Lecture – 22 Date: 31.03.2016

- Electromagnetic Fields (Contd.)
- Displacement Current
- Maxwell's Equations
- Time Varying Potentials
- Time Harmonic Fields
- Wave Propagation in Lossy Dielectrics

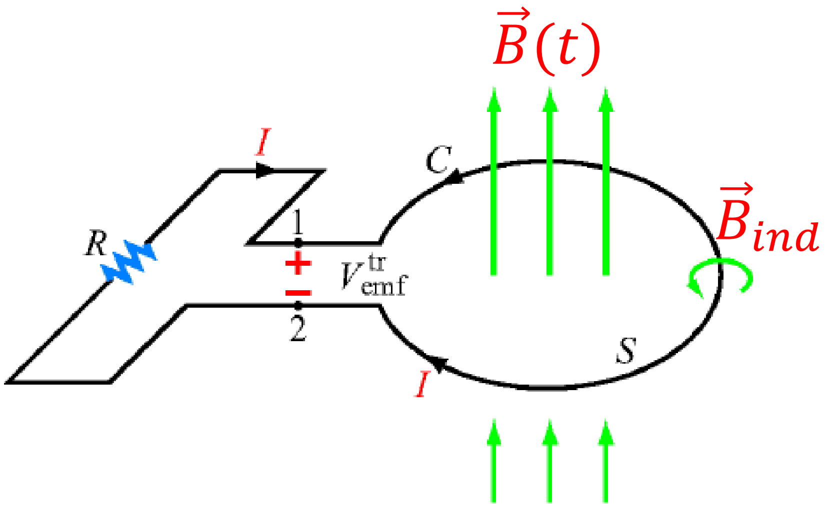
Stationary Loop in Time-Varying $\vec{B}$ (contd.)

Summary:

$$V_{emf}^{tr} = -\int_S \frac{\partial \vec{B}}{\partial t} . \overline{ds}$$

$$V_{emf}^{tr} = \oint_C \vec{E} . \overline{dl}$$

Its assumed that the contour C is closed path ↔ Approximation



$$\oint_C \vec{E} . \overline{dl} = -\int_S \frac{\partial \vec{B}}{\partial t} . \overline{ds} \Rightarrow \int_S (\nabla \times \vec{E}) . \overline{ds} = -\int_S \frac{\partial \vec{B}}{\partial t} . \overline{ds} \Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}(t)}{\partial t}$$

The time varying magnetic field induces an electric field $\vec{E}$ whose curl is equal to the negative of the time derivative of $\vec{B}$.

Moving Conductor in a Static $\overrightarrow{B}$ (contd.)

- In general, if any segment of a closed circuit with contour C moves with a velocity $\vec{u}$ across a static magnetic field $\vec{B}$, then the induced $motional\ emf$ is:

$$V_{emf}^{m} = \oint_{C} \left(\vec{u} \times \vec{B}\right).\overline{dl}$$

Only those segments of the circuit that cross magnetic field lines contribute to $motional\ emf$.

Moving Conductor in a Time-Varying $\vec{B}$

- For a general case of a single turn conducting loop moving in time-varying magnetic field, the induced emf is the sum of a $transformer\ emf$ and $motional\ emf$.

$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t} . \overline{ds} + \oint_C \left(\vec{u} \times \vec{B} \right) . \overline{dl}$$

- induced emf also equals: $V_{emf} = -\frac{d\Psi}{dt} = -\frac{d}{dt} \int_S \vec{B} . \overline{ds}$

Both expressions are equivalent and choice between these two depends on the type of problem.

Displacement Current

- You can recall that the Ampere's law in differential form is given by:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- Integration of the above expression gives:

$$\int_S \left(\nabla \times \vec{H}\right).\overline{ds} = \int_S \vec{J}.\overline{ds} + \int_S \frac{\partial \vec{D}}{\partial t}.\overline{ds}$$

- Simplification gives:

$$\oint_C \vec{H}.\overline{dl} = I_c + \int_S \frac{\partial \vec{D}}{\partial t}.\overline{ds}$$

Conduction Current

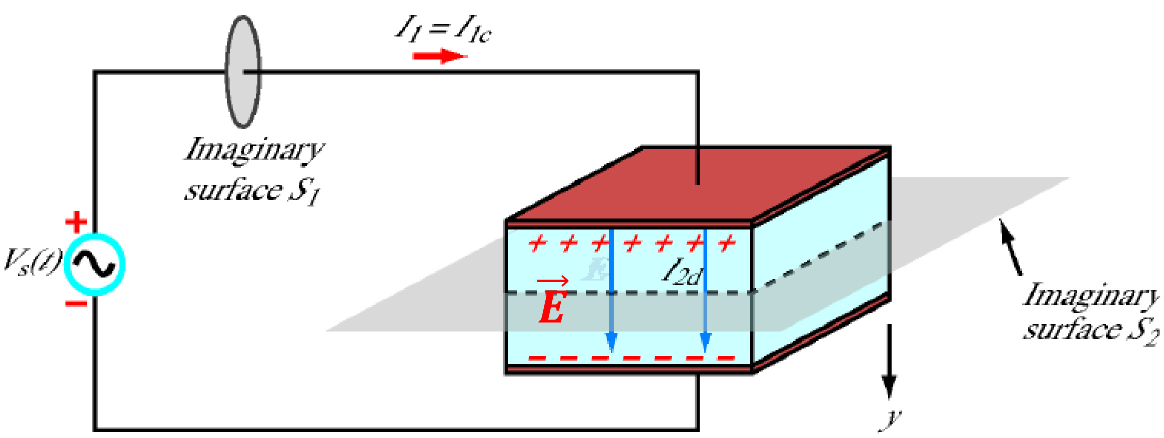
- The second term has the unit of current because it is proportional to the time derivative of the electric flux density $\vec{D}$ called the electric displacement.
- This term is therefore called the *Displacement Current*, I_d.

$$I_d = \int_S \vec{J}_d.\overline{ds} = \int_S \frac{\partial \vec{D}}{\partial t}.\overline{ds}$$

$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is called displacement current density

Displacement Current (contd.)

- Therefore: $\oint_C \vec{H}.\vec{dl} = I_c + I_d = I$ ⟵ I is the total current
- In electrostatics, $\frac{\partial \vec{D}}{\partial t} = 0$ and therefore $I_d = 0$ and $I = I_c$.
- The concept of displacement current was introduced by James Clerk Maxwell when he formulated the unified theory of electricity and magnetism under time-varying conditions.
- Let us consider the following parallel-plate capacitor to understand the physical meaning of *displacement current*.



Let us find I_c and I_d through each of the two imaginary surfaces: (1) cross section of the conducting wire, S_1; (2) cross section of the capacitor, S_2.

Displacement Current (contd.)

- The simple circuit consists of a capacitor and an ac source given by:

$$V_s(t) = V_0 \cos \omega t$$

- We know from Maxwell's hypothesis that the total current flowing through any surface consists, in general, of a conduction current and a displacement current.
- In the perfect conducting wire, $\vec{E} = \vec{D} = 0$; hence, $I_{1d} = 0$.
- As for I_{1c}, we know:

$$I_{1c} = C\frac{dV_c}{dt} = C\frac{d}{dt}\left(V_0 \cos \omega t\right) = -CV_0 \omega \sin \omega t$$

- With no displacement current in the wire, the total current in the wire is:

$$I_1 = I_{1c} = -CV_0 \omega \sin \omega t$$

- Now in the perfect dielectric with permittivity ε between the capacitor plates, $\sigma = 0$.
- Therefore, $I_{2c} = 0$ because no conduction happens.

Displacement Current (contd.)

- To determine I_{2d}, we need to determine $\vec{E}$ in the dielectric spacing:

$$\vec{E} = \hat{a}_y \frac{V_C}{d} = \hat{a}_y \frac{V_0}{d} \cos \omega t$$

d is the spacing between the plates, and $\hat{a}_y$ is the direction from the higher potential plate to the lower potential plate at $t = 0$.

- Therefore displacement current in the dielectric is:

$$I_{2d} = \int_A \frac{\partial \vec{D}}{\partial t} . \overline{ds} \Rightarrow I_{2d} = \int_A \left[\frac{\partial}{\partial t} \left(\hat{a}_y \frac{\varepsilon V_0}{d} \cos \omega t \right) \right] . \left(\hat{a}_y ds \right)$$

$$\therefore I_{2d} = -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t = -C V_0 \omega \sin \omega t$$

- It is apparent that the expression for displacement current in the dielectric is identical to the conduction current in the wire.
- The fact that these two are equal ensures the continuity of the total current flowing through the circuit.

Displacement Current (contd.)

- Even though the displacement current doesn't transport free charges, it nonetheless behaves like a real current.
- Caution, in this example we considered the wire as perfect conductor whereas the dielectric as perfect as well.
- In practice, none of them are perfect and therefore the total current at all the time is sum of conductions and displacement currents.

Example – 1

- The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\epsilon_r = 1$ is given by $I_c = 2sin\omega t\ (mA)$. If $\omega = 10^9 \frac{rad}{s}$, find the displacement current.

Example – 2

- (a) Show that the ratio of the amplitudes of the conduction current density and displacement current density is $\frac{\sigma}{\omega\epsilon}$ for the applied field $E = E_m cos\omega t$, assume $\mu = \mu_0$. (b) What is this amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$.

Maxwell's Equations

- Generalized forms of Maxwell's equations:

Differential Form	Integral Form	Remarks
$\nabla . \vec{D} = \rho_v$	$\oint_S \vec{D}.\overline{ds} = \int_v \rho_v dv$	Gauss's Law
$\nabla . \vec{B} = 0$	$\oint_S \vec{B}.\overline{ds} = 0$	Nonexistence of isolated magnetic charge
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E}.\overline{dl} = -\frac{\partial}{\partial t}\int_S \vec{B}.\overline{ds}$	Faraday's Law
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H}.\overline{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).\overline{ds}$	Ampere's Circuital Law

Maxwell's Equations (contd.)

- Other equations that go hand-in-hand with Maxwell's equations is the Lorentz force equation:

$$\vec{F} = Q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

- Continuity equation is another that is closely associated with Maxwell's equations:

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

- The concept of linearity, isotropy, and homogeneity of a material applies to time-varying fields as well.
- In a linear, homogeneous, and isotropic medium:

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P} \qquad \vec{B} = \mu\vec{H} = \mu_0\left(\vec{H} + \vec{M}\right) \qquad \vec{J} = \sigma\vec{E} + \rho_v\vec{u}$$

- The boundary conditions remain valid for time-varying fields as well.

$$\vec{E}_{1t} - \vec{E}_{2t} = 0 \qquad \left(\vec{E}_1 - \vec{E}_2\right)\times\hat{a}_n = 0 \qquad \vec{H}_{1t} - \vec{H}_{2t} = \vec{K} \qquad \left(\vec{H}_1 - \vec{H}_2\right)\times\hat{a}_n = \vec{K}$$

$$\vec{D}_{1n} - \vec{D}_{2n} = \rho_s \qquad \left(\vec{D}_1 - \vec{D}_2\right).\hat{a}_n = \rho_s \qquad \vec{B}_{1n} - \vec{B}_{2n} = 0 \qquad \left(\vec{B}_1 - \vec{B}_2\right).\hat{a}_n = 0$$

- However, for a perfect conductor in a time-varying field:

$$\vec{E} = 0, \qquad \vec{H} = 0, \qquad \vec{J} = 0 \quad \Rightarrow \quad \vec{B}_n = 0 \qquad \vec{E}_t = 0$$

Example – 3

- Electric field intensity throughout an enclosed region of free space is $E_y = A(sin20x)(sinbz)\{sin(12 \times 10^9 t)\}\frac{V}{m}$. Beginning with the $\nabla \times \vec{E}$ relationship, use Maxwell's equation to find a numerical value for b, assuming $b > 0$.

Time-Varying Potentials

- For the static EM fields, the electric scalar potential was expressed as:

$$V = \int_v \frac{\rho_v dv}{4\pi\varepsilon R}$$

- Whereas, the magnetic vector potential was expressed as:

$$\vec{A} = \int_v \frac{\mu \vec{J} dv}{4\pi R}$$

Let us examine, what happens to these potentials when the field vary with time.

- Recall that, $\vec{A}$ was defined from the fact that $\nabla . \vec{B} = 0$, which still holds for time-varying case. Therefore:

$$\vec{B} = \nabla \times \vec{A}$$

- We know from Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Therefore:

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right) \Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

- We know, that the curl of the gradient of a scalar field is zero: $\nabla \times -\nabla V = 0$, therefore:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \Rightarrow \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Time-Varying Potentials (contd.)

Thus we can determine $\vec{E}$ and $\vec{B}$ provided V and $\vec{A}$ are known.

- However, determination of V and $\vec{A}$ require expressions that are suitable for time varying fields.
- We know that $\nabla . \vec{D} = \rho_v$ is valid for time-varying conditions. We can write:

$$\nabla.\vec{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}\left(\nabla.\vec{A}\right)$$

$$\nabla^2 V + \frac{\partial}{\partial t}\left(\nabla.\vec{A}\right) = -\frac{\rho_v}{\varepsilon}$$

- Furthermore:

$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \nabla \times \vec{A} = \nabla \times \vec{B} = \mu \vec{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} - \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\nabla^2 \vec{A} - \nabla\left(\nabla.\vec{A}\right) = -\mu \vec{J} + \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

Time-Varying Potentials (contd.)

- We know that a vector field is uniquely defined when its curl and divergence are specified. The curl of $\vec{A}$ has been specified as $\vec{B}$, therefore the divergence for $\vec{A}$ can be expressed as:

$$\nabla.\vec{A} = -\mu\varepsilon\frac{\partial V}{\partial t}$$

This expression relates V and $\vec{A}$ and is called *Loretnz condition for potentials.*

- Therefore:

$$\nabla.\vec{A} = -\mu\varepsilon\frac{\partial V}{\partial t}$$

$$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$$

EM Wave Equations

- *Lorentz condition* uncouples and also creates symmetry between V and $\vec{A}$ and therefore aid the analysis of wave equations.
- Actually, V and $\vec{A}$ satisfy *Poisson's equations* for time-varying potentials.

Time-Varying Potentials (contd.)

- From those expressions, it can be deduced that the solutions for V and $\vec{A}$ are:

$$V = \int_v \frac{[\rho_v]dv}{4\pi\varepsilon R}$$

$$\vec{A} = \int_v \frac{\mu[\vec{J}]dv}{4\pi R}$$

Where $[\rho_v]$ and $[\vec{J}]$ are the retarded values. The respective V and $\vec{A}$ are called the *retarded electric scalar potential* and the *retarded magnetic vector potential*.

- It means that the time t in $\rho_v(x, y, z, t)$ or $\vec{J}(x, y, z, t)$ is replaced by retarded time t' given by:

$$t' = t - \frac{R}{u}$$

- Where, $R = |\bar{r} - \bar{r}'|$ is the distance between the source point $\bar{r}'$ and the observation point $\bar{r}$.

- Whereas: $$u = \frac{1}{\sqrt{\varepsilon\mu}}$$

u is the velocity of wave propagation. In free space, $u = c \cong 3 \times 10^8\ m/s$ is the speed of light in vacuum.

Example – 4

- Show that another form of Faraday's law is: $\vec{E} = -\dfrac{\partial \vec{A}}{\partial t}$

where $\vec{A}$ is the magnetic vector potential.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right) = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$\therefore \vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

Example – 5

- Assuming source free region, derive the diffusion equation:

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

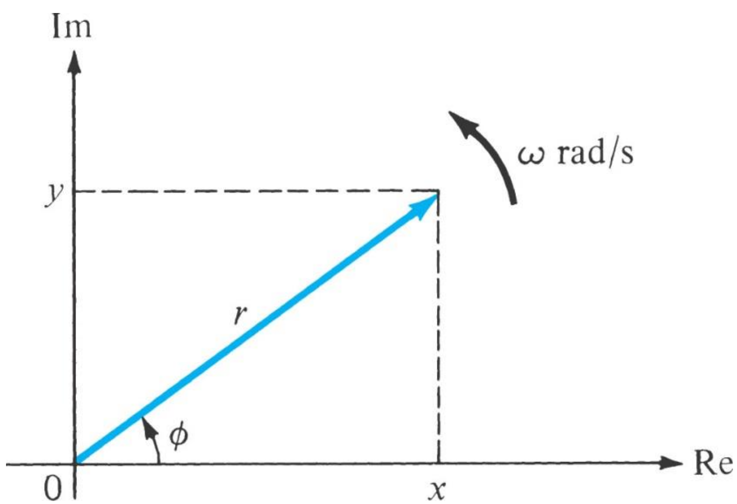
Time – Harmonic Fields

- So far, our time dependence of EM fields have been arbitrary.
- Let us consider the specific scenario where the fields are time-harmonic ↔ Generally, time-varying electric and magnetic fields and their sources (ρ_v and $\vec{J}$) depend on spatial coordinates (x, y, z) and the time variable t. However, if their time variation is sinusoidal with angular frequency ω, then these quantities can be represented by a phasor that depends on (x, y, z) only.
- Time-harmonic field is one that varies periodically or sinusoidally with time → Sinusoidal analysis is of practical value → This can be extended to most waveforms by Fourier analysis.
- Sinusoids are easily expressed in phasors, which are more convenient to work with.
- A phasor is a complex number that contains the amplitude and phase information of a sinusoidal oscillation.

$$z = x + jy = r\angle\phi \quad \rightarrow \quad z = re^{j\phi} = r(\cos\phi + j\sin\phi)$$

Time – Harmonic Fields (contd.)

- The two forms of representing z are illustrated below:



- To introduce the time element, we let:

$$\phi = \omega t + \theta$$

where, θ may be a function of time or space coordinates or constant

$$re^{j\phi} = re^{j\theta} re^{j\omega t}$$

$$\mathrm{Re}(re^{j\phi}) = r\cos(\omega t + \theta)$$

$$\mathrm{Im}(re^{j\phi}) = r\sin(\omega t + \theta)$$

- Thus a sinusoidal current $I(t) = I_0 \cos(\omega t + \theta)$ equals the real part of $I_0 e^{j\theta} e^{j\omega t}$.
- The current $I'(t) = I_0 \sin(\omega t + \theta)$, which is the imaginary part of $I_0 e^{j\theta} e^{j\omega t}$ can also be represented as the real part of $I_0 e^{j\theta} e^{j\omega t} e^{-j90^\circ}$.
- However, **be consistent** while representing the real and imaginary part of a quantity.

Time – Harmonic Fields (contd.)

- The complex term $I_0 e^{j\theta}$ which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called the phasor current, denoted by I_s.

$$I_s = I_0 e^{j\theta} = I_0 \angle \theta$$

- Therefore $I(t) = I_0 \cos(\omega t + \theta)$ can be expressed as:

$$I(t) = \mathrm{Re}\left(I_s e^{j\omega t}\right)$$

- In general, a phasor could be a scalar or a vector.
- If a vector $\vec{A}(x, y, z, t)$ is a time-harmonic field, then the phasor form of $\vec{A}$ is $\vec{A}_s(x, y, z)$; the two quantities are related as:

$$\vec{A} = \mathrm{Re}\left(\vec{A}_s e^{j\omega t}\right)$$

- For example, if $\vec{A} = A_0 \cos(\omega t - \beta x)\, \hat{a}_y$, then we can express $\vec{A}$ as:

$$\vec{A} = \mathrm{Re}\left(A_0 e^{-j\beta x} \hat{a}_y e^{j\omega t}\right) = \mathrm{Re}\left(\vec{A}_s e^{j\omega t}\right)$$

Where: $\vec{A}_s = A_0 e^{-j\beta x} \hat{a}_y$

- Notice that: $$\frac{\partial \vec{A}}{\partial t} \rightarrow j\omega \vec{A}_s$$
- Similarly: $$\int \vec{A} \partial t \rightarrow \frac{\vec{A}_s}{j\omega}$$

Time – Harmonic Fields (contd.)

- Time-Harmonic Maxwell's equations assuming time factor $e^{j\omega t}$

Differential Form	Integral Form
$\nabla . \vec{D}_s = \rho_{vs}$	$\oint_S \vec{D}_s . \overline{ds} = \int_v \rho_{vs} dv$
$\nabla . \vec{B}_s = 0$	$\oint_S \vec{B}_s . \overline{ds} = 0$
$\nabla \times \vec{E}_s = -j\omega \vec{B}_s$	$\oint_L \vec{E}_s . \overline{dl} =- -j\omega \int_S \vec{B}_s . \overline{ds}$
$\nabla \times \vec{H}_s = \vec{J}_s + j\omega \vec{D}_s$	$\oint_L \vec{H}_s . \overline{dl} = \int_S (\vec{J}_s + j\omega \vec{D}_s) . \overline{ds}$

Example – 6

- Given $\vec{A} = 4sin\omega t\hat{a}_x + 3cos\omega t\hat{a}_y$ and $\vec{B}_s = j10ze^{-jz}\hat{a}_x$, express $\vec{A}$ in phasor form and $\vec{B}_s$ in instantaneous form.

$$\vec{A} = 4\cos(\omega t - 90°)\hat{a}_x + 3\cos\omega t\hat{a}_y \Rightarrow \vec{A} = \mathrm{Re}\left[4e^{j(\omega t - 90°)}\hat{a}_x + 3e^{j\omega t}\hat{a}_y\right]$$

$$\therefore \vec{A} = \mathrm{Re}\left[\left(4e^{-j90°}\hat{a}_x + 3\hat{a}_y\right)e^{j\omega t}\right] \Rightarrow \therefore \vec{A}_s = 4e^{-j90°}\hat{a}_x + 3\hat{a}_y = -j4\hat{a}_x + 3\hat{a}_y$$

$$\vec{B}_s = 10ze^{-jz}\hat{a}_x \Rightarrow \vec{B}_s = 10ze^{j90°}e^{-jz}\hat{a}_x$$

$$\therefore \vec{B} = \mathrm{Re}\left[\vec{B}_s e^{j\omega t}\right] = \mathrm{Re}\left[10ze^{j(\omega t - z + 90°)}\hat{a}_x\right]$$

$$\therefore \vec{B} = 10z\cos(\omega t - z + 90°)\hat{a}_x = -10z\sin(\omega t - z)\hat{a}_x$$

Example – 7

- The electric field phasor of an EM wave in free space is:

$$\vec{E}_s(y) = 10e^{-j4y}\hat{a}_x \quad V/m$$

Find (a) ω such that $\vec{E}_s$ satisfies Maxwell's equations.,
(b) the corresponding magnetic field $\vec{H}_s$.

Introduction – EM Wave Propagation

- Let us consider the Maxwell's equations in free space (i.e., $\rho_v = \vec{J} = 0$).

$$\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad \nabla . \vec{E} = 0 \qquad \nabla . \vec{H} = 0$$

- **First equation states that:** If $\vec{E}$ is changing with time at some point, then $\vec{H}$ has curl at that point; therefore $\vec{H}$ varies spatially in a direction normal to its orientation direction.
- Also, if $\vec{E}$ is changing with time, then $\vec{H}$ will in general also change with time, although not necessarily in the same way.
- **Next we see from second equation:** a time varying $\vec{H}$ generates $\vec{E}$, which having curl, varies spatially in the direction normal to its orientation.
- We now once more have a changing $\vec{E}$, our original hypothesis, but this field is present at a small distance away from the point of original disturbance.

Clearly demonstrates the propagation of Electric and Magnetic field and in turn transfer of energy.

Introduction (contd.)

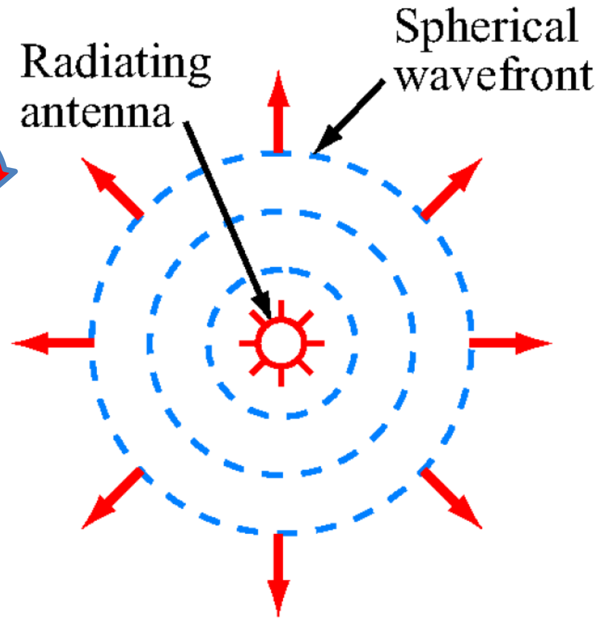
- The velocity with which this effect moves away from the original point is the velocity of light.
- We postulate the existence of $uniform\ plane\ wave$, in which both fields $\vec{E}$ and $\vec{H}$, lie in the transverse plane → that is, the plane whose normal is the direction of propagation.

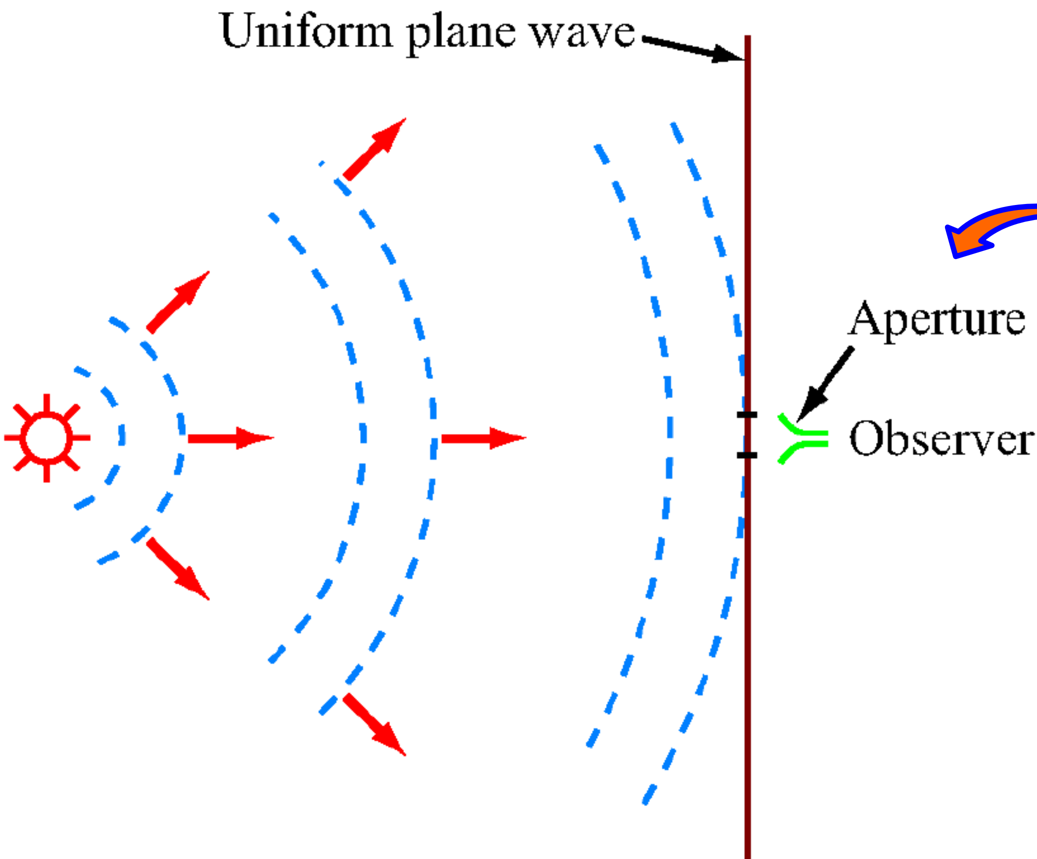
A $uniform\ plane\ wave$ is characterized by electric and magnetic fields that have uniform properties at all points across an infinite plane.

A $plane\ wave$ has no electric or magnetic field components along its direction of propagation

Introduction (contd.)

For example, a wave produced by a localized source, such as an antenna, expands outwardly in the form of spherical wave.

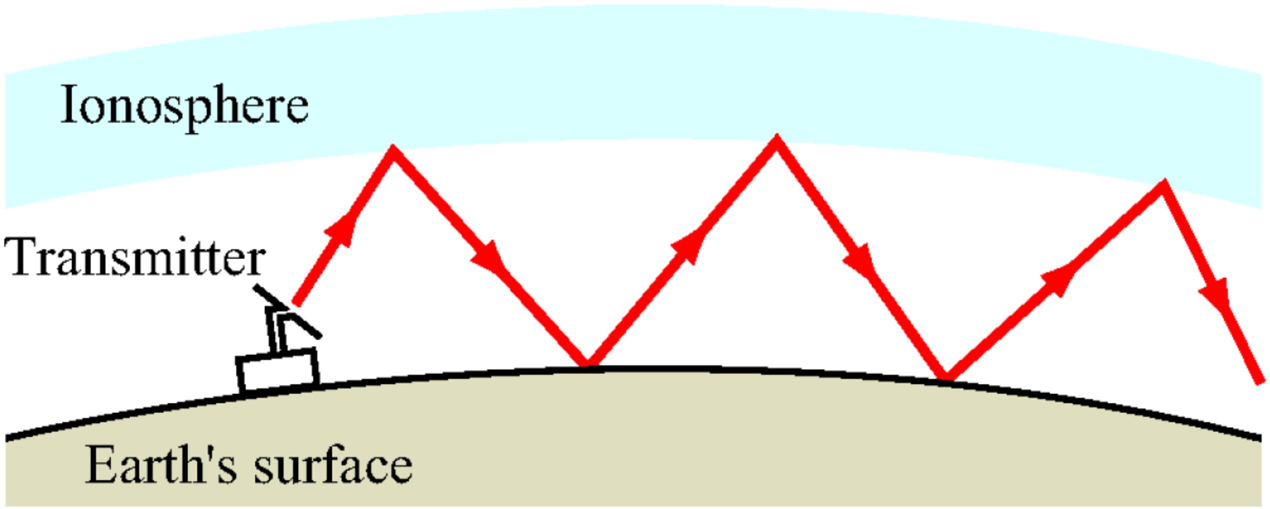




However, it looks a part of a uniform plane wave, with an identical properties at all points in the plane tangent to the wavefront, to an observer very far.

Introduction (contd.)

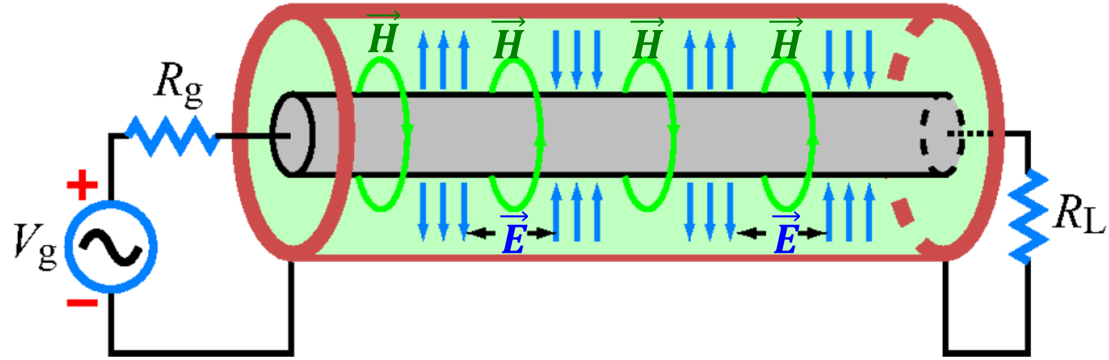
- When a wave propagates through a homogeneous medium without interacting with obstacles or material interfaces, it is called *unbounded* and when a wave propagates along a material structure, it is called guided.
- Earth’s surface and ionosphere constitute parallel boundaries of a natural structure capable of guiding short-wave radio transmission in the HF band (3 to 30MHz).



Indeed, the ionosphere is a good reflector at HF band.

Introduction (contd.)

- Similarly, a transmission line such as coaxial can guide a wave. For example, when an ac source excites an incident wave that travels down the coaxial line toward the load.



- Unless the load is matched to the line, part (or all) of the incident wave is reflected back toward the source.
- At any point on the line, the instantaneous total voltage $v(z,t)$ is the sum of the reflected and incident waves, both of which vary sinusoidally with time.
- Associated with the voltage difference between the inner and outer conductors is a radial electric field $\vec{E}(z,t)$ that exists in the dielectric material. $\vec{E}(z,t)$ is also sinusoidal as $v(z,t)$ varies sinusoidally.
- Furthermore, the current flowing through the inner conductor induces an azimuthal magnetic field $\vec{H}(z,t)$.
- The coupled $\vec{E}(z,t)$ & $\vec{H}(z,t)$ constitute an EM field and models the wave propagation on a transmission line.
- So, propagation can be talked in terms of $v(z,t)$ & $i(z,t)$ or $\vec{E}(z,t)$ & $\vec{H}(z,t)$.

Wave Propagation in Lossy Dielectrics

- Let us develop formulations for wave propagation in lossy dielectrics – it provides the general case of wave propagation.
- A lossy dielectric is a medium in which an EM wave, as it propagates, loses power owing to imperfect dielectric.
- In other words, a lossy dielectric is partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from perfect dielectric in which $\sigma = 0$.
- The Maxwell's equations in a linear, isotropic, homogeneous, lossy dielectric medium that is charge free is given by:

$$\nabla \times \vec{H}_s = (\sigma + j\omega\varepsilon)\vec{E}_s$$

$$\nabla \times \vec{E}_s = -j\omega\mu\vec{H}_s$$

$$\nabla . \vec{E}_s = 0$$

$$\nabla . \vec{H}_s = 0$$

The time factor $e^{j\omega t}$ has been suppressed in above expressions.

Wave Propagation in Lossy Dielectrics (contd.)

$$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$$

Take curl on both sides

$$\nabla \times \nabla \times \vec{E}_s = -j\omega\mu\left(\nabla \times \vec{H}_s\right)$$

$$\nabla\left(\nabla . \vec{E}_s\right) - \nabla^2 \vec{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}_s$$

=0

$$\Rightarrow \nabla^2 \vec{E}_s - j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}_s = 0$$

$$\nabla^2 \vec{E}_s - \gamma^2 \vec{E}_s = 0$$

Where, $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

γ is called the propagation constant

- We can similarly find expression for magnetic field: $\nabla^2 \vec{H}_s - \gamma^2 \vec{H}_s = 0$
- These expressions are called vector Helmholtz's equations.
- In cartesian coordinates, for example, **each of these two vector equations** are equivalent to three scalar wave equations → one for each components of $\vec{E}_s$ or $\vec{H}_s$ along $\hat{a}_x$, $\hat{a}_y$, and $\hat{a}_z$.

Wave Propagation in Lossy Dielectrics (contd.)

$$\frac{\partial^2 E_{sx}}{\partial x^2} + \frac{\partial^2 E_{sx}}{\partial y^2} + \frac{\partial^2 E_{sx}}{\partial z^2} = \gamma^2 E_{sx}$$

$$\frac{\partial^2 H_{sx}}{\partial x^2} + \frac{\partial^2 H_{sx}}{\partial y^2} + \frac{\partial^2 H_{sx}}{\partial z^2} = \gamma^2 H_{sx}$$

$$\frac{\partial^2 E_{sy}}{\partial x^2} + \frac{\partial^2 E_{sy}}{\partial y^2} + \frac{\partial^2 E_{sy}}{\partial z^2} = \gamma^2 E_{sy}$$

$$\frac{\partial^2 H_{sy}}{\partial x^2} + \frac{\partial^2 H_{sy}}{\partial y^2} + \frac{\partial^2 H_{sy}}{\partial z^2} = \gamma^2 H_{sy}$$

$$\frac{\partial^2 E_{sz}}{\partial x^2} + \frac{\partial^2 E_{sz}}{\partial y^2} + \frac{\partial^2 E_{sz}}{\partial z^2} = \gamma^2 E_{sz}$$

$$\frac{\partial^2 H_{sz}}{\partial x^2} + \frac{\partial^2 H_{sz}}{\partial y^2} + \frac{\partial^2 H_{sz}}{\partial z^2} = \gamma^2 H_{sz}$$

The component fields of any time-harmonic EM wave must individually satisfy these six partial differential equations. In many cases, the EM wave will not contain all six components. An example of this is the $plane\ wave$.

Wave Propagation in Lossy Dielectrics (contd.)

- If we assume that the wave propagates along $+\hat{a}_z$ and that $\vec{E}_s$ has only an x-component, then:

$$\vec{E}_s = E_{xs}(z)\hat{a}_x$$

- Substitution of this into Helmholtz equation results in:

$$\left(\nabla^2 - \gamma^2\right)E_{xs}(z) = 0$$

- Therefore:

$$\frac{\partial^2 E_{xs}(z)}{\partial x^2} + \frac{\partial^2 E_{xs}(z)}{\partial y^2} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

=0

- Hence:

$$\left[\frac{d^2}{dz^2} - \gamma^2\right]E_{xs}(z) = 0$$

Scalar wave equation

It is a linear homogeneous differential equation whose solution is:

$$E_{xs}(z) = E_0^+ e^{-\gamma z} + E_0^- e^{\gamma z}$$

- Where, the first component is the wave propagating in $+z$ *direction* and the second term is the wave propagating in $-z$ *direction*.
- We assumed, wave only propagating in $+z$ *direction*. Therefore, $E_0^- = 0$.

Wave Propagation in Lossy Dielectrics (contd.)

- Since γ is a complex quantity, we can express it as: $\gamma = \alpha + j\beta$ ➔ $\gamma^2 = \alpha^2 - \beta^2 + 2j\alpha\beta = j\omega\mu(\sigma + j\omega\varepsilon)$
- Simplification gives: $\text{Re}\,\gamma^2 = \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$ ➔ $\beta^2 - \alpha^2 = \omega^2\mu\varepsilon$
- Furthermore: $|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2}$
- From the above two expressions we can obtain:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

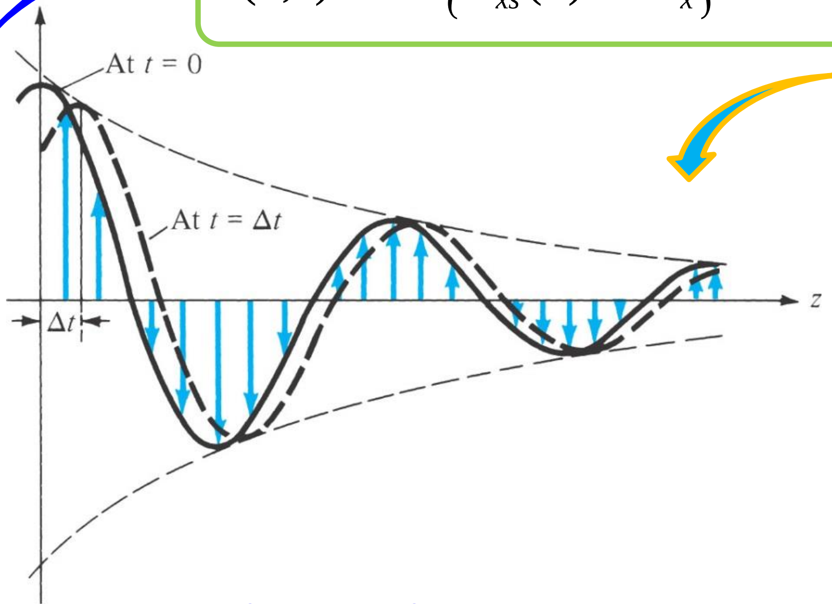
$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

- Therefore the simplified solution of wave equation is: $E_{xs}(z) = E_0^+ e^{-\gamma z} = E_0^+ e^{-(\alpha + j\beta)z}$
- Inserting the time factor in the solution yields:

$$\vec{E}(z,t) = \text{Re}\left\{E_{xs}(z)e^{j\omega t}\hat{a}_x\right\} = \text{Re}\left\{E_0^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{a}_x\right\}$$

Wave Propagation in Lossy Dielectrics (contd.)

$$\vec{E}(z,t) = \mathrm{Re}\left\{E_{xs}(z)e^{j\omega t}\hat{a}_x\right\} = \mathrm{Re}\left\{E_0^+ e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{a}_x\right\}$$



An electric field with an x-component traveling in +z direction at $t = 0$ and $t = \Delta t$; arrows indicate instantaneous values of Electric Field.

- It is apparent that as the wave propagates along $+\hat{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, and therefore α is known as the attenuation constant or attenuation coefficient of the medium → It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter → For free space, $\sigma = 0$ and therefore $\alpha = 0$ → the wave doesn't attenuate in free space.
- The quantity β is a measure of phase shift per unit length in radians per meter and is called the phase constant or wave number.

Wave Propagation in Lossy Dielectrics (contd.)

- The solution for magnetic field is: $\vec{H}(z,t) = \text{Re}\left\{H_0^+ e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a}_y\right\}$
- Where: $H_0^+ = \dfrac{E_0^+}{\eta}$

η is a complex quantity known as the *intrinsic impedance* of the medium.

$$\eta = \eta = |\eta| e^{j\theta_\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Derive it !

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$$

$$0 \le \theta_\eta \le 45^\circ$$

- Therefore the magnetic field expression is:

$$\vec{H}(z,t) = \text{Re}\left\{\frac{E_0}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a}_y\right\} \Rightarrow \vec{H}(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \hat{a}_y$$

It is evident that $\vec{E}$ and $\vec{H}$ are out of phase by θ_η.

Wave Propagation in Lossy Dielectrics (contd.)

- In terms of β, the wave velocity u and wavelength λ are:

$$u = \frac{\omega}{\beta}$$

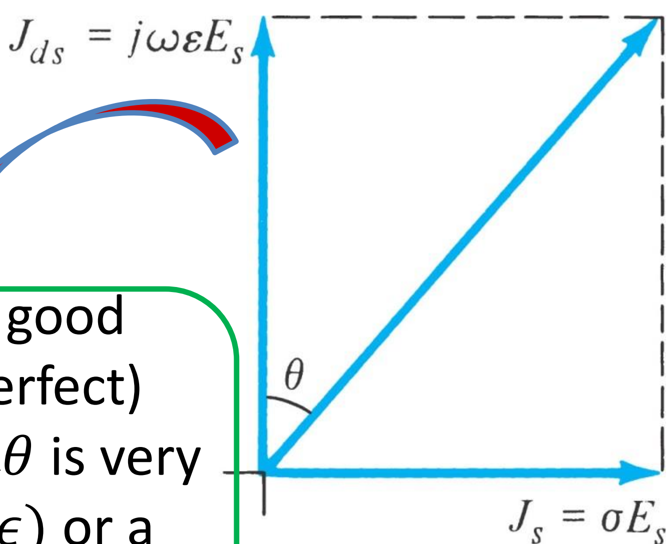
$$\lambda = \frac{2\pi}{\beta}$$

- Furthermore, the ratio of the magnitude of conduction current density $\vec{J}_c$ to that of the displacement current density $\vec{J}_d$ is:

$$\frac{|\vec{J}_{cs}|}{|\vec{J}_{ds}|} = \frac{|\sigma \vec{E}_s|}{|j\omega\varepsilon \vec{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

$tan\theta$ is known as the *loss tangent* and θ is the *loss angle* of the medium.

A medium is good (lossless or perfect) dielectric if $tan\theta$ is very small ($\sigma \ll \omega\epsilon$) or a good conductor if $tan\theta$ is large ($\sigma \gg \omega\epsilon$)



Wave Propagation in Lossy Dielectrics (contd.)

- In general, for propagation of wave, characteristics of any medium doesn't only depend on the parameters $\sigma, \epsilon, and\ \mu$ but also on frequency of operation.
- A medium that is regarded as good conductor at low frequency may be a good dielectric at high frequencies.
- We have:
  - From definition of intrinsic impedance: $\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$
  - From definition of loss tangent: $\frac{\sigma}{\omega\varepsilon} = \tan\theta$
  - Therefore: $\theta = 2\theta_\eta$
- Furthermore: $\nabla \times \vec{H}_s = (\sigma + j\omega\varepsilon)\vec{E}_s$ → $\nabla \times \vec{H}_s = j\omega\varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]\vec{E}_s$

$$\varepsilon_c = \varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]$$

$$\Rightarrow \nabla \times \vec{H}_s = j\omega\varepsilon_c\vec{E}_s$$

Wave Propagation in Lossy Dielectrics (contd.)

$$\varepsilon_c = \varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]$$

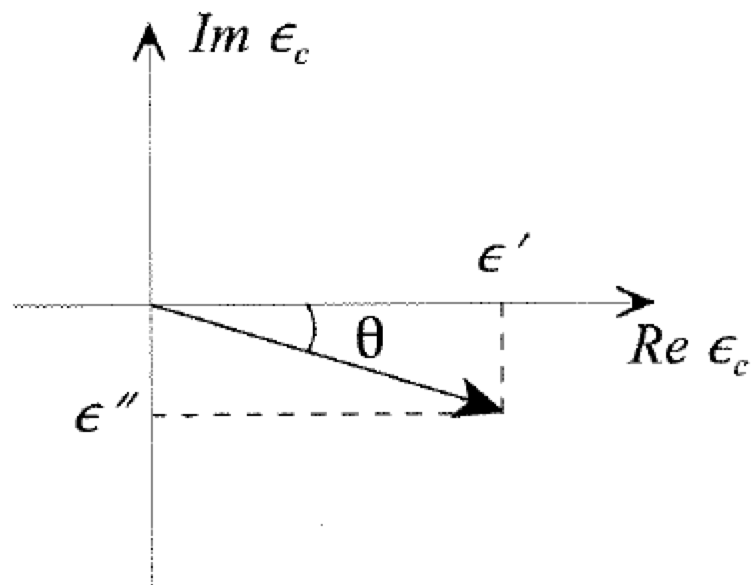
$$\varepsilon_c = \varepsilon' - j\varepsilon''$$

$$\varepsilon' = \varepsilon \qquad \varepsilon'' = \frac{\sigma}{\omega}$$

ε_c is called the complex permittivity of the medium.

- The loss tangent is:

$$\tan\theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}$$



Example – 8

- If the magnetic field phasor of a plane wave traveling in a medium with intrinsic impedance $\eta = 100\Omega$ is given by $\vec{H}_s = \left(10\hat{a}_y + 20\hat{a}_z\right)e^{-j4x}\,\frac{mA}{m}$. Find the associated electric field phasor.

- It is clear that the wave travels in $x - direction$.
- Therefore:

$$\vec{E}_s = -\eta(\hat{a}_x \times \vec{H}_s)$$

$$\vec{E}_s = -100\left[\hat{a}_x \times (10\hat{a}_y + 20\hat{a}_z)\right]e^{-j4x} \times 10^{-3}$$

$$\therefore\ \vec{E}_s = \left(-\hat{a}_z + 2\hat{a}_y\right)e^{-j4x}\,\frac{V}{m}$$

Example – 9

- In the previous example, determine the electric field if the magnetic field is given by $\vec{H}_s = \hat{a}_y\left(10e^{-j3x} - 20e^{j3x}\right)\frac{mA}{m}$.

- This magnetic field is composed of two components, one with amplitude of 10 mA/m belonging to a wave traveling along $+\hat{a}_x$ and another with amplitude of 20 mA/m belonging to a separate wave traveling in the opposite direction $-\hat{a}_x$. Hence, we need to treat these two components separately.

$$\vec{H}_s = \vec{H}_{1s} + \vec{H}_{2s} = \hat{a}_y 10e^{-j3x}\frac{mA}{m} - \hat{a}_y\, 20e^{j3x}\frac{mA}{m}$$

- Then use: $\vec{E}_s = -\eta(\hat{a}_x \times \vec{H}_s)$

$$\therefore\ \vec{E}_s = \hat{a}_z\left(e^{-j3x} + 2e^{j3x}\right)\frac{V}{m}$$