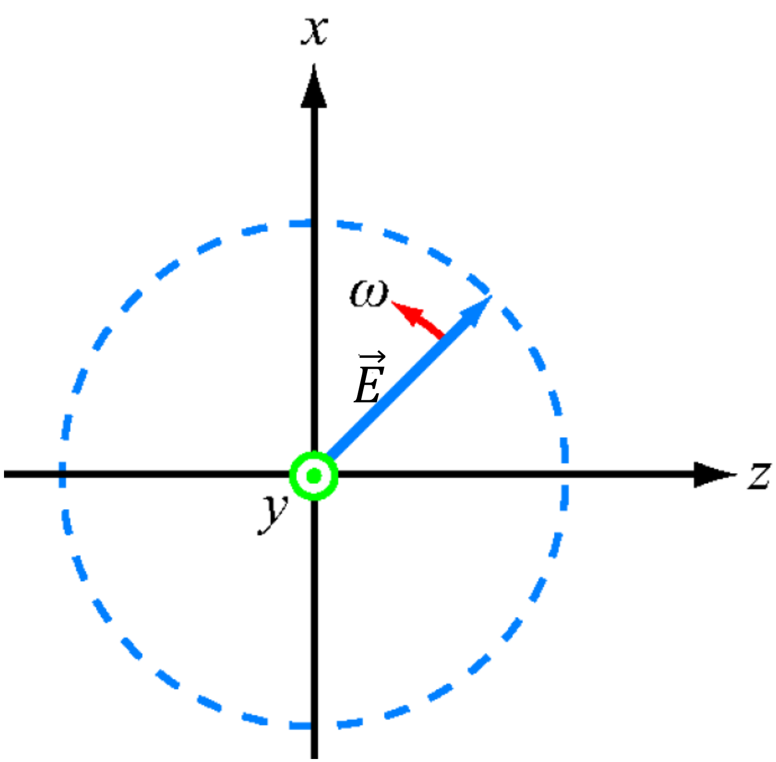
- The trace of $\vec{E}(t)$ as a function of time is:



For $RHCP$, the fingers of the right hand point in the direction of rotation of $\vec{E}$ when the thumb point in the direction of propagation.

Example – 1

- An RHC polarized plane wave with electric field magnitude of $3\ mV/m$ is traveling in the $+y - direction$ in a dielectric medium with $\varepsilon = 4\varepsilon_0, \mu = \mu_0$ and $\sigma = 0$. If the frequency is 100MHz, obtain the expression for $\vec{E}(y,t)$ and $\vec{H}(y,t)$.



- Let us assign the z-component of $\vec{E}_s(y)$ a phase angle of zero and the x-component a phase shift of $\delta_p = -\frac{\pi}{2}$.
- Then:

$$\vec{E}_s(y) = \left(\hat{a}_x E_{sx} + \hat{a}_z E_{sz}\right) e^{-j\beta y}$$

$$\vec{E}_s(y) = 3\left(\hat{a}_x e^{-j\pi/2} + \hat{a}_z\right) e^{-j\beta y} \quad \text{mV/m}$$

$$\therefore \vec{E}_s(y) = 3\left(-j\hat{a}_x + \hat{a}_z\right) e^{-j\beta y} \quad \text{mV/m}$$

Example – 1 (contd.)

- Similarly: $\vec{H}_s(y) = \frac{1}{\eta}\left[\hat{a}_y \times \vec{E}_s(y)\right]$ ⇒ $\vec{H}_s(y) = \frac{3}{\eta}\left(\hat{a}_x + j\hat{a}_z\right)e^{-j\beta y}$ mA/m

- The instantaneous fields are:

$$\vec{E}(y,t) = \mathrm{Re}\left[\vec{E}_s(y)e^{j\omega t}\right] = \mathrm{Re}\left[3\left(-j\hat{a}_x + \hat{a}_z\right)e^{-j\beta y}e^{j\omega t}\right] \text{ mV/m}$$

$$\therefore \vec{E}(y,t) = 3\left[\hat{a}_x \sin(\omega t - \beta y) + \hat{a}_z \cos(\omega t - \beta y)\right] \text{ mV/m}$$

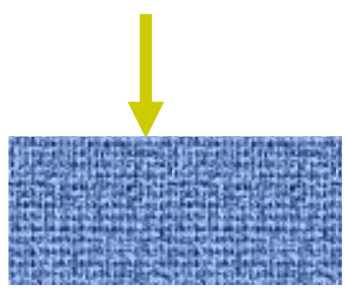
$$\vec{H}(y,t) = \mathrm{Re}\left[\vec{H}_s(y)e^{j\omega t}\right] = \mathrm{Re}\left[\frac{3}{\eta}\left(\hat{a}_x + j\hat{a}_z\right)e^{-j\beta y}e^{j\omega t}\right] \text{ mA/m}$$

$$\therefore \vec{H}(y,t) = \frac{3}{\eta}\left[\hat{a}_x \cos(\omega t - \beta y) - \hat{a}_z \sin(\omega t - \beta y)\right] \text{ mA/m}$$

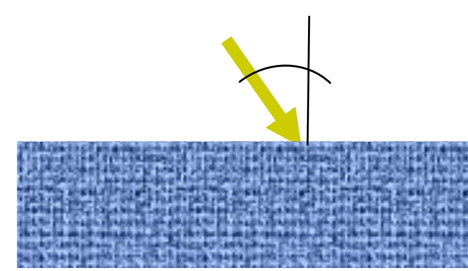
Wave Incidence

- For many applications, [such as fiber optics, wire line transmission, wireless transmission], it's necessary to know what happens to a wave when it meets a different medium.
  - How much is *transmitted*?
  - How much is *reflected* back?

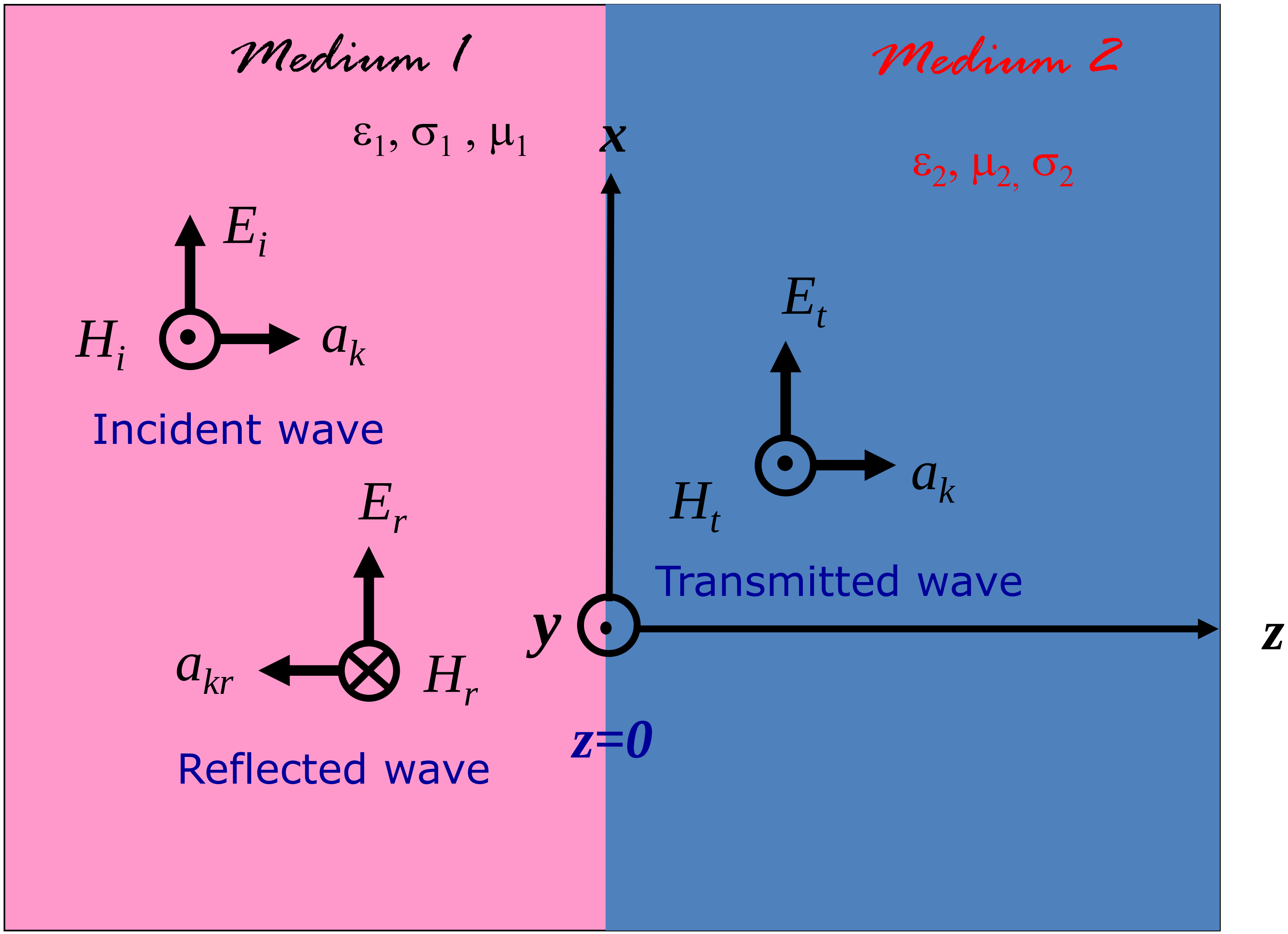
➢ **Normal incidence:** *Wave arrives at 0º from normal*



➢ **Oblique incidence:** *Wave arrives at another angle*



Reflection at Normal Incidence

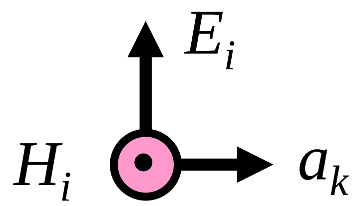


Reflection at Normal Incidence (contd.)

Incident wave

$$\vec{E}_{is}(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x$$

$$\vec{H}_{is}(z) = H_{io} e^{-\gamma_1 z} \hat{a}_y = \frac{E_{io}}{\eta_1} e^{-\gamma_1 z} \hat{a}_y$$

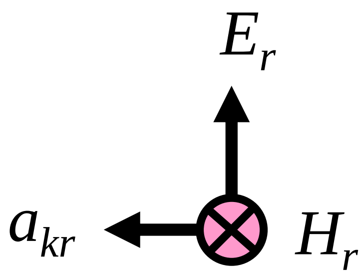


Incident wave

Reflected wave

$$\vec{E}_{rs}(z) = E_{ro} e^{\gamma_1 z} \hat{a}_x$$

$$\vec{H}_{rs}(z) = H_{ro} e^{\gamma_1 z} \left(-\hat{a}_y\right) = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y$$

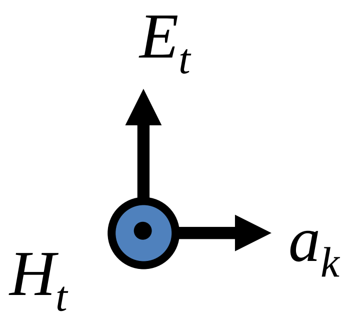


Reflected wave

Transmitted wave

$$\vec{E}_{ts}(z) = E_{to} e^{-\gamma_2 z} \hat{a}_x$$

$$\vec{H}_{ts}(z) = H_{to} e^{-\gamma_2 z} \hat{a}_y = \frac{E_{to}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y$$



Transmitted wave

Reflection at Normal Incidence (contd.)

- The total waves in medium 1: $\boxed{\vec{E}_1 = \vec{E}_i + \vec{E}_r}$ $\boxed{\vec{H}_1 = \vec{H}_i + \vec{H}_r}$
- The total waves in medium 2: $\boxed{\vec{E}_2 = \vec{E}_t}$ $\boxed{\vec{H}_2 = \vec{H}_t}$
- At the interface $z = 0$, the boundary conditions require that the tangential components of $\vec{E}$ and $\vec{H}$ fields must be continuous.
- Since the waves are transverse, $\vec{E}$ and $\vec{H}$ fields are entirely tangential to the surface.
- Therefore, at $z = 0$: $\vec{E}_{1tan} = \vec{E}_{2tan}$ and $\vec{H}_{1tan} = \vec{H}_{2tan}$ imply that -

$$\boxed{\vec{E}_{is}(0) + \vec{E}_{rs}(0) = \vec{E}_{ts}(0)} \Rightarrow \boxed{E_{io} + E_{ro} = E_{to}}$$

$$\boxed{\vec{H}_{is}(0) + \vec{H}_{rs}(0) = \vec{H}_{ts}(0)} \Rightarrow \boxed{\frac{1}{\eta_1}\left(E_{io} + E_{ro}\right) = \frac{E_{to}}{\eta_2}}$$

Reflection at Normal Incidence (contd.)

- Simplification results in:

$$E_{ro} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{io}$$

$$E_{to} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{io}$$

These expressions aid us in the definitions of *reflection coefficient* Γ and *transmission coefficient* τ.

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \Rightarrow E_{ro} = \Gamma E_{io}$$

$$\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1} \Rightarrow E_{to} = \tau E_{io}$$

- It is important to note that:
  - $1 + \Gamma = \tau$
  - Both Γ and τ are dimensionless and may be complex
  - $0 \le |\Gamma| \le 1$

Reflection at Normal Incidence (contd.)

- Therefore the total fields in the two medium are:

$$\boxed{\vec{E}_{1s} = E_{io}\left[e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right]\hat{a}_x}$$

$$\boxed{\vec{H}_{1s} = \frac{E_{io}}{\eta_1}\left[e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}\right]\hat{a}_y}$$

$$\boxed{\vec{E}_{2s} = E_{io}\tau e^{-\gamma_2 z}\hat{a}_x}$$

$$\boxed{\vec{H}_{2s} = \frac{E_{io}}{\eta_2}\tau e^{-\gamma_2 z}\hat{a}_y}$$

- Special Cases:
  - $\eta_1 = \eta_2$ $\quad \Gamma = 0$ $\quad \tau = 1$ (total transmission, no reflection)
  - $\eta_1 = 0$ $\quad \Gamma = 1$ $\quad \tau = 2$ (total reflection, no inversion of $\vec{E}$)
  - $\eta_2 = 0$ $\quad \Gamma = -1$ $\quad \tau = 0$ (total reflection, inversion of $\vec{E}$)

Reflection at Normal Incidence (contd.)

Special Case – I

- Medium 1: perfect dielectric (lossless): $\sigma_1 = 0$, $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$, $\alpha_1 = 0, \gamma = j\beta_1$
- Medium 2: perfect conductor: $\sigma_2 = \infty$, $\eta_2 = 0$, $\alpha_2 = \beta_2 = \infty$

$\eta_2 = 0$ $\quad \Gamma = -1$ $\quad \tau = 0$ $\quad$ (total reflection, inversion of $\vec{E}$

$$\vec{E}_{1s} = \vec{E}_{is} + \vec{E}_{rs} \Rightarrow \vec{E}_{1s} = -E_{io}\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\hat{a}_x \Rightarrow \vec{E}_{1s} = -2jE_{io}\sin\beta_1 z\hat{a}_x$$

$$\vec{H}_{1s} = \frac{E_{io}}{\eta_1}\left[e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}\right]\hat{a}_y \Rightarrow \vec{H}_{1s} = \frac{2E_{io}}{\eta_1}\cos\beta_1 z\hat{a}_y$$

$$\vec{E}_{2s} = E_{io}\tau e^{-\gamma_2 z}\hat{a}_x = 0$$

$$\vec{H}_{2s} = \frac{E_{io}}{\eta_2}\tau e^{-\gamma_2 z}\hat{a}_y = 0$$

Reflection at Normal Incidence (contd.)

- The instantaneous electric field: $\vec{E}_1(z,t) = \mathrm{Re}\left(\vec{E}_{1s} e^{j\omega t}\right) = 2E_{io} \sin \beta_1 z \sin \omega t \hat{a}_x$

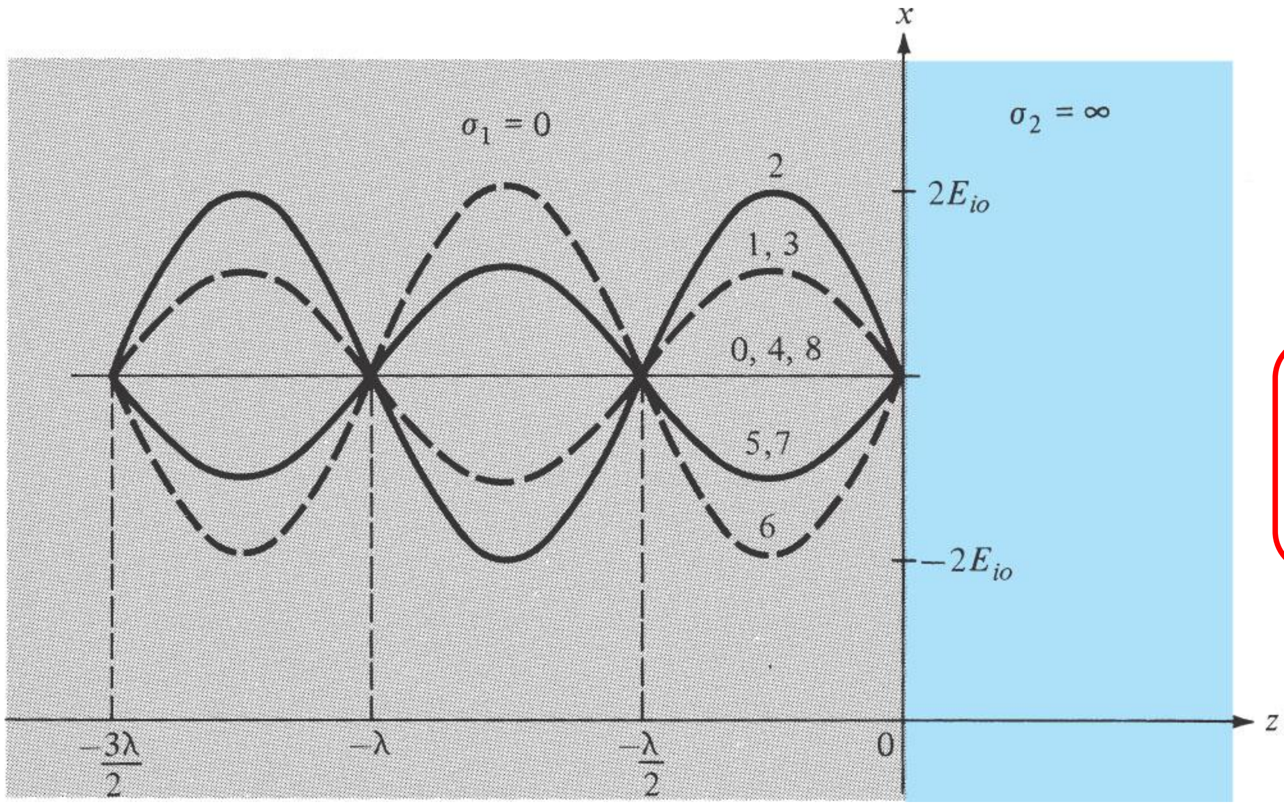
- Similar steps result in: $\vec{H}_1(z,t) = \dfrac{2E_{io}}{\eta_1} \cos \beta_1 z \cos \omega t \hat{a}_y$

Note that the position dependence of the instantaneous electric and magnetic fields is not a function of time → standing wave!!!

It is expected considering that there is total reflection and in a lossless dielectric the waves consist of two travelling waves ($\vec{E}_i$ and $\vec{E}_r$) of equal amplitudes but in opposite directions.

Reflection at Normal Incidence (contd.)

$$\vec{E}_1(z,t) = \mathrm{Re}\left(\vec{E}_{1s}e^{j\omega t}\right) = 2E_{io}\sin\beta_1 z\sin\omega t\hat{a}_x$$



The wave doesn't travel but oscillate

Standing waves $\vec{E} = 2E_{io}sin\beta_1 zsin\omega t\hat{a}_x$. The curves 0, 1, 2, 3, 4, . . ., are, respectively, at times $t = 0$, $T/8$, $T/4$, $3T/8$, $T/2$, . . . ; $\lambda = 2\pi/\beta_1$.

Reflection at Normal Incidence (contd.)

- The locations of the minimums (nulls) and maximums (peaks) in the standing wave electric field pattern are found by:

$$\left|\vec{E}_1(z,t)\right|_{\min} = 0 \xrightarrow{\text{when}} \sin\beta_1 z = 0 \Rightarrow \beta_1(-z) = n\pi$$

$$z = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}$$

$$\left|\vec{E}_1(z,t)\right|_{\max} = 2E_{io} \xrightarrow{\text{when}} \sin\beta_1 z = 1 \Rightarrow \beta_1(-z) = (2n+1)\frac{\pi}{2}$$

$$z = -\frac{(2n+1)\pi}{2\left(\frac{2\pi}{\lambda_1}\right)} = -\frac{(2n+1)}{4}\lambda_1$$

Reflection at Normal Incidence (contd.)

Special Case – II: Two Perfect Dielectrics

- Medium 1: perfect dielectric (lossless): $\sigma_1 = 0$, $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$, $\alpha_1 = 0, \gamma = j\beta_1$
- Medium 2: perfect dielectric (lossless): $\sigma_2 = 0$, $\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$, $\alpha_2 = 0, \gamma = j\beta_2$

If $\eta_2 > \eta_1$ $\quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ $\quad$ $\mathbf{0 < \Gamma < 1}$ $\quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1}$ $\quad$ $\mathbf{1 < \tau < 2}$

If $\eta_2 < \eta_1$ $\quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ $\quad$ $\mathbf{-1 < \Gamma < 0}$ $\quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1}$ $\quad$ $\mathbf{0 < \tau < 1}$

Reflection at Normal Incidence (contd.)

- Therefore:

$$\vec{E}_{1s} = E_{io}(e^{-j\gamma_1 z} + \Gamma e^{+j\gamma_1 z})\hat{a}_x = E_{io}e^{-j\beta_1 z}(1+\Gamma e^{+2j\beta_1 z})\hat{a}_x$$

$$\vec{H}_{1s} = \frac{E_{io}}{\eta_1}(e^{-j\gamma_1 z} - \Gamma e^{+j\gamma_1 z})\hat{a}_y = \frac{E_{io}}{\eta_1}(1-\Gamma e^{+2j\beta_1 z})\hat{a}_y$$

$$\vec{E}_{2s} = E_{io}\tau e^{-j\gamma_2 z}\hat{a}_x = E_{io}\tau e^{-j\beta_2 z}\hat{a}_x$$

$$\vec{H}_{2s} = \frac{E_{io}}{\eta_2}e^{-j\gamma_2 z}\hat{a}_y = \frac{E_{io}}{\eta_2}e^{-j\beta_2 z}\hat{a}_y$$

Standing wave exists only in medium 1.

- The magnitude of the electric field in medium 1 can be analyzed to determine the locations of the maximum and minimum values of the electric field standing wave pattern.

$$\left|\vec{E}_{1s}\right| = E_{io}\left|(1+\Gamma e^{+2j\beta_1 z})\right|$$

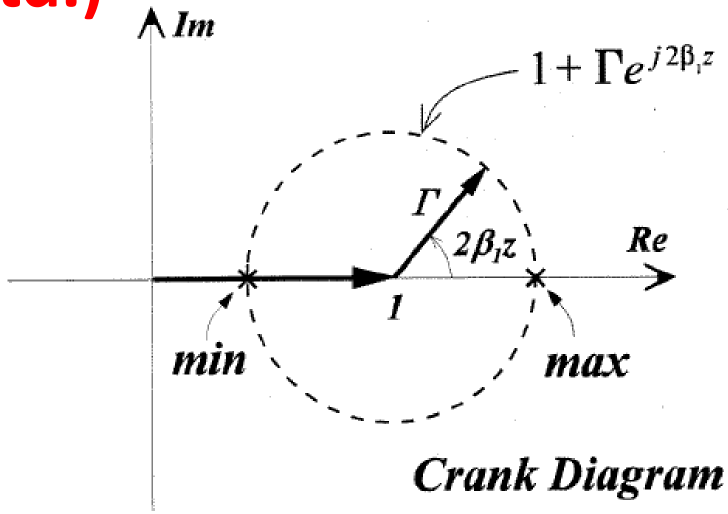
$$\left|(1+\Gamma e^{+2j\beta_1 z})\right| = \left|1\angle 0^\circ + \Gamma\angle 2\beta_1 z\right|$$

This can be described in the complex plane using *crank diagram*

Reflection at Normal Incidence (contd.)

- The distance from the origin to the respective point on the circle in the crank diagram represents the magnitude of:

$$\boxed{1+\Gamma e^{+2j\beta_1 z}}$$



Crank Diagram

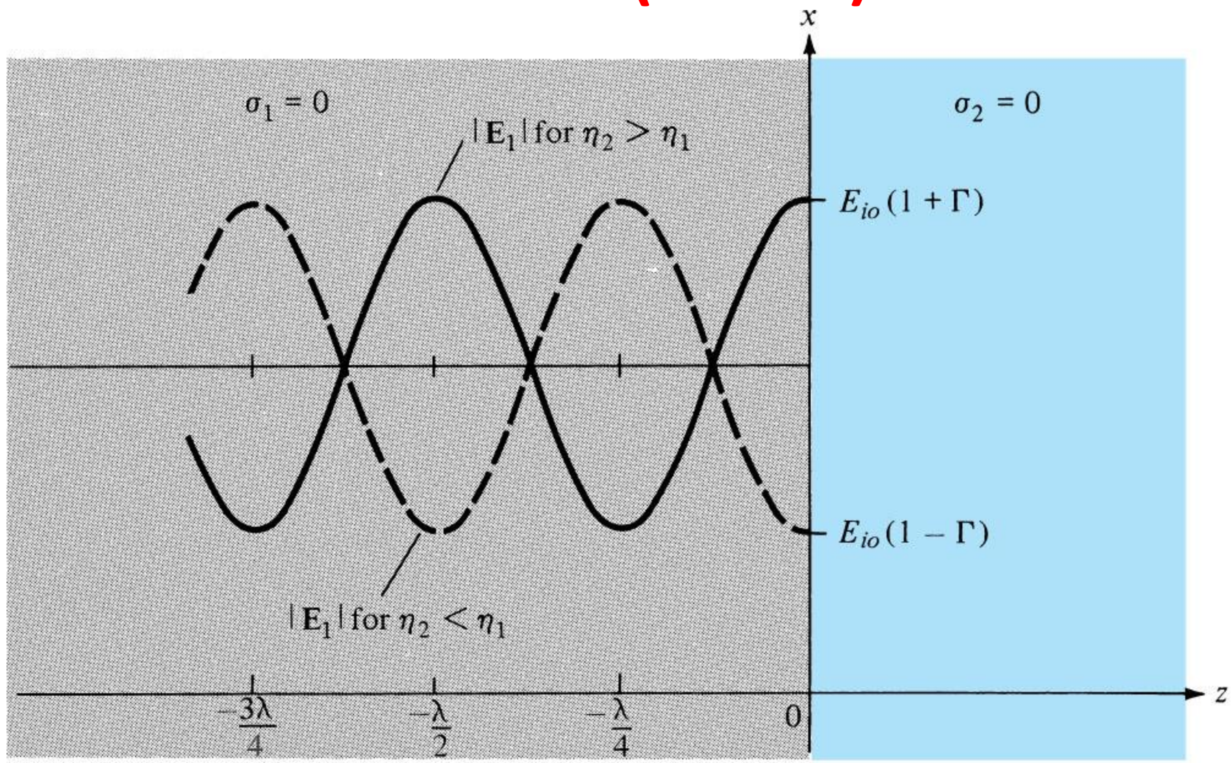
- If $\eta_2 > \eta_1$, (Γ is positive), then the maximum and minimum of the function are:

$$\left|\left(1+\Gamma e^{+2j\beta_1 z}\right)\right|_{\max} = 1+\Gamma \quad \text{when} \quad 2\beta_1(-z) = n(2\pi) \Rightarrow z = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}$$

$$\left|\left(1+\Gamma e^{+2j\beta_1 z}\right)\right|_{\min} = 1-\Gamma \quad \text{when} \quad 2\beta_1(-z) = (2n+1)\pi \Rightarrow z = -\frac{(2n+1)\pi}{\beta_1} = -\frac{(2n+1)}{4}\lambda_1$$

$$\left|\vec{E}_{1s}\right|_{\max} = E_{io}\left(1+|\Gamma|\right) \qquad \left|\vec{E}_{1s}\right|_{\min} = E_{io}\left(1-|\Gamma|\right)$$

Reflection at Normal Incidence (contd.)



- If $\eta_2 > \eta_1$ then Γ is negative.
- The positions of the maximums and minimums are reversed, but the equations for the maximum and minimum electric field magnitude in terms of $|\Gamma|$ are the same.

$$\left|\vec{E}_{1s}\right|_{\max} = E_{io}\left(1+|\Gamma|\right)$$

$$\left|\vec{E}_{1s}\right|_{\min} = E_{io}\left(1-|\Gamma|\right)$$

Reflection at Normal Incidence (contd.)

- The standing wave ratio (s) in a medium where standing waves exist is defined as the ratio of the maximum electric field magnitude to the minimum electric field magnitude.

$$s = \frac{\left|\vec{E}_{1s}\right|_{\max}}{\left|\vec{E}_{1s}\right|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

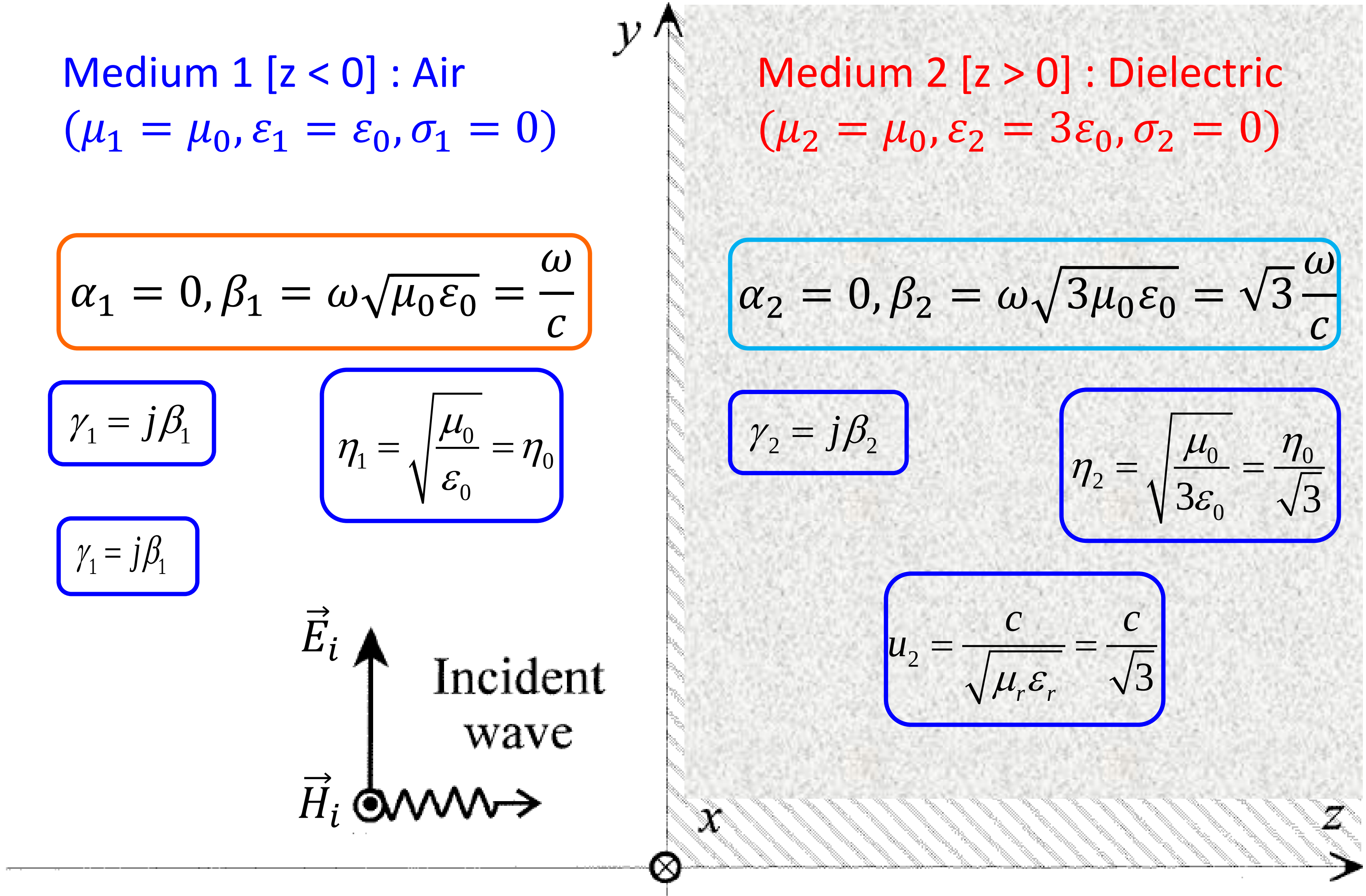
- The standing wave ratio (purely real) ranges from a minimum value of 1 (no reflection, $|\Gamma| = 0$) to ∞ (total reflection, $|\Gamma| = 1$).
- The standing wave ratio is sometimes defined in dB as:

$$s(dB) = 20\log_{10} s$$

Example – 2

- A uniform plane wave in air is normally incident on an infinite lossless dielectric material having $\varepsilon = 3\varepsilon_0$ and $\mu = \mu_0$. If the incident wave is $\vec{E}_{is} = 10\cos(\omega t - z)\hat{a}_y\ V/m$, find (a) ω and λ of the waves in both the mediums, (b) $\vec{H}_{is}$, (c) Γ and τ, (d) the total electric field and time-average power in both mediums.

Example – 2 (contd.)



Medium 1 [z < 0] : Air

$(\mu_1 = \mu_0, \varepsilon_1 = \varepsilon_0, \sigma_1 = 0)$

Medium 2 [z > 0] : Dielectric

$(\mu_2 = \mu_0, \varepsilon_2 = 3\varepsilon_0, \sigma_2 = 0)$

$$\alpha_1 = 0, \beta_1 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c}$$

$$\gamma_1 = j\beta_1$$

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0$$

$$\gamma_1 = j\beta_1$$

$$\alpha_2 = 0, \beta_2 = \omega\sqrt{3\mu_0\varepsilon_0} = \sqrt{3}\frac{\omega}{c}$$

$$\gamma_2 = j\beta_2$$

$$\eta_2 = \sqrt{\frac{\mu_0}{3\varepsilon_0}} = \frac{\eta_0}{\sqrt{3}}$$

$$u_2 = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{3}}$$

Example – 2 (contd.)

(a) $\vec{E}_{is} = 10\cos(\omega t - z)\hat{a}_y \Rightarrow \vec{E}_{is} = 10e^{-jz}\hat{a}_y = E_o e^{-j\beta_1 z}\hat{a}_y \Rightarrow E_{io} = 10 \quad \beta_1 = 1$

$$\beta_1 = \frac{2\pi}{\lambda_1} = \frac{\omega}{u_1} = \frac{\omega}{c} = 1 \text{ rad/m}$$

$$\beta_2 = \frac{2\pi}{\lambda_2} = \frac{\omega}{u_2} = \sqrt{3}\frac{\omega}{c} = \sqrt{3}\beta_1 \quad \text{rad/m}$$

$$\lambda_1 = \frac{2\pi}{\beta_1} = 2\pi = 6.28 \text{ m}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = 3.63 \text{ m}$$

$$\omega = \beta_1 u_1 = \beta_2 u_2 = 3\times 10^8 \text{ rad/sec} \Rightarrow 47.8\ MHz$$

(b) $$\vec{H}_{is} = \frac{E_{io}}{\eta_0} e^{-j\beta_1 z}(-\hat{a}_x) = -\frac{10}{377} e^{-jz}\hat{a}_x = -0.0266 e^{-jz}\hat{a}_x$$

$$\vec{H}_{is} = \mathrm{Re}\left\{-0.0266 e^{-jz}\hat{a}_x\right\} = -0.0266\cos(\omega t - z)\hat{a}_x \text{ A/m}$$

Example – 2 (contd.)

(c)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \Rightarrow \Gamma = \frac{\frac{\eta_0}{\sqrt{3}} - \eta_0}{\frac{\eta_0}{\sqrt{3}} + \eta_0} \Rightarrow \Gamma = -0.268 \qquad \tau = 1 + \Gamma = 0.732$$

(d)

$$\boxed{\vec{E}_{1s} = E_{io}(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z})\hat{a}_y} \Rightarrow \boxed{\vec{E}_1 = [10\cos(\omega t - z) - 2.68\cos(\omega t + z)]\hat{a}_y \quad \text{V/m}}$$

$$\boxed{\vec{E}_{2s} = \tau E_o e^{-j\beta_2 z}\hat{a}_y} \Rightarrow \boxed{\vec{E}_2 = \left[7.32\cos(\omega t - \sqrt{3}z)\right]\hat{a}_y} \ \text{V/m}$$

The time average power density in medium 1 is due to the +z directed incident wave and the –z directed reflected wave. The time-average power density in medium 2 is due to the +z directed transmitted wave.

$$\boxed{\vec{P}_{ave,1} = \frac{\left|\vec{E}_{is}\right|^2}{2\eta_1}\hat{a}_z + \frac{\left|\vec{E}_{rs}\right|^2}{2\eta_1}(-\hat{a}_z)}$$

Oblique Incidence – Introduction

- One can't expect plane waves to be incident normally on a plane in all types of applications.
- Therefore one must consider the general problem of a plane wave propagating along a specified axis that is arbitrarily located relative to a rectangular coordinate system.
- The most general form of a plane wave in a lossless media is given by:

$$\boxed{\vec{E}(r,t) = \vec{E}_o \cos(\vec{\beta}.\bar{r} - \omega t)}$$

Where: $\boxed{\vec{\beta} = \beta_x \hat{a}_x + \beta_y \hat{a}_y + \beta_z \hat{a}_z}$ $\boxed{\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}$ $\boxed{\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2}$

One can deduce Maxwell's equations in the following form:

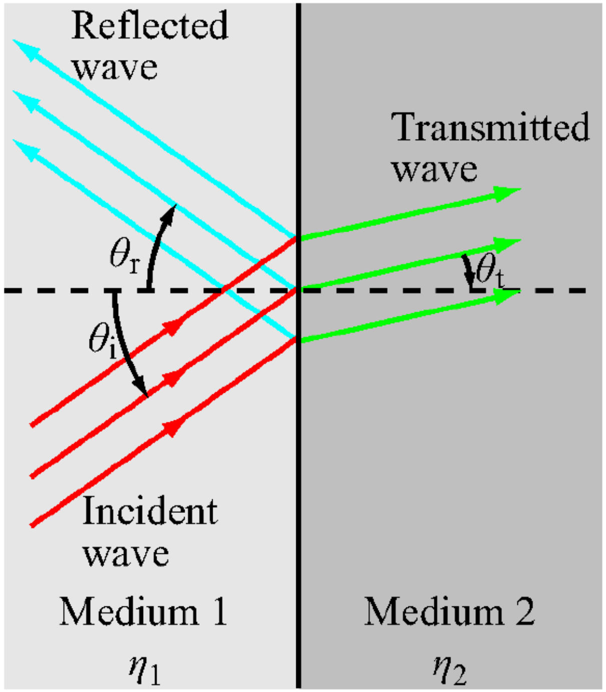
$$\boxed{\vec{\beta} \times \vec{E} = \omega\mu\vec{H}} \quad \boxed{\vec{\beta} \times \vec{H} = -\omega\varepsilon\vec{E}} \quad \boxed{\vec{\beta}.\vec{H} = 0} \quad \boxed{\vec{\beta}.\vec{E} = 0}$$

They show two things: (i) $\vec{E}, \vec{H}$ and $\vec{\beta}$ are orthogonal, (ii) $\vec{E}$ and $\vec{H}$ lie on the same plane

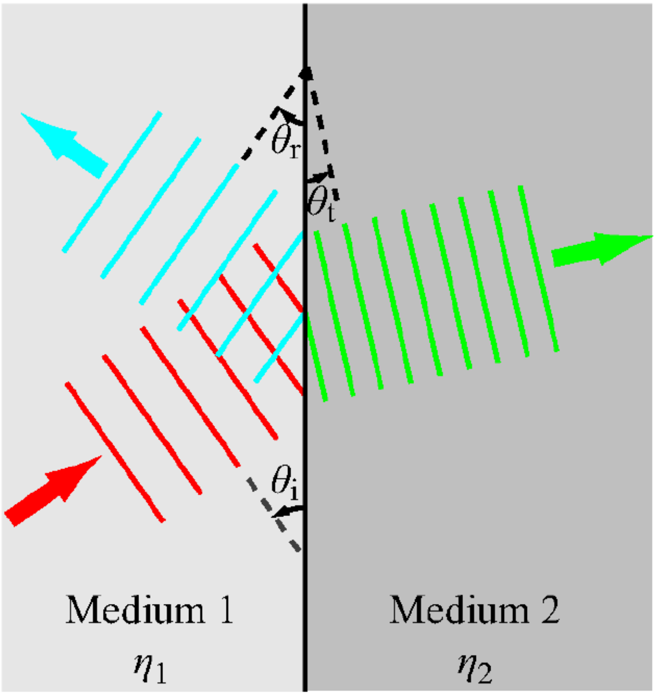
Oblique Incidence (contd.)

- Furthermore: $\boxed{\vec{\beta}.\bar{r} = \beta_x x + \beta_y y + \beta_z z = cons\tan t}$

- The corresponding magnetic field is: $\boxed{\vec{H} = \frac{1}{\omega\mu}\vec{\beta}\times\vec{E} = \frac{\hat{a}_\beta \times \vec{E}}{\eta}}$



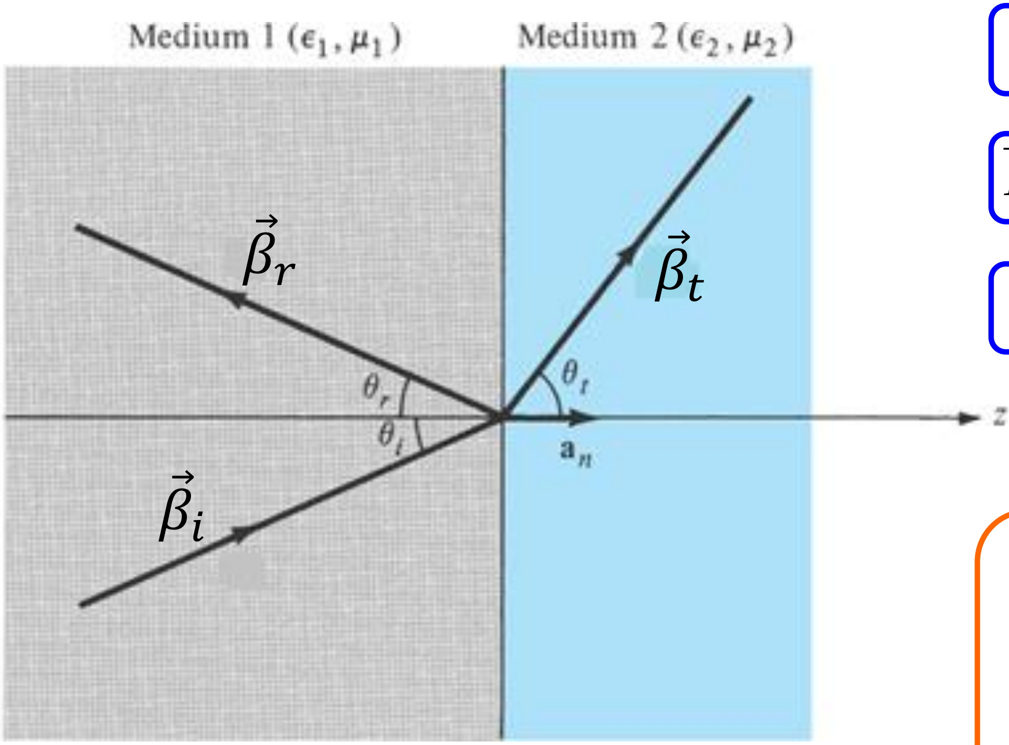
Ray representation of oblique incidence



Wavefront representation of oblique incidence

Reflection at Oblique Incidence

- The plane defined by the propagation vector $\vec{\beta}$ and a unit normal vector $\hat{a}_n$ to the boundary is called the plane of incidence.
- For example, The angle between $\vec{\beta}_i$ and $\hat{a}_n$ is the *angle of incidence*.



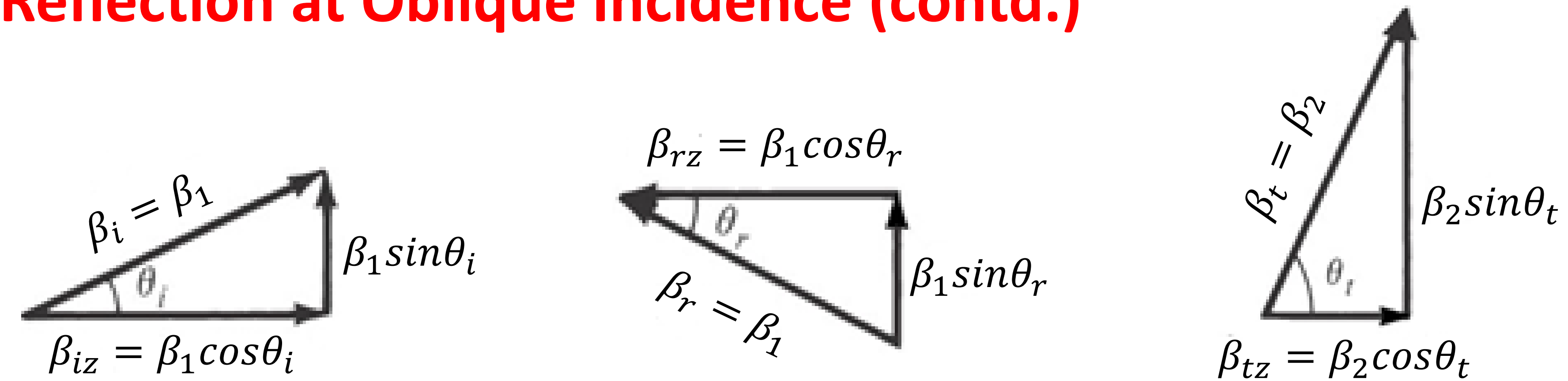
$$\vec{E}_i = \vec{E}_{io} \cos(\beta_{ix}x + \beta_{iy}y + \beta_{iz}z - \omega_i t)$$

$$\vec{E}_r = \vec{E}_{ro} \cos(\beta_{rx}x + \beta_{ry}y + \beta_{rz}z - \omega_r t)$$

$$\vec{E}_t = \vec{E}_{to} \cos(\beta_{tx}x + \beta_{ty}y + \beta_{tz}z - \omega_t t)$$

Where: β_i, β_r and β_t will have normal and tangential components to the plane of incidence.

Reflection at Oblique Incidence (contd.)



- From boundary condition we can write: the tangential component of $\vec{E}$ must be continuous at $z = 0$.

$$\vec{E}_{i\tan}(z=0) + \vec{E}_{r\tan}(z=0) = \vec{E}_{t\tan}(z=0)$$

This boundary condition can be satisfied if:

$$\omega_i = \omega_r = \omega_t = \omega$$

$$\beta_{ix} = \beta_{rx} = \beta_{tx} = \beta_x$$

$$\beta_{iy} = \beta_{ry} = \beta_{ty} = \beta_y$$

First condition implies that the frequency remains unchanged.

Reflection at Oblique Incidence (contd.)

- From second and third conditions we can write:

$$\beta_1 sin\theta_i = \beta_1 sin\theta_r$$

$$\beta_1 sin\theta_i = \beta_2 sin\theta_t$$

Where, θ_r is the *angle of reflection* and θ_t is the *angle of transmission*.

- We know for lossless media: $\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$ $\beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$

$$\theta_i = \theta_r$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{u_2}{u_1} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}$$

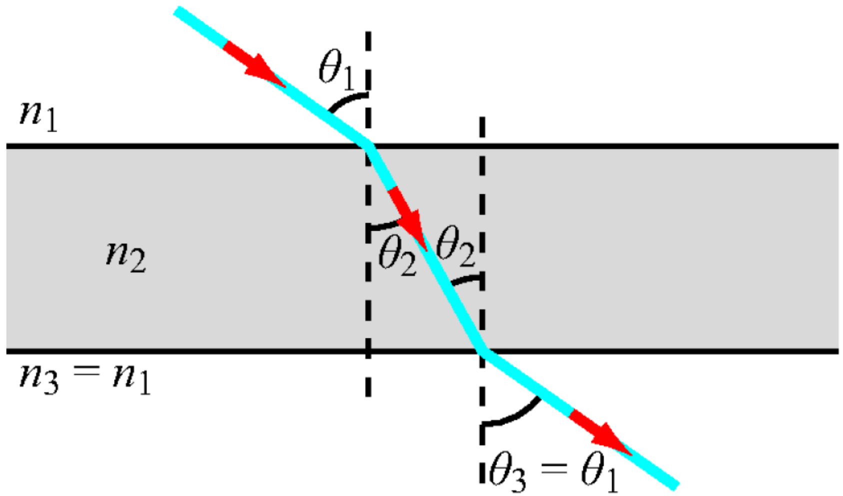
Snell's Law

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

n_1 and n_2 are the refractive indices of the media

Example – 3

A dielectric slab with index of refraction n_2 is surrounded by a medium with index of refraction n_1 as shown. If $\theta_i < \theta_c$, show that the emerging beam is parallel to the incident beam.



At the upper surface:

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$$

Similarly at the lower surface:

$$\sin\theta_3 = \frac{n_2}{n_3}\sin\theta_2 \quad \Rightarrow \quad \sin\theta_3 = \frac{n_2}{n_1}\sin\theta_2$$

$$\boxed{\Rightarrow \sin\theta_3 = \left(\frac{n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right)\sin\theta_1 = \sin\theta_1}$$

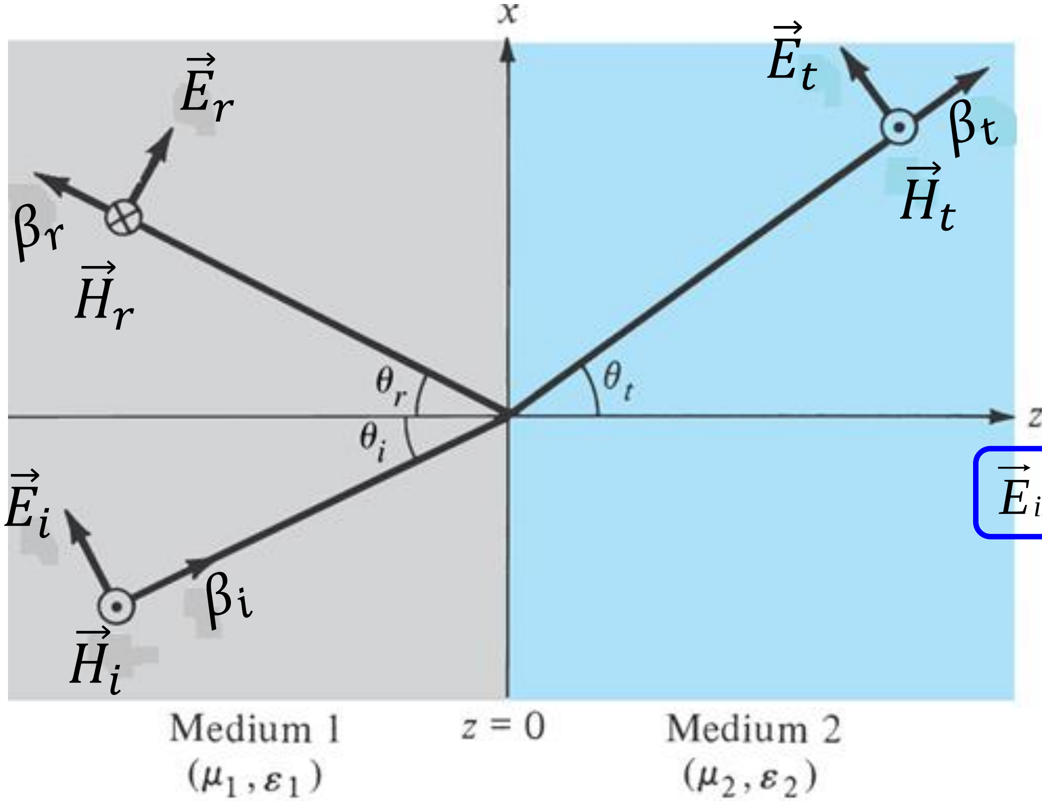
The slab displaces the beam's position but the beam's direction remains unchanged.

Reflection at Oblique Incidence (contd.)

- For normal incidence, the reflection and transmission coefficients Γ and τ at a boundary between two media are independent of the polarization of the incident wave, as both the $\vec{E}$ and $\vec{H}$ of a normally incident plane wave are tangential to the boundary regardless to the wave polarization.
- This is not the case for wave travelling at an angle $\theta_i \neq 0$ with respect to the normal to the interface.
- A wave of arbitrary polarization may be described as the superposition of two orthogonally polarized waves, one with its $\vec{E}$ parallel to the plane of incidence (parallel polarization or transverse magnetic (TM) polarization) and the other with $\vec{E}$ perpendicular to the plane of incidence (perpendicular polarization or transverse electric (TE) polarization).

Parallel Polarization

- Consider this figure: $\vec{E}$ field lies in the xz-plane, the plane of incidence.
- It illustrates the case of "Parallel Polarization".



- In medium 1 the incident waves are:

$$\boxed{\vec{E}_{is} = \vec{E}_{io}(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}}$$

$$\boxed{\vec{H}_{is} = \frac{\vec{E}_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\hat{a}_y}$$

Parallel Polarization (contd.)

- In medium 1 the reflected waves are:

$$\vec{E}_{rs} = \vec{E}_{ro}(\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_{rs} = -\frac{\vec{E}_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\hat{a}_y$$

- The transmitted fields in medium 2 are given by:

$$\vec{E}_{ts} = \vec{E}_{to}(\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_{ts} = \frac{\vec{E}_{to}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\hat{a}_y$$

- We know: $\theta_i = \theta_r$ and tangential components of electric and magnetic fields are continuous at the boundary z=0.
- Therefore:

$$(E_{io} + E_{ro})\cos\theta_i = E_{to}\cos\theta_t$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{1}{\eta_2}E_{to}$$

Parallel Polarization (contd.)

- Simplification gives: $$\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$ $$\tau_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

Fresnel's Equations for parallel polarization

- For $\theta_i = \theta_t = 0$, we get: $$\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma$$ $$\tau_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1} = \tau$$
- Furthermore, the expressions for reflection coefficient and transmission coefficient can be written in terms of *angle of incidence*.

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \left(\frac{u_1^2}{u_2^2}\right)\sin^2\theta_i}$$

- In addition: $$1 + \Gamma_{\parallel} = \tau_{\parallel}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)$$

Parallel Polarization (contd.)

- The reflection coefficient $\Gamma_{||}$ equals zero when there is no reflection (only the parallel component is not reflected), and the *incident angle* at which this happens is called *Brewster's Angle* $\theta_{B||}$.
- The *Brewster's Angle* is also known as *polarizing angle*.
- At this angle, the perpendicular component of $\vec{E}$ will be reflected.
- Brewster's concept is utilized in *laser tube* used in surgical procedures.

- For Brewster's Angle, set $\Gamma_{||} = 0$:

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B||}$$

$$\eta_2^2\left(1-\sin^2\theta_t\right) = \eta_1^2\left(1-\sin^2\theta_{B||}\right)$$

$$\sin^2\theta_{B||} = \frac{1-\dfrac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}{1-\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2}$$

For a lossless and nonmagnetic medium:

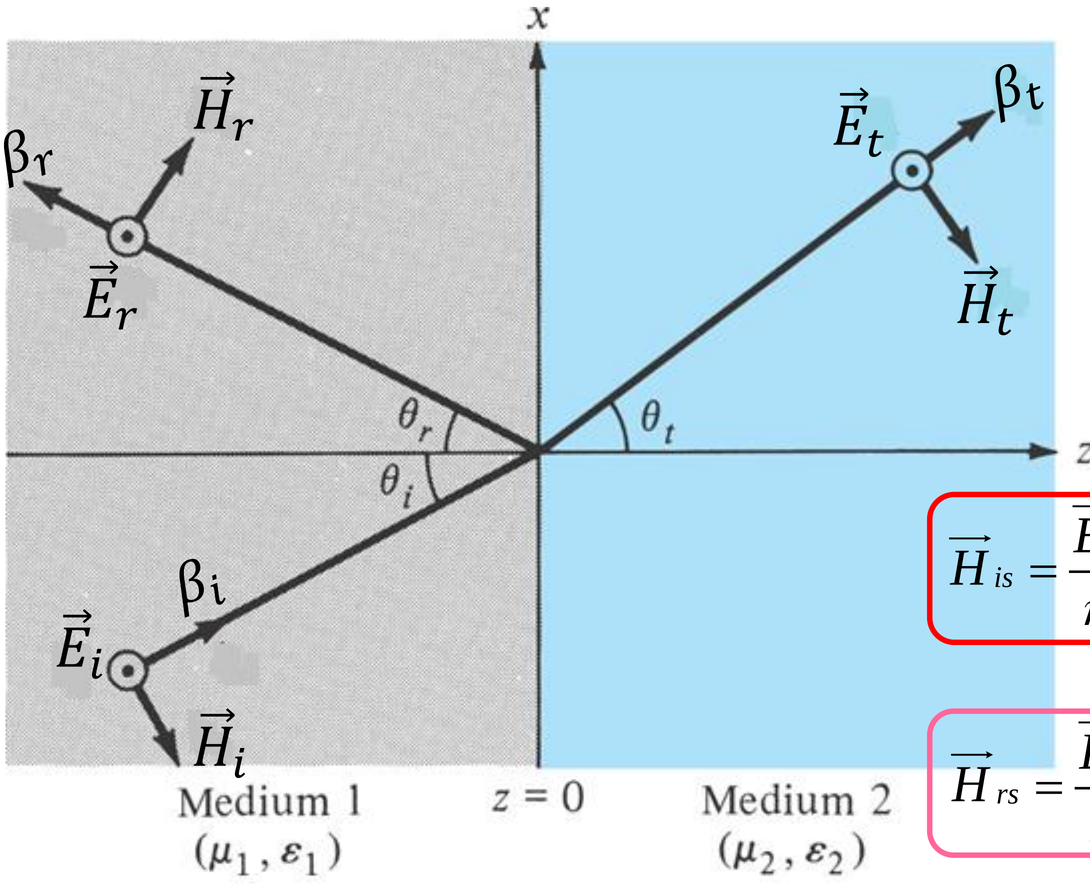
$$\sin^2\theta_{B||} = \frac{1}{1+\dfrac{\varepsilon_1}{\varepsilon_2}}$$

$$\sin\theta_{B||} = \sqrt{\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}}$$

There is a Brewster Angle for any combination of ε_1 and ε_2.

Perpendicular Polarization

- The $\vec{E}$ field is perpendicular to the plane of incidence (the xz-plane).
- In this situation we get "Perpendicular Polarization".
- Here, $\vec{H}$ field is parallel to the plane of incidence.



$$\vec{E}_{is} = \vec{E}_{io} e^{-j\beta_1 (x \sin\theta_i + z\cos\theta_i)} \hat{a}_y$$

$$\vec{E}_{rs} = \vec{E}_{io} e^{-j\beta_1 (x \sin\theta_r - z\cos\theta_r)} \hat{a}_y$$

$$\vec{H}_{is} = \frac{\vec{E}_{io}}{\eta_1} (-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i) e^{-j\beta_1 (x \sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_{rs} = \frac{\vec{E}_{ro}}{\eta_2} (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) e^{-j\beta_1 (x \sin\theta_r - z\cos\theta_r)}$$

Perpendicular Polarization (contd.)

- The transmitted fields in medium 2 are given by:

$$\vec{E}_{ts} = \vec{E}_{to} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \hat{a}_y$$

$$\vec{H}_{ts} = \frac{\vec{E}_{to}}{\eta_2}(-\hat{a}_x \cos\theta_t + \hat{a}_z \sin\theta_t) e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

- Again, $\theta_i = \theta_r$ and tangential components of electric and magnetic fields are continuous at the boundary z=0.
- Therefore:

$$(E_{io} + E_{ro}) = E_{to}$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro})\cos\theta_i = \frac{1}{\eta_2} E_{to} \cos\theta_t$$

- Simplification gives:

$$\underbrace{\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \qquad \tau_\perp = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}}_{\text{Fresnel's Equations for perpendicular polarization}}$$

- For $\theta_i = \theta_t = 0$, we get:

$$\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma \qquad \tau_\perp = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1} = \tau$$

Perpendicular Polarization (contd.)

- Simplification for Brewster's Angle in Perpendicular Polarization gives:

$$\eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t \Rightarrow \eta_2^2\left(1-\sin^2\theta_{B\perp}\right) = \eta_1^2\left(1-\sin^2\theta_t\right)$$

$$\sin^2\theta_{B\perp} = \frac{1-\dfrac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}}{1-\left(\dfrac{\mu_1}{\mu_2}\right)^2}$$

For nonmagnetic media, $\mu_1 = \mu_2 = \mu_0$ and therefore:

$$\sin^2\theta_{B\perp} \to \infty$$

Brewster's Angle doesn't exist as sine of an angle is never greater than unity

- If $\mu_1 \neq \mu_2$ and $\varepsilon_1 = \varepsilon_2$ then:

$$\sin^2\theta_{B\perp} = \frac{1}{1+\dfrac{\mu_1}{\mu_2}} \Rightarrow \sin\theta_{B\perp} = \sqrt{\frac{\mu_2}{\mu_2+\mu_1}}$$

Theoretically possible but rarely occurs in practice

Reflection at Oblique Incidence (contd.)

- The Brewster's Angle is also called Polarizing Angle.
- This is because if a wave composed of both the perpendicular and parallel polarization components is incident on a nonmagnetic surface at the Brewster angle $\theta_{B||}$, the parallel polarized component totally transmitted into the second medium and only the perpendicularly polarized component is reflected by the surface.
- Natural light, including sunlight and light generated by most manufactured sources, is $unpolarized$ because it consists of equal parallel and perpendicular rays. When they are incident upon a surface at the Brewster angle, the reflected wave is strictly perpendicularly polarized. Hence the surface acts as a polarizer.

Example – 4

- A wave in air is incident upon a soil surface at $\theta_i = 50°$. If soil has $\varepsilon_r = 4$ and $\mu_r = 1$, determine the following:

$\Gamma_{\parallel}$ $\tau_{\parallel}$ $\Gamma_{\perp}$ $\tau_{\perp}$ The Brewster angle