Lecture – 23 Date: 04.04.2016

- Plane Waves in Free Space and Good Conductors
- Power and Poynting Vector

Wave Propagation in Lossy Dielectrics

- Wave propagating in z-direction and having only x-component is given by:

$$E_{xs}(z) = E_0^+ e^{-\gamma z} = E_0^+ e^{-(\alpha + j\beta)z}$$

- Where:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

- Inserting the time factor yields:

$$\vec{E}(z,t) = \mathrm{Re}\left\{E_{xs}(z)e^{j\omega t}\hat{a}_x\right\} = \mathrm{Re}\left\{E_0^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{a}_x\right\}$$

- The solution for magnetic field is:

$$\vec{H}(z,t) = \mathrm{Re}\left\{H_0^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{a}_y\right\}$$

- Where:

$$H_0^+ = \frac{E_0^+}{\eta}$$

η is a complex quantity known as the *intrinsic impedance* of the medium.

$$\eta = \eta = |\eta| e^{j\theta_\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$$

$$0 \le \theta_\eta \le 45^\circ$$

Wave Propagation in Lossy Dielectrics (contd.)

- In terms of β, the wave velocity u and wavelength λ are:

$$u = \frac{\omega}{\beta}$$

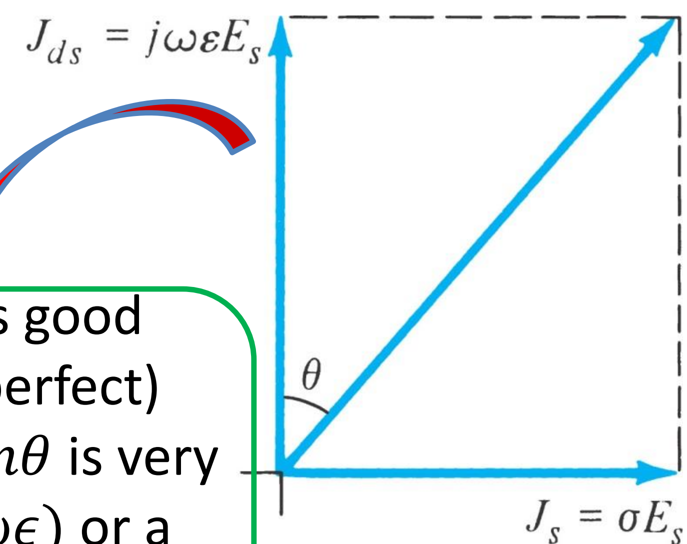
$$\lambda = \frac{2\pi}{\beta}$$

- Furthermore, the ratio of the magnitude of conduction current density $\vec{J}_c$ to that of the displacement current density $\vec{J}_d$ is:

$$\frac{|\vec{J}_{cs}|}{|\vec{J}_{ds}|} = \frac{|\sigma \vec{E}_s|}{|j\omega\varepsilon \vec{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

$tan\theta$ is known as the *loss tangent* and θ is the *loss angle* of the medium.

A medium is good (lossless or perfect) dielectric if $tan\theta$ is very small ($\sigma \ll \omega\epsilon$) or a good conductor if $tan\theta$ is large ($\sigma \gg \omega\epsilon$)



Plane Waves in Lossless Dielectrics

- In a lossless dielectrics, $\sigma \ll \omega\varepsilon$.
- In such a scenario: $\sigma \approx 0,\ \varepsilon = \varepsilon_0\varepsilon_r,\quad \mu = \mu_0\mu_r$.
- Therefore:

$$\alpha = 0 \qquad \beta = \omega\sqrt{\mu\varepsilon} \qquad u = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0^\circ$$

Thus $\vec{E}$ and $\vec{H}$ are in time phase with each other.

Plane Waves in Free Space

- In this case: $\sigma = 0, \ \varepsilon = \varepsilon_0, \quad \mu = \mu_0$.
- Therefore: $\alpha = 0$ $\beta = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c}$ $u = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$ $\lambda = \frac{2\pi}{\omega c}$ $\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}}\angle 0°$
- The fact that EM waves travel in free space with the speed of light is significant.
- It provides evidence that light is the manifestation of an EM wave.
- We have:

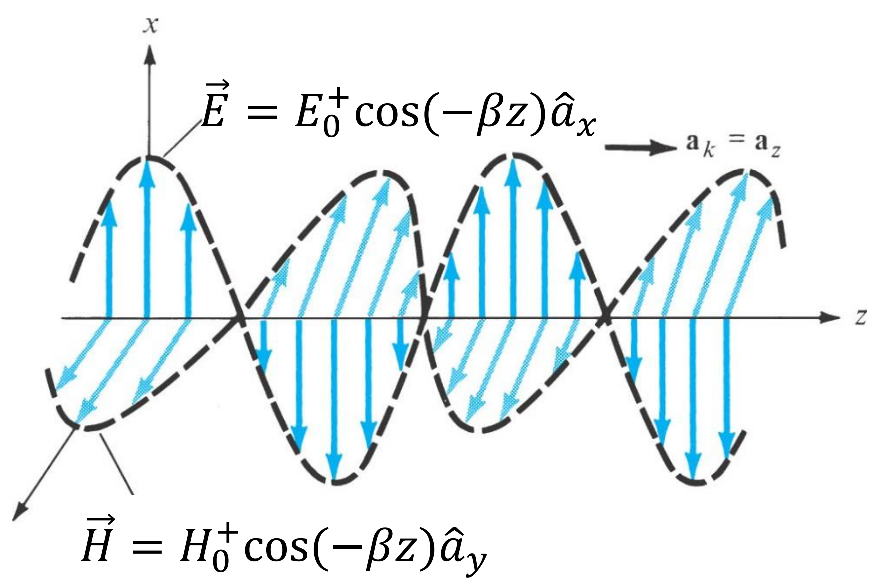
$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \simeq 377\Omega$$

- Furthermore:

$$\vec{E} = E_0^+ \cos(\omega t - \beta z)\hat{a}_x$$

$$\vec{H} = \frac{E_0^+}{\eta_0} \cos(\omega t - \beta z)\hat{a}_y$$

- The plots of $\vec{E}$ and $\vec{H}$ are shown below.



Plane Waves in Free Space (contd.)

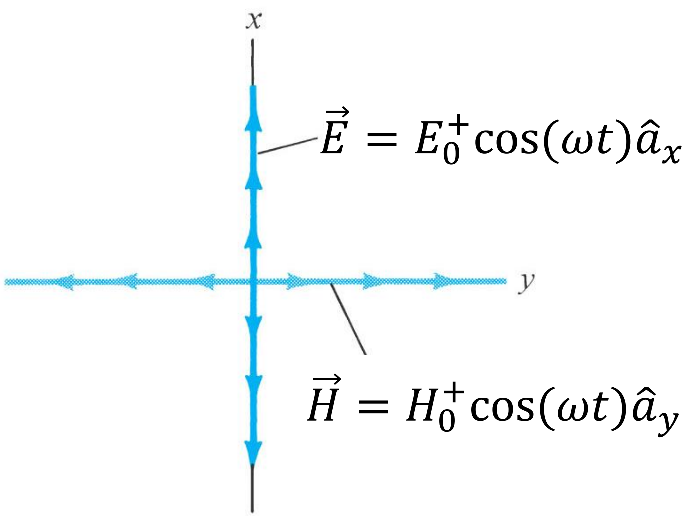
$$\hat{a}_k \times \hat{a}_E = \hat{a}_H \qquad \hat{a}_k \times \hat{a}_H = -\hat{a}_E \qquad \hat{a}_E \times \hat{a}_H = \hat{a}_k$$

- In general, if $\hat{a}_E$, $\hat{a}_H$ and $\hat{a}_k$ are unit vectors along $\vec{E}$, $\vec{H}$ and the direction of propagation, then:
- Both $\vec{E}$ and $\vec{H}$ fields are everywhere normal to the direction of wave propagation.
- It means that the fields lie in a plane that is transverse or orthogonal to the direction of propagation.
- They form an EM wave that has no electric or magnetic field components along the direction of propagation → such a wave is called transverse electromagnetic (TEM) wave.
- A combination of $\vec{E}$ and $\vec{H}$ is called a uniform plane wave because fields have same magnitude throughout any transverse plane.
- The direction in which the electric field points is the **polarization** of a TEM wave → Essentially, polarization of a uniform plane wave describes the locus traced by the tip of the $\vec{E}$ vector (in the plane orthogonal to the direction of propagation) at a given point in space as a function of time.

Plane Waves in Free Space (contd.)

$$\vec{E}(z,t) = E_0^+ e^{-\alpha z} \cos(\omega t - \beta z)\hat{a}_x$$

It is polarized in x-direction



It illustrates a uniform plane wave

- In practice, a uniform plane wave can't exist because it stretches to infinity and would represent an infinite energy → however these waves are characteristically simple and fundamentally important.
- These serve as approximations for practical waves such as those from radio antenna at distances sufficiently far from radiating sources.
- The on-going discussion are applicable for any other isotropic medium.

Plane Waves in Good Conductors

- In a good conductor, displacement current is negligible in comparison to conduction current $(J_{conduction} \gg J_{displacement}) \leftrightarrow$ Because, for a perfect or good conductor, $\sigma \gg \omega\varepsilon$.
- **Although this inequality is frequency dependent,** most good conductors (such as copper and aluminum) have conductivities on the order of $10^7\ mho/m$ and negligible polarization such that we never encounter the frequencies at which the displacement current becomes comparable to the conduction current.
- For a good conductor: $\sigma = \infty,\ \varepsilon = \varepsilon_0,\ \ \mu = \mu_0\mu_r.$
- Therefore: $\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$ → $\alpha = \beta = \sqrt{\pi f \mu\sigma}$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$\lambda = \frac{2\pi}{\beta}$$

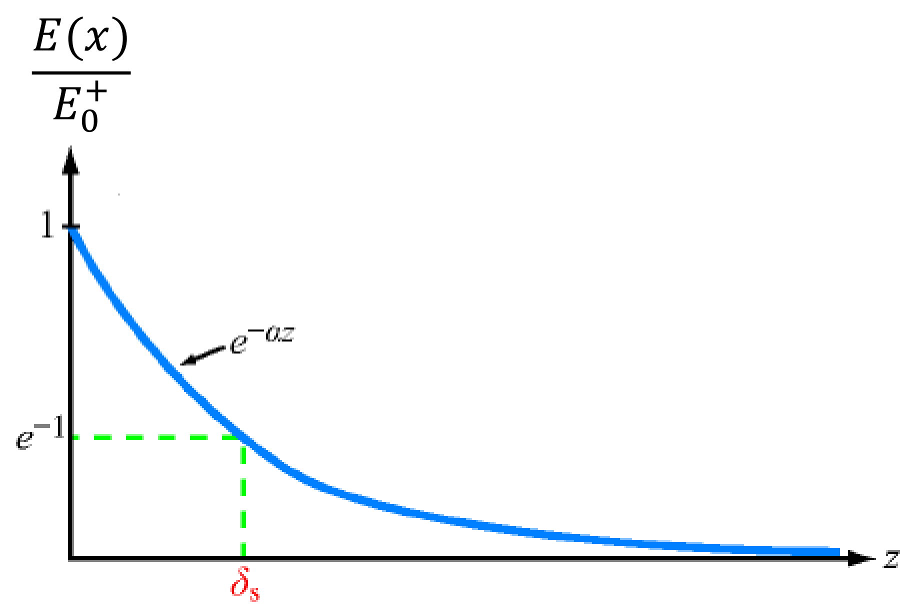
Plane Waves in Good Conductors (contd.)

- Furthermore: $\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45°$

Thus $\vec{E}$ leads $\vec{H}$ by 45°

- If: $\vec{E} = E_0^+ e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_x$
- Then: $\vec{H} = \frac{E_0^+}{\sqrt{\omega\mu/\sigma}} e^{-\alpha z}\cos(\omega t - \beta z - 45°)\hat{a}_y$

- The amplitude of $\vec{E}$ or $\vec{H}$ is attenuated by the factor $e^{-\alpha z}$ as it travels along the medium.
- The rate of attenuation in a good conductor is characterized by distance called *skin depth* (δ) $\leftrightarrow$ a distance over which plane wave is attenuated by a factor e^{-1} (about 37% of the original value) in a good conductor.



Plane Waves in Good Conductors (contd.)

- *skin depth* is a measure of the depth to which an EM wave can penetrate the medium.

$$E_0^+ e^{-\alpha\delta} = E_0^+ e^{-1} \Rightarrow \therefore \delta = \frac{1}{\alpha}$$

Valid for any material medium

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu\sigma}}$$

For a partially conducting medium, the skin depth can be considerably large.

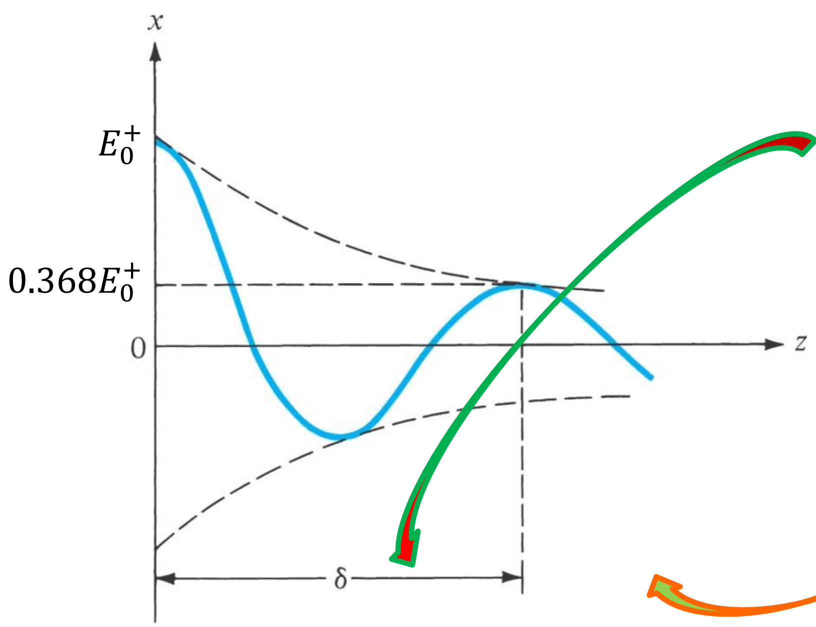
- For a good conductor: $\eta = \frac{1}{\sigma\delta}\sqrt{2}e^{j\pi/4} = \frac{1+j}{\sigma\delta}$

- For good conductors, $\alpha = \beta = \frac{1}{\delta}$, therefore: $\vec{E} = E_0^+ e^{-z/\delta}\cos(\omega t - \frac{z}{\delta})\hat{a}_x$

Plane Waves in Good Conductors (contd.)

$$\vec{E} = E_0^+ e^{-z/\delta} \cos(\omega t - \frac{z}{\delta})\hat{a}_x$$

It shows that *skin depth* (δ) is the measure of exponential damping the wave experiences as it travels through the conductor.



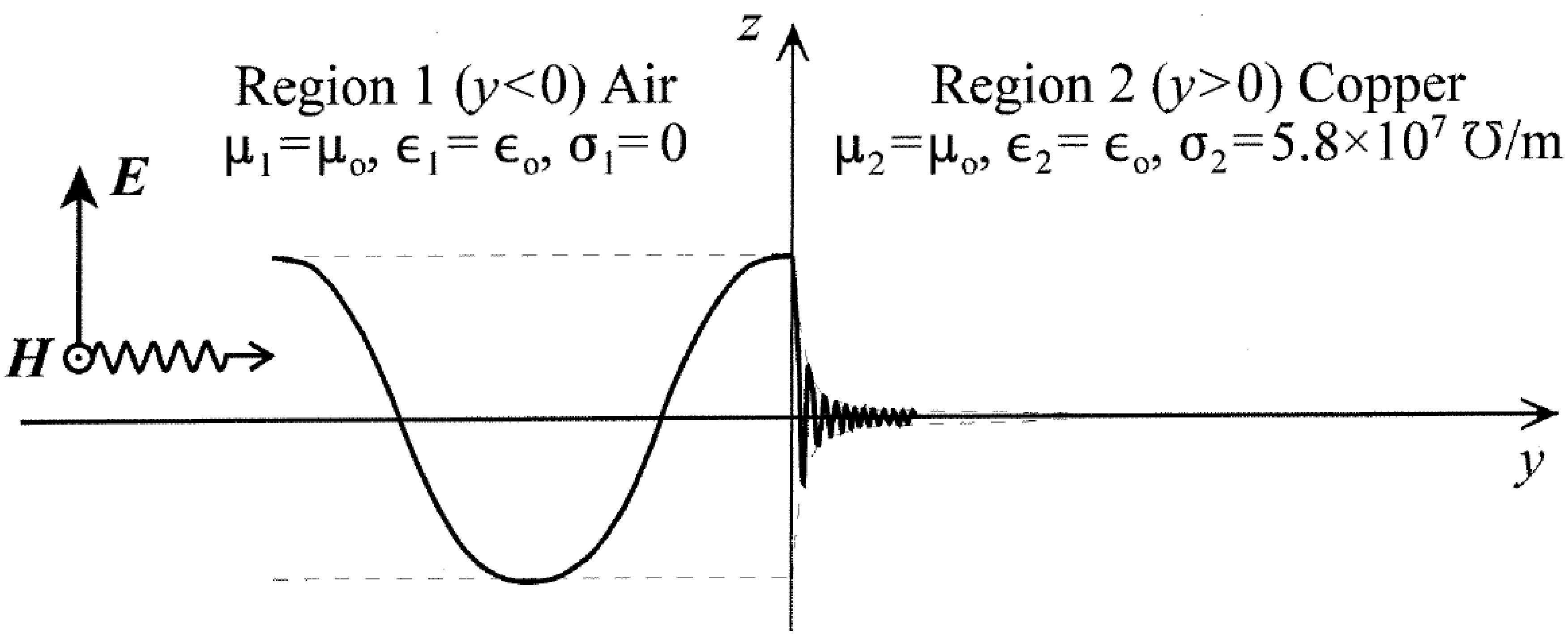
$$\delta = 66.1/\sqrt{f}\,(\text{mm})$$

For Copper !

It demonstrates that the fields dampen and will hardly propagate through good conductors

Example – 1

- Uniform plane wave ($f = 1MHz$) at an air/copper interface.



Determine $\alpha_1, \alpha_2, \beta_1, \beta_2, u_1, u_2, \lambda_1, and\ \lambda_2$.

$$\alpha_1 = 0, \quad \beta_1 = \frac{\omega}{c}$$

$$\alpha_2 = \beta_2 = \frac{1}{\delta}$$

Example – 1 (contd.)

- In the air,

$$c = 3\times10^{8} m/s$$

$$\beta_1 = \frac{\omega}{c} = \frac{2\pi\times10^{6}}{3\times10^{8}} = 0.0209 rad/m$$

$$\lambda_1 = \frac{c}{f} = \frac{3\times10^{8}}{10^{6}} = 300m$$

- In the copper,

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = \frac{1}{\sqrt{\pi f \left(4\pi\times10^{-7}\right)\left(5.8\times10^{7}\right)}} = \frac{0.066}{\sqrt{f}}$$

at 1 MHZ:

$$\delta = 0.066mm$$

$$\alpha_2 = \beta_2 = \frac{1}{\delta} = 15.2\times10^{3} Np/m$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = 2\pi\delta = 0.415mm$$

$$u_2 = \lambda_2 f = 415m/s$$

Example – 2

- A plane wave $\vec{E} = E_0 \cos(\omega t - \beta z)\hat{a}_x$ is incident on a good conductor at $z \geq 0$. Find the current density in the conductor.

- Since, $\vec{J} = \sigma \vec{E}$, the wave equation changes to:

$$\nabla^2 \vec{J}_s - \gamma^2 \vec{J}_s = 0$$

- Furthermore, the incident $\vec{E}$ has only an x-component that varies with z. Therefore, $\vec{J} = J_x(z,t)\hat{a}_x$ and:

$$\frac{d^2}{dz^2} J_{sx} - \gamma^2 J_{sx} = 0$$

The solution is:

$$J_{sx} = Ae^{-\gamma z} + Be^{+\gamma z}$$

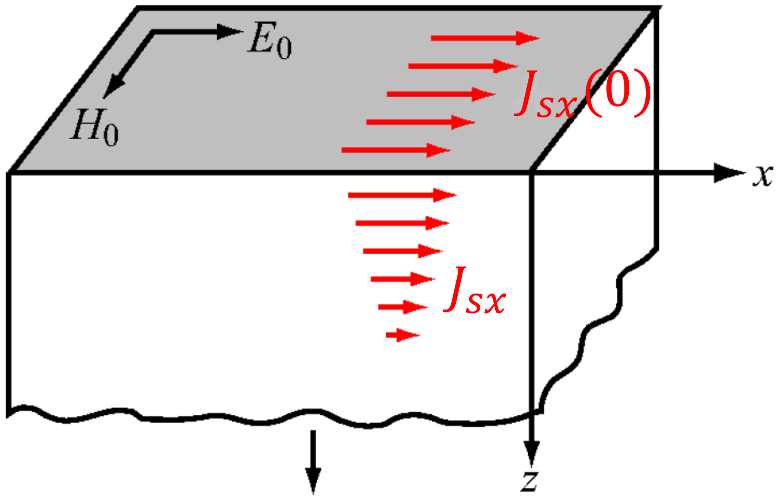
Example – 2 (contd.)

- **B** is zero considering that wave is propagating in $+z\ direction$.
- Furthermore, in a good conductor $\sigma \gg \omega\varepsilon$ so that $\alpha = \beta = \frac{1}{\delta}$. Therefore,

$$\gamma = \alpha + j\beta = \alpha(1+j) = \frac{(1+j)}{\delta}$$

- Therefore: $J_{sx} = Ae^{-z(1+j)/\delta}$ → $J_{sx} = J_{sx}(0)e^{-z(1+j)/\delta}$

Where, $J_{sx}(0)$ is the current density on the conductor surface.



This depicts the scenario

Example – 3

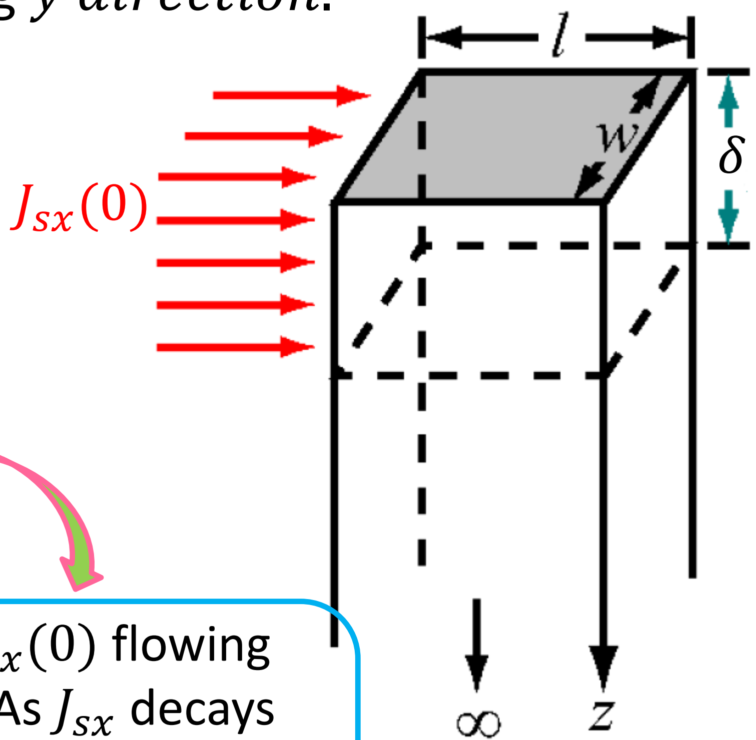
- Given the current density of previous problem $J_{sx} = J_{sx}(0)e^{-z(1+j)/\delta}$, find the magnitude of total current through a strip of the conductor of infinite depth along $z\ direction$ and width w along $y\ direction$.

$$I_s = \int_0^w \int_0^\infty J_{sx}\,dy\,dz$$

$$I_s = J_{sx}(0)\int_0^w dy \int_0^\infty e^{-z(1+j)/\delta}\,dz$$

$$I_s = \frac{J_{sx}(0)w\delta}{1+j}$$

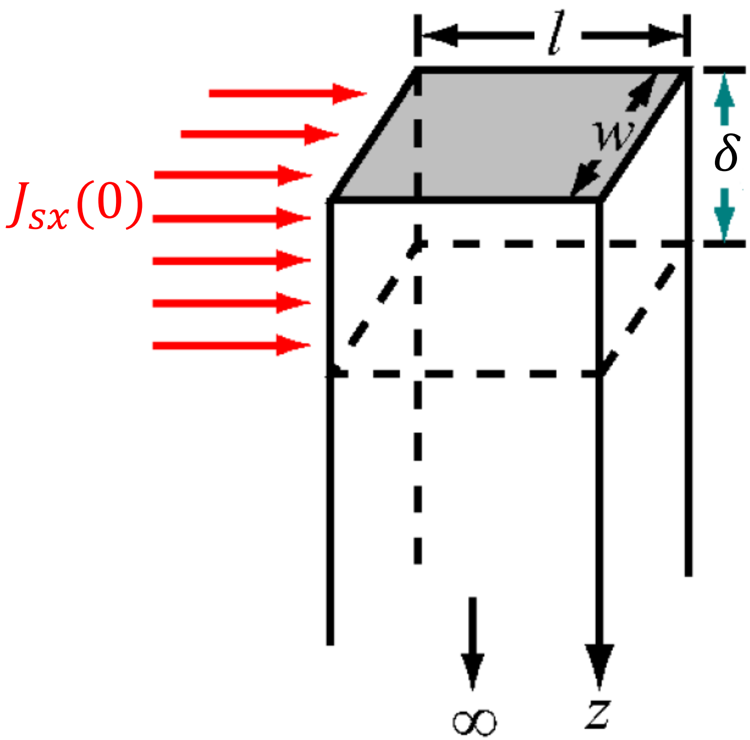
$$|I_s| = \frac{J_{sx}(0)w\delta}{\sqrt{2}}$$



It actually resembles a uniform current density $J_{sx}(0)$ flowing through a thin surface width w and depth δ. → As J_{sx} decays exponentially with depth z, a conductor of finite thickness d can be considered electrically equivalent to one of infinite depth as long as d exceeds a few skin depth (δ).

Example – 4

- In the previous example, what is the voltage across a length l at the surface. What is the impedance of the conductor in consideration?



$$V_s = E_0 l = \frac{J_{sx}(0)}{\sigma} l$$

$$Z = \frac{V_s}{I_s} = \frac{J_{sx}(0)}{\sigma} l \times \frac{1+j}{J_{sx}(0) w \delta}$$

$$Z = \frac{V_s}{I_s} = \frac{1+j}{\sigma\delta} \times \frac{l}{w}$$

$$Z = Z_s \frac{l}{w}$$

Z_s is surface impedance and the real part of this is called ac resistance.

Plane Waves in Good Conductors (contd.)

Electromagnetic Shielding

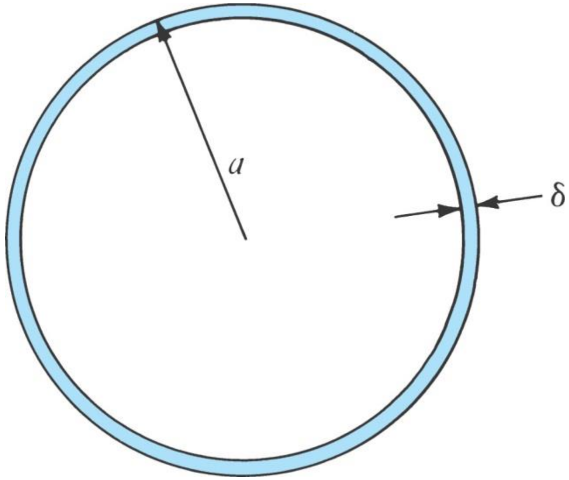
The previous example shows that we may enclose a volume with a thin layer of good conductor to act as an electromagnetic shield. Depending on the application, the electromagnetic shield may be necessary to prevent waves radiating out of the shielded volume or to prevent waves from penetrating into the shielded volume.

Plane Waves in Good Conductors (contd.)

- Given a plane wave incident on a highly-conducting surface, the electric field (and thus the current density) is found to be concentrated at the surface of the conductor.
- The same phenomenon occurs for a current carrying conductor such as a wire.
- The effect is frequency dependent, just as it is in the incident plane wave example.
- This phenomenon is known as the $skin\ effect$.
- Therefore, one can say, The process whereby the field intensity in a conductor rapidly decreases is called $skin\ effect$.
- $skin\ effect$ is the tendency of the charges to migrate from the bulk of the conducting material to the surface, resulting in higher resistances (for ac!)
- The fields and associated currents are confined to a very thin layer $(the\ skin)$ of the conductor surface.

Plane Waves in Good Conductors (contd.)

- For a wire of radius a, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness δ.



- $skin\ effect$ is used to advantage in many applications.
- For example, because the $skin\ depth$ in silver is very small, the difference in performance between a pure silver and silver-plated brass component is negligible, so silver plating is often used to reduce the material cost of waveguide components.
- Furthermore, hollow tubular conductors are used instead of solid conductors in outdoor television antennas.

Plane Waves in Good Conductors (contd.)

- The *skin depth* is useful in calculating the ac resistance.
- The resistance $\left(R = \frac{l}{\sigma S}\right)$ is called the dc resistance R_{dc}.
- The *skin resistance* R_s is the real part of η.

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

Resistance of a unit width and unit length of the conductor having cross-sectional area $1 \times \delta$.

- Therefore, for a given width w and length l, the ac resistance is:

$$R_{ac} = \frac{l}{w\sigma\delta} = \frac{R_s l}{w}$$

- For a conductor wire of radius a:

$$\frac{R_{ac}}{R_{dc}} = \frac{\frac{l}{w\sigma\delta}}{\frac{l}{\sigma S}} = \frac{\frac{l}{\sigma(2\pi a)\delta}}{\frac{l}{\sigma(\pi a^2)}} = \frac{a}{2\delta}$$

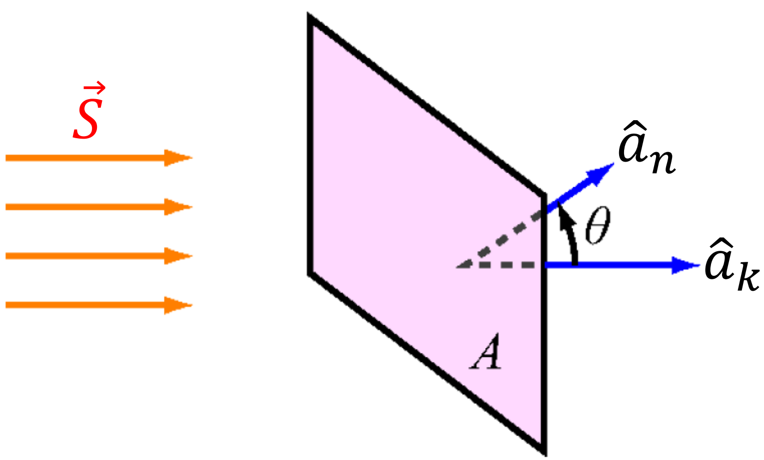
Since, $\delta \ll a$ at high frequencies, R_{ac} is far greater than R_{dc}. In general, the ratio of the ac and dc resistance starts at 1.0 for dc and very low frequencies and increases as the frequency increases.

Power and Poynting Vector

- For any wave with an electric field $\vec{E}$ and magnetic field $\vec{H}$, the direction of wave propagation is also the direction of power per unit area (or power density) carried by the wave. It is represented by $Poynting\ Vector\ \vec{S}$.

$$\vec{S} = \vec{E} \times \vec{H} \qquad W/m^2$$

Instantaneous Poynting Vector – direction and density of power flow at a point



- The total power flowing through this aperture is:

$$P = \int_A \vec{S}.\hat{a}_n dA = SAcos\theta$$

Power and Poynting Vector (contd.)

- Except for the fact that units of $\vec{S}$ are per unit area, the $Poynting\,Vector$ is the vector analogue of the scalar expression for the instantaneous power $P(z,t)$ flowing through a transmission line:

$$P(z,t) = v(z,t)i(z,t)$$

From LC we can recall

$$P_{av}(z) = \frac{1}{2}\mathrm{Re}\left[V_s(z)I_s^*(z)\right]$$

- In a similar manner, power density (W/m^2) associated with a time-harmonic EM field in terms of $\vec{E}$ and $\vec{H}$ phasors is:

$$\vec{P}_{ave} = \frac{1}{2}\mathrm{Re}\left[\vec{E}_s \times \vec{H}_s^*\right]$$

Example – 5

- Determine the expressions for the time-average power density for an EM plane wave in terms of electric field only and magnetic field only; given (a) a lossy medium, (b) a lossless medium.

(a)

$$\vec{P}_{ave} = \frac{1}{2}\text{Re}\left[\vec{E}_s \times \vec{H}_s^*\right] \rightarrow \vec{P}_{ave} = \frac{1}{2}\text{Re}\left[\vec{E}_s \vec{H}_s^* \hat{a}_k\right] \rightarrow \vec{P}_{ave} = \frac{\hat{a}_k}{2}\text{Re}\left[\vec{E}_s \vec{H}_s^*\right]$$

$$\vec{E}_s = \vec{H}_s \eta$$

$$\eta = |\eta| e^{j\theta_\eta} \rightarrow \vec{H}_s^* = \frac{\vec{E}_s^*}{\eta^*} = \frac{\vec{E}_s^*}{|\eta| e^{-j\theta_\eta}}$$

$$\vec{P}_{ave} = \frac{\hat{a}_k}{2}\text{Re}\left[\frac{\vec{E}_s \vec{E}_s^*}{|\eta| e^{-j\theta_\eta}}\right] \rightarrow \vec{P}_{ave} = \frac{\hat{a}_k}{2}\text{Re}\left[\frac{\vec{E}_s \vec{E}_s^*}{|\eta| e^{-j\theta_\eta}}\right] = \frac{\left|\vec{E}_s\right|^2}{2|\eta|}\cos\theta_\eta \hat{a}_k$$

Example – 5 (contd.)

$$\vec{P}_{ave} = \frac{\hat{a}_k}{2}\mathrm{Re}\left[\vec{H}_s|\eta|e^{j\theta_\eta}\vec{H}_s^*\right]$$

$$\Rightarrow$$

$$\vec{P}_{ave} = \frac{\hat{a}_k}{2}\mathrm{Re}\left[\frac{\vec{E}_s\vec{E}_s^*}{|\eta|e^{-j\theta_\eta}}\right] = \frac{|\eta|\left|\vec{H}_s\right|^2}{2}\cos\theta_\eta\hat{a}_k$$

(b) Lossless Medium → η − $real,\ \theta_\eta = 0$

$$\vec{P}_{ave} = \frac{\left|\vec{E}_s\right|^2}{2|\eta|}\hat{a}_k$$

$$\vec{P}_{ave} = \frac{|\eta|\left|\vec{H}_s\right|^2}{2}\hat{a}_k$$