Lecture – 21 **Date: 28.03.2016**

- Example
- Magnetic Boundary Conditions
- Inductance (Self and Mutual)
- Electromagnetic Field

Example – 1

- Consider an **infinite cylinder** made of **magnetic** material. This cylinder is centered along the z-axis, has a **radius of $2m$**, and a **permeability** of $4\mu_0$.

  Inside the cylinder there exists a **magnetic flux density**:

$$\vec{B} = \frac{8\mu_0}{\rho}\hat{a}_\phi \qquad (\rho \leq 1)$$

  Determine the **magnetization current** $\vec{K}_b$ flowing **on the surface** of this cylinder, as well as the magnetization current $\vec{J}_b$ flowing **within the volume** of this cylinder.

Magnetic Boundary Conditions

- Consider the **interface** between two **different materials** with dissimilar **permeabilities**:

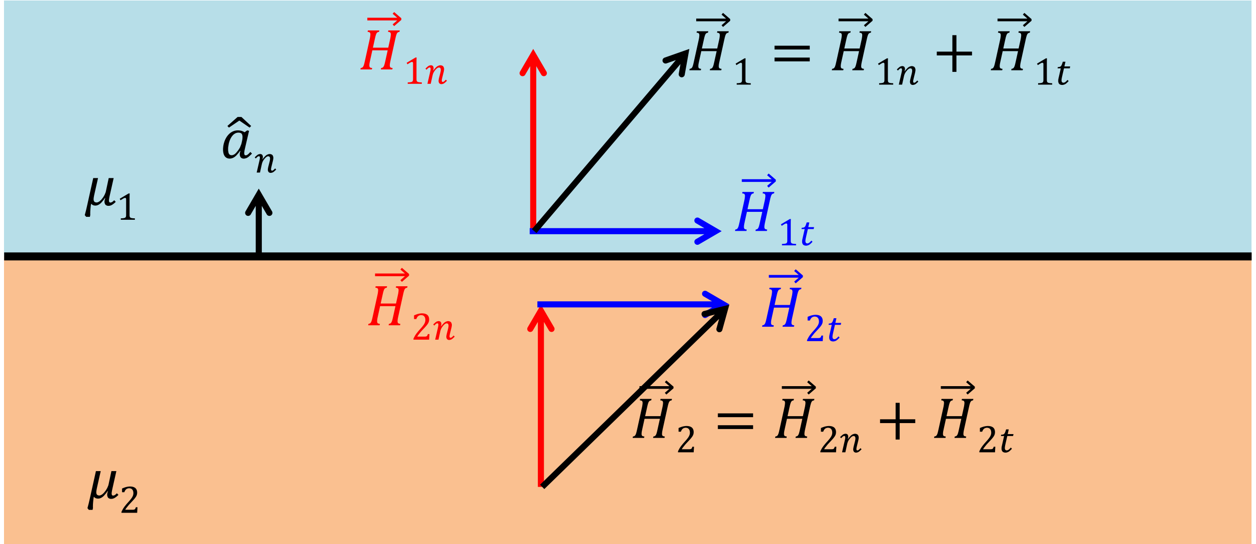
Say that a magnetic field and a magnetic flux density is present in **both** regions.

Q: How are the fields in **region 1** (i.e., $\vec{H}_1$ and $\vec{B}_1$) related to the fields in **region 2** (i.e., $\vec{H}_2$ and $\vec{B}_2$)

A: They must satisfy the **magnetic boundary conditions** !

Magnetic Boundary Conditions (contd.)

- First, let's write the fields **at the interface** in terms of their **normal** $\vec{H}_n$ and **tangential** $\vec{H}_t$ vector components:



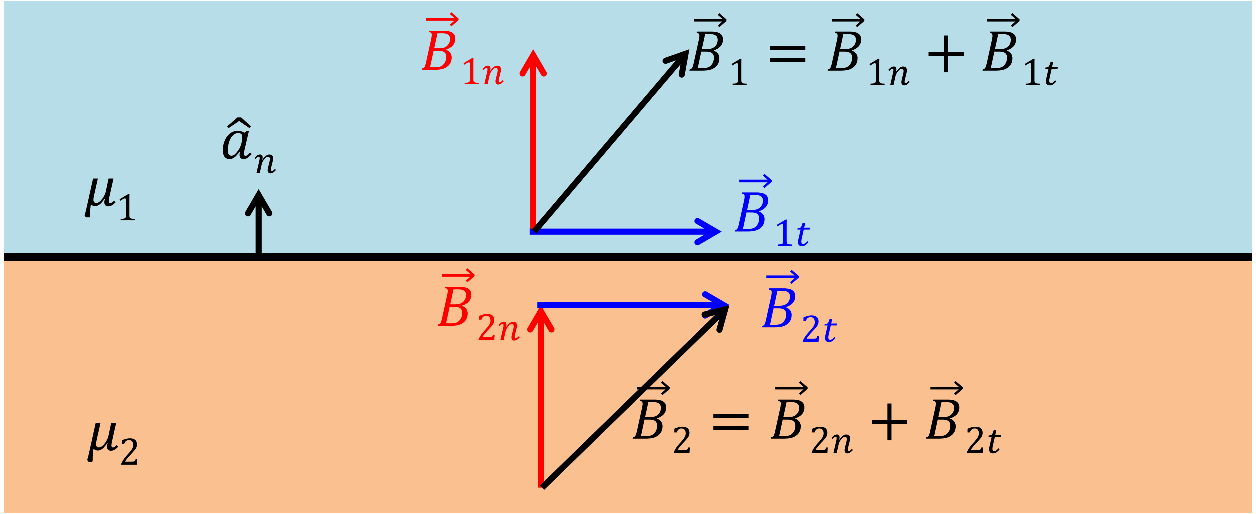
- Our first boundary condition states that the **tangential** component of the magnetic field is **continuous** across a boundary. In other words:

$$\vec{H}_{1t} = \vec{H}_{2t}$$

The **tangential** component of the magnetic field on **one** side of the material boundary is **equal** to the tangential component on the **other** side !

Magnetic Boundary Conditions (contd.)

- Furthermore: $\frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2}$ → Tangential component of magnetic flux density is discontinuous
- **Interface having bound surface charge density $\vec{K}_b$ will follow:** $\vec{H}_{1t} - \vec{H}_{2t} = \vec{K}_b$
- We can likewise consider the **magnetic flux densities** on the material interface in terms of their **normal** and **tangential** components:

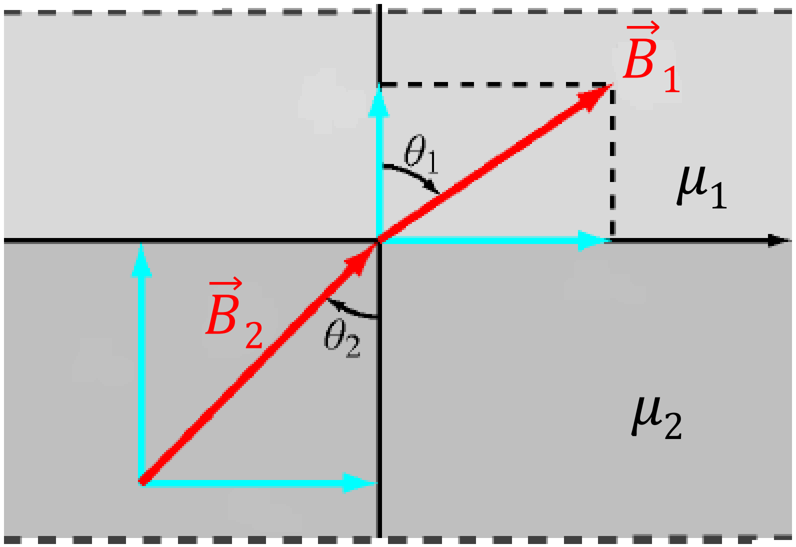


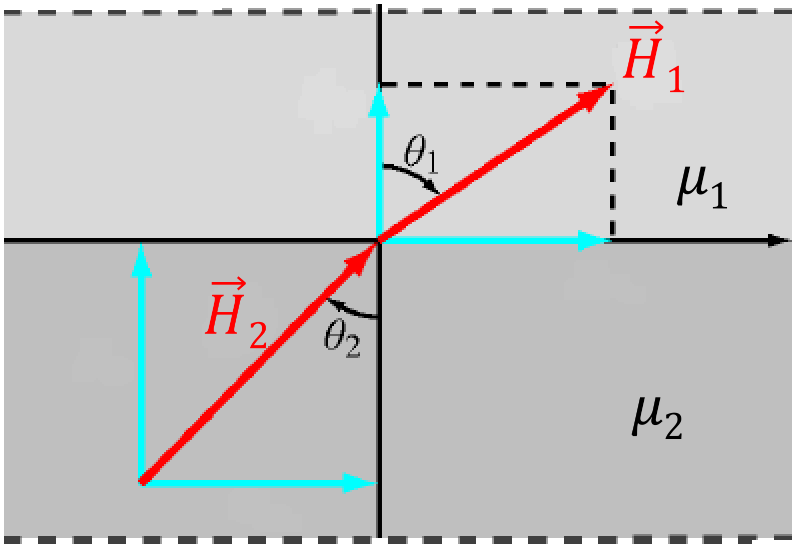
- **Furthermore:** $\mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$

- The second magnetic boundary condition states that the **normal** vector component of the **magnetic flux density** is **continuous** across the material boundary. In other words: $\vec{B}_{1n} = \vec{B}_{2n}$

Magnetic Boundary Conditions (contd.)

- If the fields make angle θ with the normal to the interface then:





$$B_1\cos\theta_1 = \vec{B}_{1n} = \vec{B}_{2n} = B_2\cos\theta_2$$

$$\frac{B_1}{\mu_1}\sin\theta_1 = \vec{H}_{1t} = \vec{H}_{2t} = \frac{B_2}{\mu_2}\sin\theta_2$$

- Simplification gives: $$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\mu_1}{\mu_2}$$

Law of refraction for magnetic flux lines at a boundary with no surface current

Example – 2

- Given that $\vec{H}_1 = -2\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z \; A/m$ in region $y - x - 2 \leq 0$, where $\mu_1 = 5\mu_0$, calculate:

  (a) $\vec{M}_1$ and $\vec{B}_1$

  (b) $\vec{H}_2$ and $\vec{B}_2$ in region $y - x - 2 \geq 0$, where $\mu_2 = 2\mu_0$

Example – 3

Region -1, described by $3x + 4y \geq 10$, is free space, whereas region-2, described by $3x + 4y \leq 10$, is a magnetic material for which $\mu = 10\mu_0$. Assuming that the boundary between the material and free space is current free, find $\vec{B}_2$ if $\vec{B}_1 = 0.1\hat{a}_x + 0.4\hat{a}_y + 0.2\hat{a}_z\ Wb/m^2$.

Example – 4

The interface $4x - 5z = 0$ between two magnetic media carries current $35\hat{a}_y\ A/m$. If $\vec{H}_1 = 25\hat{a}_x - 30\hat{a}_y + 45\hat{a}_z\ A/m$ in region $4x - 5z \leq 0$ where $\mu_{r1} = 5$, calculate $\vec{H}_1$ in the region $4x - 5z \geq 0$ where $\mu_{r2} = 10$.

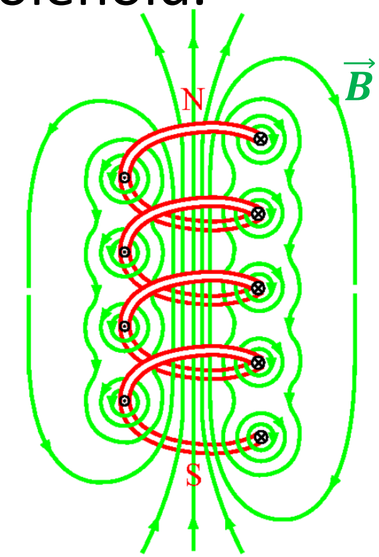
Inductors

- Generally - coil of conducting wire
  - Usually wrapped around a solid core. If no core is used, then the inductor is said to have an 'air core'.

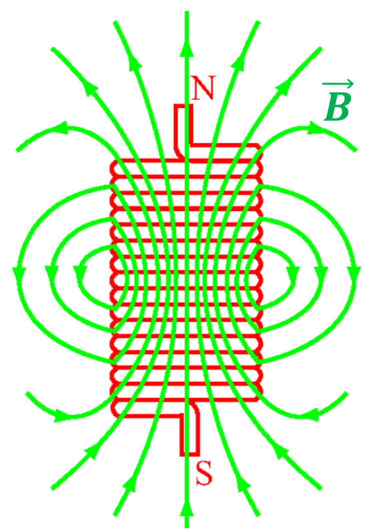
- An inductor is the magnetic analogue of an electric capacitor.
- Just as a capacitor can store energy in the electric field in the medium between its conducting surfaces, an inductor can store energy in the magnetic field near its current carrying conductors.

Inductors (contd.)

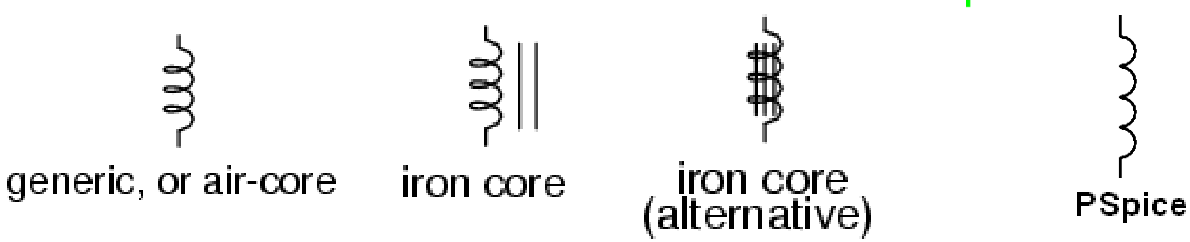
- A typical inductor consists of multiple turns of wire helically coiled around a cylindrical core called a solenoid.
- Core may be air-filled or may contain a magnetic material with permeability μ.
- If the turns are closely spaced, the solenoid will create better inductor.



The magnetic field lines resembles those of the permanent magnet



- Symbols

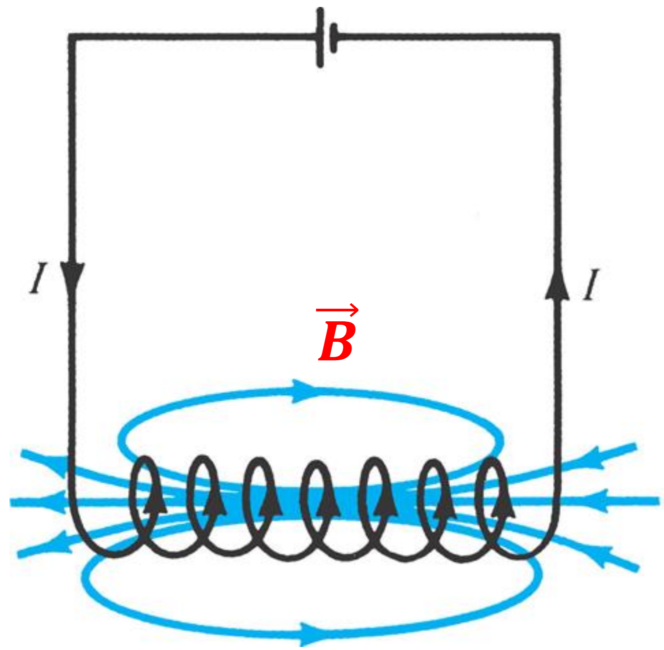


Inductors (contd.)

- In such architecture, the current is proportional to the flux linkage:

$$\lambda = LI$$

where, L is a constant of proportionality called the **inductance** of the circuit.



- A circuit (or closed conducting path) carrying current I produces a magnetic field $\vec{B}$ that causes a flux $\psi = \int \vec{B}.\overline{ds}$ to pass through each turn of the circuit. Circuit with N identical turns has flux linkage of:

$$\lambda = N\psi$$

- Inductance, L, is then defined as the ratio of the magnetic flux linkage λ to the current I through the inductor as:

$$L = \frac{\lambda}{I} = \frac{N\psi}{I}$$

The unit is Henry (i.e, Wb/A).

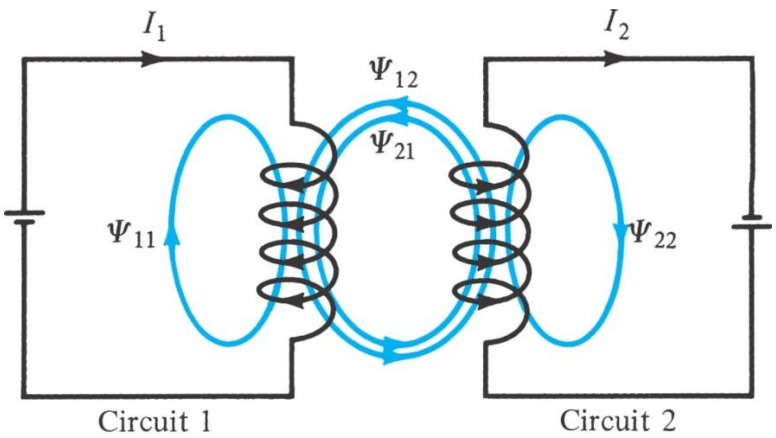
Inductors (contd.)

- Like capacitance, inductance may also be regarded as a measure of how much magnetic energy is stored in an inductor.
- The magnetic energy stored in an inductor is:

$$W_m = \frac{1}{2} L I^2$$

$$L = \frac{2W_m}{I^2}$$

- In case of two circuits carry current I_1 and I_2 as shown then a magnetic interaction exists between the circuits.



- Four component fluxes ψ_{11}, ψ_{12}, ψ_{21}, and ψ_{22} are produced. ψ_{12}, for example, is the flux passing through circuit-1 due to current I_2 in circuit-2.
- If $\vec{B}_2$ is the field due to I_2 and S_1 is the area of circuit-1 then:

$$\Psi_{12} = \int_{S_1} \vec{B}_2 . \overline{ds}$$

Inductors (contd.)

- The mutual inductance M_{12} is defined as the ratio of the flux linkage $\lambda_{12} = N_1\psi_{12}$ of circuit-1 to current I_2:

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\Psi_{12}}{I_2}$$

- Similarly, M_{21} is defined as the ratio of the flux linkage $\lambda_{21} = N_2\psi_{21}$ of circuit-2 to current I_1:

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2\Psi_{21}}{I_1}$$

- Now, the self-inductance of circuit-1 and circuit-2 is given by:

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1\psi_1}{I_1}$$

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2\psi_2}{I_2}$$

- The total energy in the magnetic field is the sum of the energies due to L_1, L_2 and M_{12} (or M_{21}):

$$W_m = W_1 + W_2 + W_{12} = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 \pm M_{12}I_1I_2$$

The +ve sign is taken if currents I_1 and I_2 flow such that the magnetic fields of the two circuits strengthen each other, otherwise the –ve sign is taken.

Inductors (contd.)

- We find the self-inductance L by taking the following steps.
  1. Choose a suitable coordinate system.
  2. Let the inductor carry current I.
  3. Determine $\vec{B}$ from Biot-Savart Law (or from Ampere's circuital law if symmetry exists) and calculate ψ from $\psi = \int \vec{B}.\overline{ds}$.
  4. Finally find L from $L = \frac{\lambda}{I} = \frac{N\psi}{I}$.

The mutual inductance between two circuits may be calculated by taking a similar procedure.

- In coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the internal inductance L_{in} while that produced by the flux external to it is called external inductance L_{ext}.
- The total inductance is given by:

$$L = L_{in} + L_{ext}$$

Example – 5

- Calculate the self-inductance per unit length of an infinitely long solenoid.

Magnetic Energy

- The potential energy in electrostatic field was derived as:

$$W_E = \frac{1}{2}\int \vec{D}.\vec{E}dv = \frac{1}{2}\int \varepsilon E^2 dv$$

One can derive similar expression for magnetostatic field

- Expressions are: $$w_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\vec{B}.\vec{H} = \frac{B^2}{2\mu}$$

- The total energy in a linear medium is:

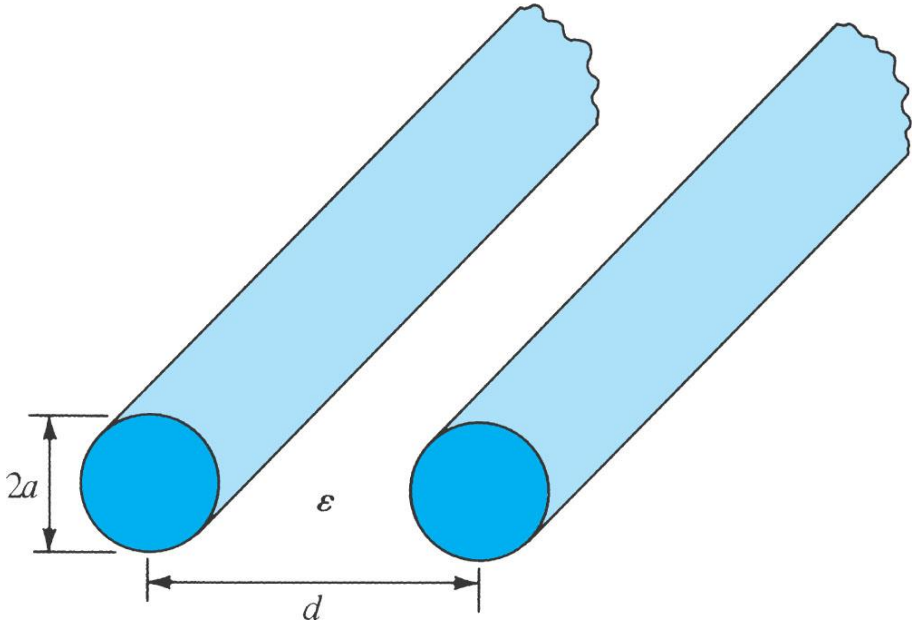
$$W_m = \int w_m dv \Rightarrow W_m = \frac{1}{2}\int \vec{B}.\vec{H}dv \Rightarrow W_m = \frac{1}{2}\int \mu H^2 dv$$

Example – 6

- Determine the self-inductance of a coaxial cable of inner radius a and outer radius b.

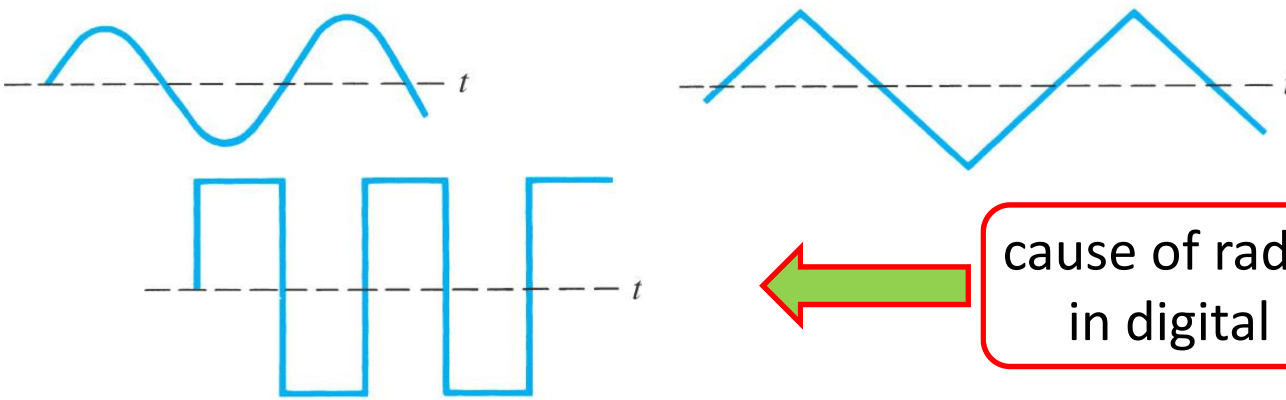
Example – 7

- Determine the inductance per unit length of a two-wire transmission line with separation distance d shown below.



Electromagnetic Fields

- Stationary Charges → Electrostatic Fields
- Steady Currents → Magnetostatic Fields
- Time Varying Currents → Electromagnetic Fields (or Waves)
- Any pulsating current will produce radiation (time-varying fields)



cause of radiated emission in digital logic boards

Say instead of a static magnetic flux density, we consider a **time-varying** $\vec{B}$ field (i.e., $\vec{B}(x, y, z, t)$).

Faraday's Law

- Recall that one of **Maxwell's** equations is:

$$\nabla \times E(x,y,z) = -\frac{\partial B(x,y,z,t)}{\partial t}$$

Yikes! The curl of the electric field is therefore **not zero** if the magnetic flux density is **time-varying**!

If the magnetic flux density is changing with time, the electric field will **not be conservative**!

Q: What the heck does this equation mean ?!?

$$\nabla \times E(x,y,z) = -\frac{\partial B(x,y,z,t)}{\partial t}$$

A: Integrate both sides over some surface S:

$$\iint_S \nabla \times \vec{E}(x,y,z).\overline{ds} = -\frac{\partial}{\partial t}\iint_S \vec{B}(x,y,z,t).\overline{ds}$$

Stoke's Theorem

$$\oint_L \vec{E}(x,y,z).\overline{dl} = -\frac{\partial}{\partial t}\iint_S \vec{B}(x,y,z,t).\overline{ds}$$

Faraday's Law (contd.)

$$\oint_L \vec{E}(x,y,z).\overline{dl} = -\frac{\partial}{\partial t}\iint_S \vec{B}(x,y,z,t).\overline{ds}$$

Note that $\oint \vec{E}(x,y,z).\overline{dl} \neq 0$

This equation is called **Faraday's Law of Induction**.

Q: Again, what does this **mean**?

A: It means that a time varying magnetic flux density $\vec{B}(x,y,z,t)$ can **induce** an electric field (and thus an electric potential difference)!

Faraday's Law describes the behavior of devices such as **generators**, **inductor**s, and **transformers** !

- Farday discovered that an **induced potential difference** (or electromotive force, emf) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} = -N\frac{d}{dt}\int_S \vec{B}.\overline{ds}$$

Faraday's Law (contd.)

- **It is apparent that an emf can be generated in a closed loop under any of the three conditions**
  - **A time varying magnetic field linking a stationary loop; the induced emf is then called the $transformer\ emf$.**
  - **A moving loop with a time-varying surface area in a static field; the induced emf is then called $motional\ emf$.**
  - **A moving loop in a time-varying field $\vec{B}$.**

- **The total emf is then given by:** $$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

- **For stationary loop:** $V_{emf}^{m} = 0$

- **For static $\vec{B}$:** $V_{emf}^{tr} = 0$

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} = -N\frac{d}{dt}\int_S \vec{B}.\overline{ds}$$

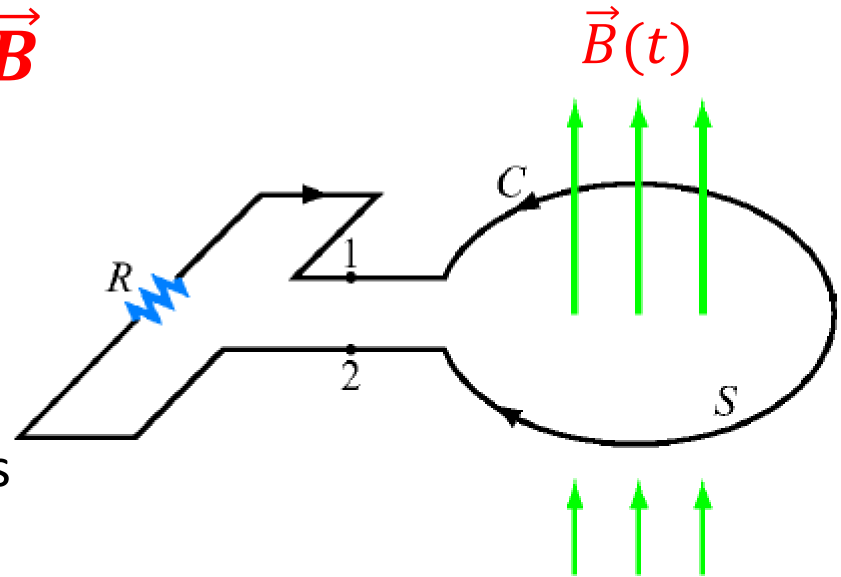
The negative sign in this expression shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as ***Lenz's Law***.

It emphasizes that the direction of current flow in the circuit is such that the induced $\vec{B}$ produced by the induced current will oppose the change in the original $\vec{B}$.

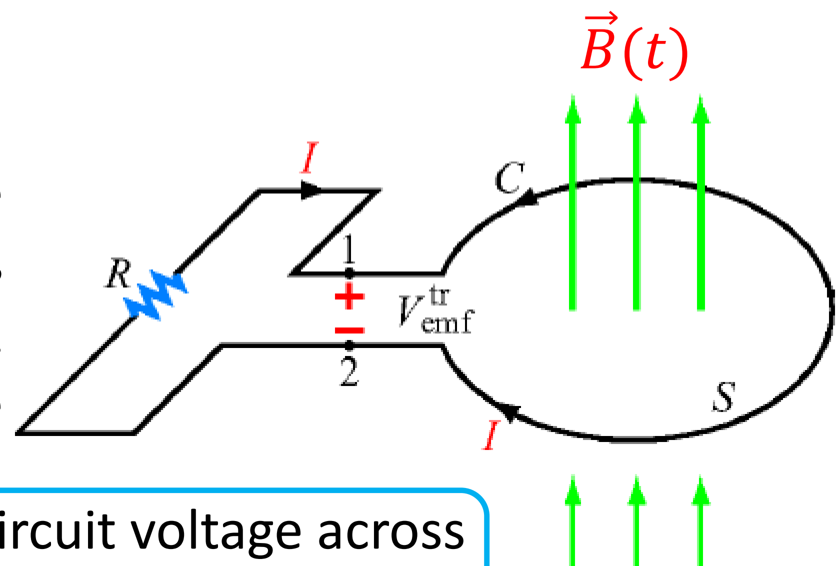
Stationary Loop in Time-Varying $\vec{B}$

- Let us consider a stationary, single-turn, conducting, circular loop with contour C and surface area S placed in a time-varying magnetic field $\vec{B}(t)$.
- As stated, emf will be induced in this loop and its given by:

$$V_{emf}^{tr} = -\int_S \frac{\partial \vec{B}}{\partial t} . \overline{ds}$$



- The $transformer\ emf$ is the voltage difference that would appear across the small opening between terminals 1 and 2, even in the absence of the resistor R.

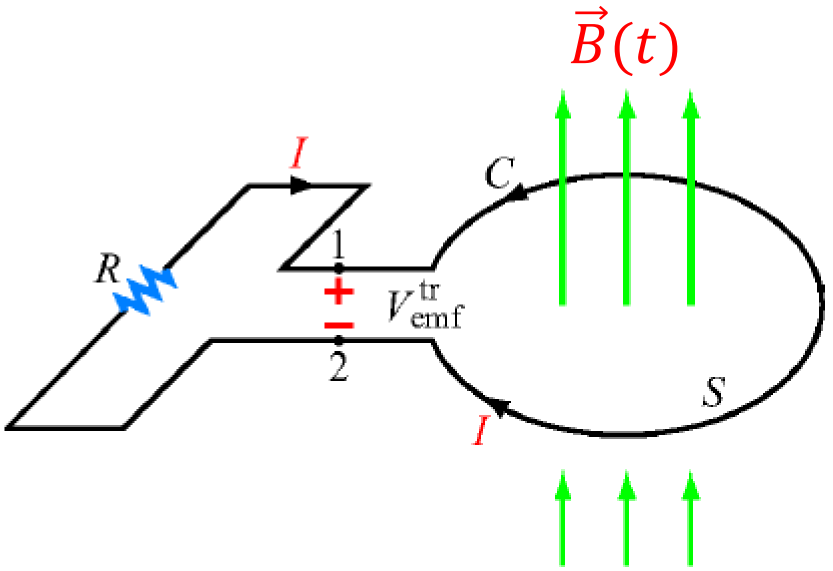


$$V_{emf}^{tr} = V_{12}$$

V_{12} is the open-circuit voltage across the open ends of the loop

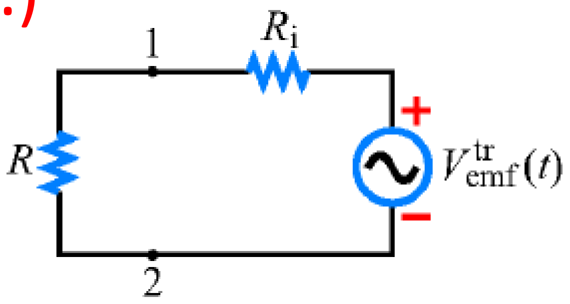
Stationary Loop in Time-Varying $\vec{B}$ (contd.)

- The direction of $\overline{ds}$, the loops differential surface normal, can be chosen either upward or downward.
- These two choices are associated with the opposite designations of the polarities of terminals 1 and 2.
- The choice of direction of $\overline{ds}$ and the polarity of emf is governed by right hand rule: If $\overline{ds}$ points along the thumb of the right hand, then the directions of the contour C indicated by the four fingers is such that it always passes across the opening from the positive terminal to the negative terminal.



$$V^{tr}_{emf} = V_{12} = -\int_S \frac{\partial \vec{B}}{\partial t} . \overline{ds}$$

Stationary Loop in Time-Varying $\vec{B}$ (contd.)

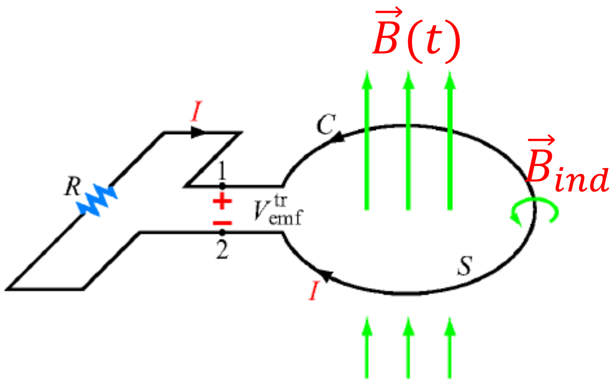


- If the loop has an internal resistance R_i, the circuit can be represented equivalently as:
- Therefore the current I flowing through the circuit is:

$$I = \frac{V_{emf}^{tr}}{R + R_i}$$

- The polarity of emf and hence the direction of I is governed by Lenz's law, **which states** that the current in the loop is always in a direction that opposes the change of magnetic flux $\psi(t)$ that produced I.

- The current I induces a magnetic field of its own, $\vec{B}_{ind}$, with a corresponding flux $\vec{\psi}_{ind}$.
- The direction of $\vec{B}_{ind}$ is governed by right hand rule: If I is in a clockwise direction, then $\vec{B}_{ind}$ points downward through S.
- **Conversely,** if I is in counter clockwise direction, then $\vec{B}_{ind}$ points upwards through S.



Stationary Loop in Time-Varying $\overrightarrow{B}$ (contd.)

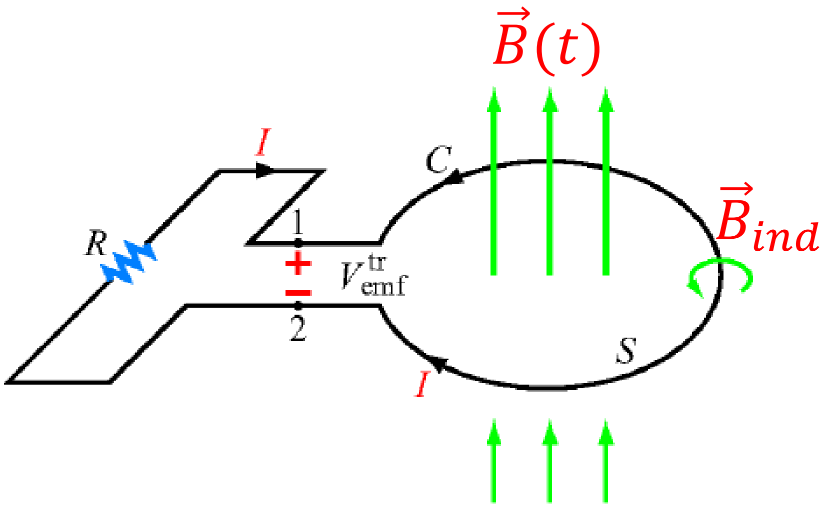
- **If the original $\vec{B}(\mathrm{t})$ is increasing,** means $\frac{d\psi}{dt} > 0$, then according to Lenz's law, I has to be in the direction shown in order for $\vec{B}_{ind}$ to be in opposition to $\vec{B}(\mathrm{t})$.
- As a consequence, terminal 2 would be at higher potential and emf would have a negative value.
- However, if $\vec{B}(\mathrm{t})$ were to remain in the same direction but decrease in magnitude, means $\frac{d\psi}{dt} < 0$, then the current would have to reverse direction, and its induced field $\vec{B}_{ind}$ would be in the same direction as $\vec{B}(\mathrm{t})$ so as to oppose the change (decrease) in $\vec{B}(\mathrm{t})$.

It is important to remember that $\vec{B}_{ind}$ serves to oppose the change in $\vec{B}(\mathrm{t})$, and not necessarily $\vec{B}(\mathrm{t})$ itself.

Stationary Loop in Time-Varying $\vec{B}$ (contd.)

Summary:

$$V_{emf}^{tr} = -\int_S \frac{\partial \vec{B}}{\partial t}.\overline{ds}$$



$$V_{emf}^{tr} = \oint_C \vec{E}.\overline{dl}$$

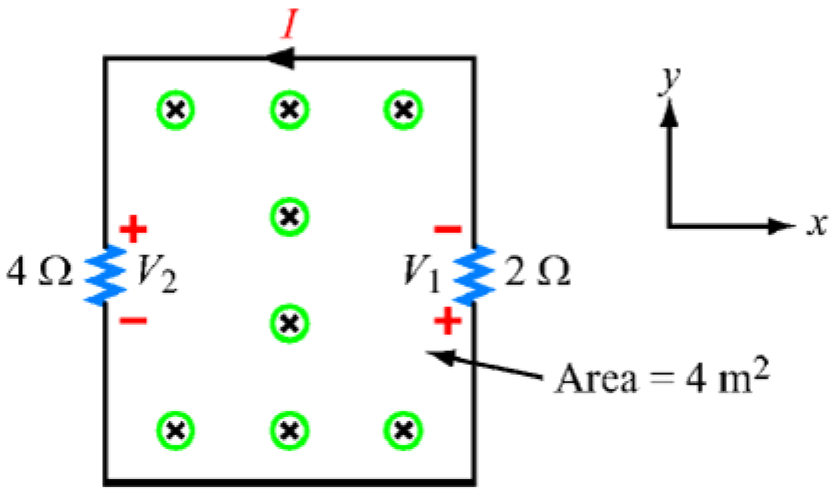
Its assumed that the contour C is closed path ↔ Approximation

$$\oint_C \vec{E}.\overline{dl} = -\int_S \frac{\partial \vec{B}}{\partial t}.\overline{ds} \;\Rightarrow\; \int_S (\nabla \times \vec{E}).\overline{ds} = -\int_S \frac{\partial \vec{B}}{\partial t}.\overline{ds} \;\Rightarrow\; \nabla \times \vec{E} = -\frac{\partial \vec{B}(t)}{\partial t}$$

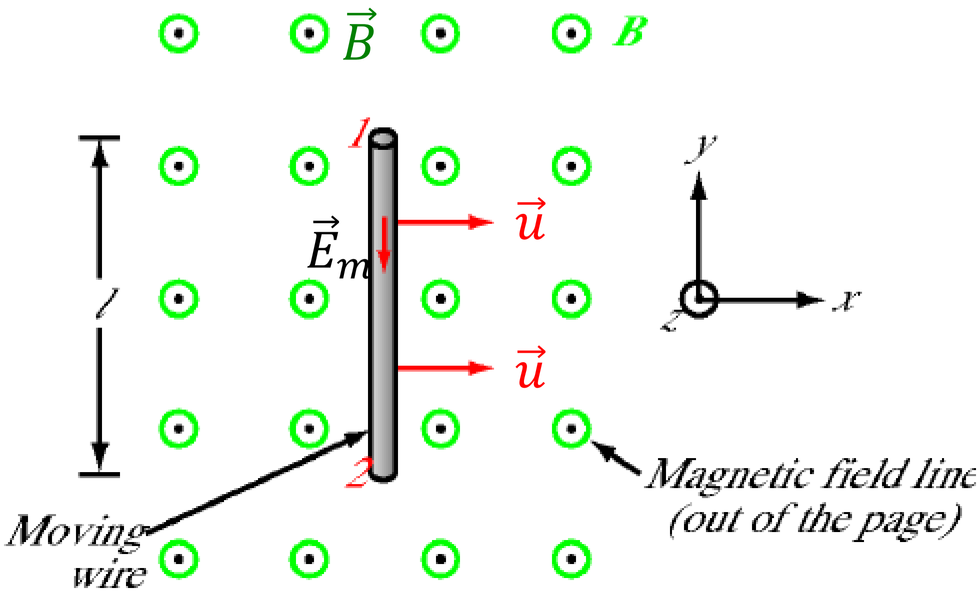
The time varying magnetic field induces an electric field $\vec{E}$ whose curl is equal to the negative of the time derivative of $\vec{B}$.

Example – 8

Determine voltages V_1 and V_2 across 2Ω and 4Ω resistors shown in the figure. The loop is located in $xy - plane$, its area is $4m^2$, the magnetic flux density is $\vec{B} = -\hat{a}_z 0.3t$ (T), and the internal resistance of the wire may be ignored.



Moving Conductor in a Static $\vec{B}$



- Let us consider a wire of length l moving across a static magnetic field $\vec{B} = \hat{a}_z B_0$ with constant velocity $\vec{u}$. The conducting wire contains free electrons.

- The magnetic force $\vec{F}_m$ acting on a particle with charge q moving with velocity $\vec{u}$ in a magnetic field $\vec{B}$ is:

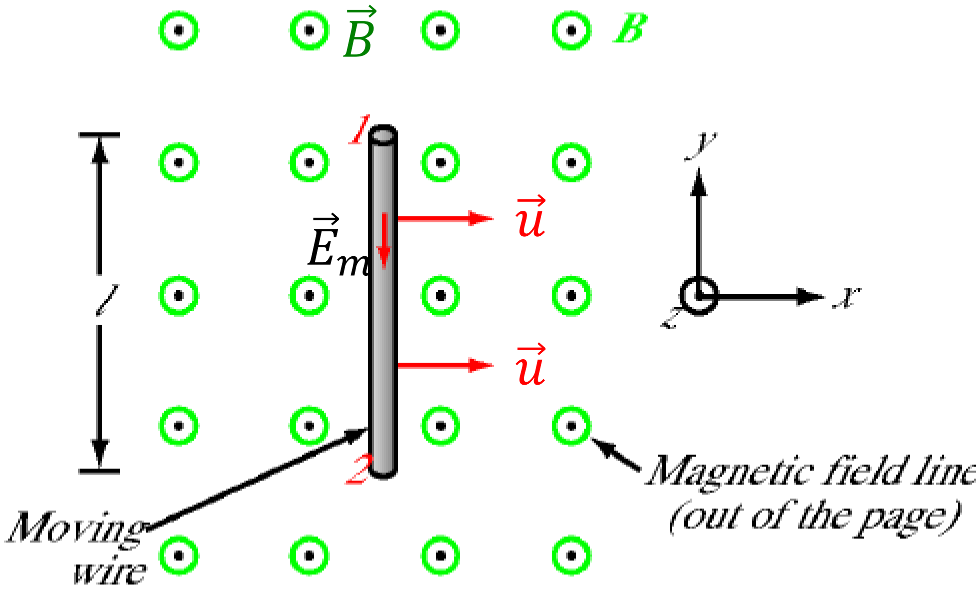
$$\vec{F}_m = q\left(\vec{u} \times \vec{B}\right)$$

- This magnetic force is equivalent to the electrical force that would be exerted on the particle by the electric field $\vec{E}_m$ given by:

$$\vec{E}_m = \frac{\vec{F}_m}{q} = \vec{u} \times \vec{B}$$

Moving Conductor in a Static $\vec{B}$ (contd.)

- The field $\vec{E}_m$ generated by the motion of the charged particle is called *motional electric field* and is orthogonal to both $\vec{u}$ and $\vec{B}$.
- For our example, $\vec{E}_m$ is along $-\hat{a}_y$.
- The magnetic force acting on the negatively charged electrons causes them to drift in the direction of $-\vec{E}_m$; i.e., toward the wire end label 1.



- The movement of electrons induces a voltage between ends 1 and 2.
- The induced voltage is called $motional\ emf$.
- $motional\ emf$ is defined as:

$$V_{emf}^{m} = V_{12} = \int_{2}^{1} \vec{E}_m . \overline{dl} = \int_{2}^{1} \left(\vec{u} \times \vec{B}\right) . \overline{dl}$$

Moving Conductor in a Static $\vec{B}$ (contd.)

- For the conducting wire:

$$\vec{u} \times \vec{B} = u\hat{a}_x \times \hat{a}_z B_0 = -\hat{a}_y u B_0$$

$$\overline{dl} = \hat{a}_y dl$$

- Therefore: $$V_{emf}^{m} = V_{12} = -uB_0 l$$

- In general, if any segment of a closed circuit with contour C moves with a velocity $\vec{u}$ across a static magnetic field $\vec{B}$, then the induced $motional\ emf$ is:

$$V_{emf}^{m} = \oint_C \left(\vec{u} \times \vec{B}\right).\overline{dl}$$

Only those segments of the circuit that cross magnetic field lines contribute to $motional\ emf$.

Example – 9

- The wire shown in the figure carries a current $I = 10A$. A 30-cm long metal rod moves with a constant velocity $\vec{u} = 5\hat{a}_z$ m/s. Find V_{12}.