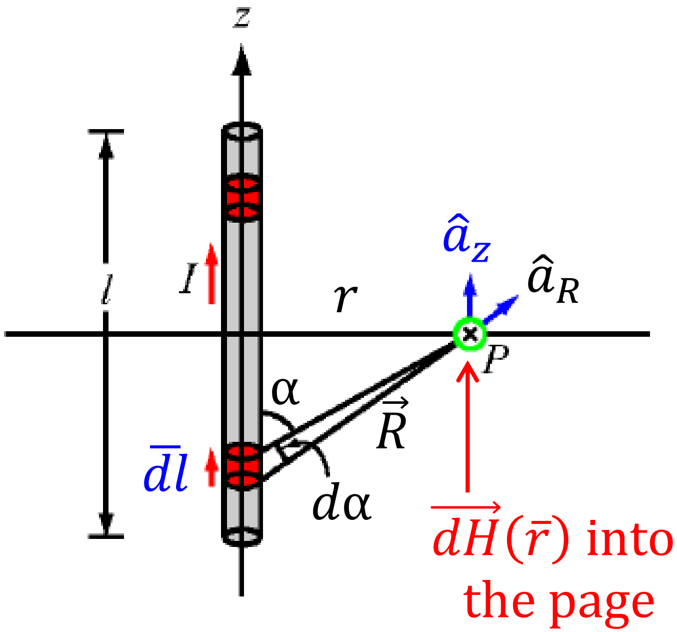
Lecture – 18

Date: 10.03.2016

- Ampere's Circuital Law
- Applications of Ampere's Law
- Magnetic Flux Density
- Magnetic Vector Potential

Example – 1

- A free-standing linear conductor of length l carries a current I along z-axis as shown in Figure. Determine the magnetic field intensity at point P located at a distance r in the x-y plane.



- It is apparent that:

$$\bar{d}l = dz\hat{a}_z \Rightarrow \bar{d}l \times \hat{a}_R = dz\sin\alpha\,\hat{a}_\phi$$

where α is the angle between $\bar{d}l$ and $\hat{a}_R$

- From Biot-Savart Law: $\vec{H}(\bar{r}) = \hat{a}_\phi \dfrac{I}{4\pi} \displaystyle\int_{z=-\frac{l}{2}}^{z=\frac{l}{2}} \frac{\sin\alpha}{R^2} dz$

- Here, both α and R are dependent on the integration variable z, but the radial distance r is not.

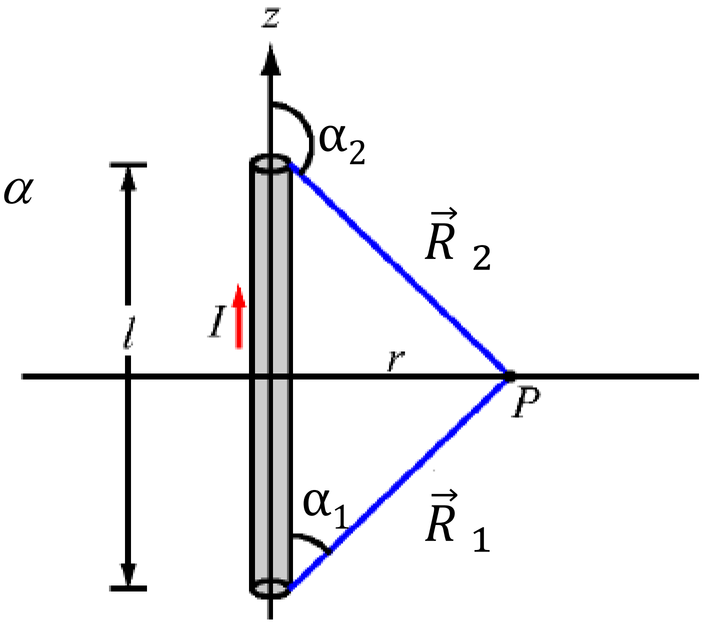
Example – 1 (contd.)

- Lets use the following transformation:

$$\vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\sin\alpha}{R^2}dz \qquad z = -r(\cot\alpha) \qquad dz = r\left(\text{cosec}^2\,\alpha\right)d\alpha$$

- Therefore:

$$\boxed{\vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\sin\alpha}{R^2}dz}$$



Where α_1 and α_2 are the limiting angles at $z = -\frac{l}{2}$ and $z = \frac{l}{2}$ respectively.

$$\boxed{\therefore \vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi r}\left(\cos\alpha_1 - \cos\alpha_2\right)}$$

- This expression is usually valid for any straight filamentary conductor of finite length.
- The conductor need not lie on the z-axis but it must be straight.
- $\vec{H}$ is always along the unit vector $\hat{a}_\phi$ (i.e, along concentric circular paths) irrespective of the length of the wire or the point of interest P.

Example – 1 (contd.)

- As a special case: when the conductor is semi-finite (with respect to P) so that its bottom end is at the origin (i.e., 0, 0, 0) while the top end is at (0, 0, ∞) then,

$$\therefore \vec{H}(\bar{r}) = \frac{I}{4\pi r}\hat{a}_\phi$$

$\alpha_1 = 90^o$ and $\alpha_2 = 180^o$

- Another special case: when the conductor is infinite (with respect to P) so that its bottom end is at (i.e., 0, 0, -∞) while the top end is at (0, 0, ∞) then,

$$\therefore \vec{H}(\bar{r}) = \frac{I}{2\pi r}\hat{a}_\phi$$

$\alpha_1 = 0^o$ and $\alpha_2 = 180^o$

$$\vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi r}(\cos\alpha_1 - \cos\alpha_2)$$

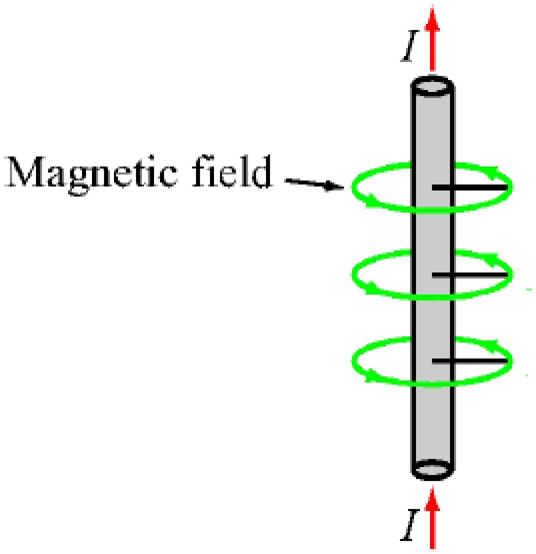
- not always easy to find the unit vector $\hat{a}_\phi$.
- simple approach is to determine $\hat{a}_\phi$ from:

$$\hat{a}_\phi = \hat{a}_l \times \hat{a}_R$$

where, $\hat{a}_l$ is the unit vector along the line current and $\hat{a}_R$ is a unit vector along the perpendicular line from the train current to the field point.

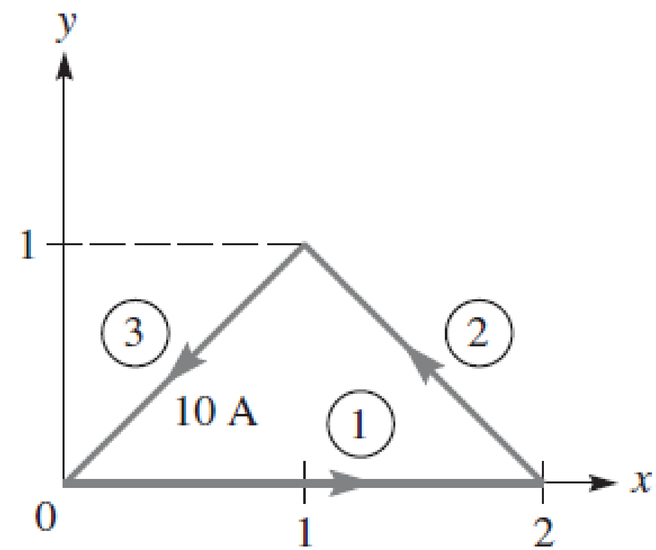
Example – 1 (contd.)

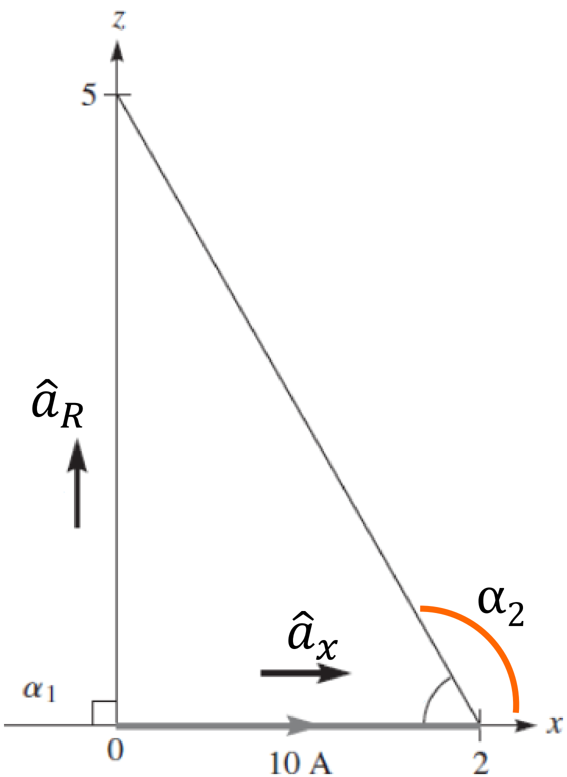
- This result is very useful expression to memorize. It states that in the neighbourhood of a linear conductor carrying a current I, the induced magnetic field forms concentric circles around the wire and its intensity is directly proportional to I and inversely proportional to distance r.



Example – 2

- The conducting triangular loop in the figure carries a current of 10A. Find $\vec{H}$ at (0, 0, 5) due to side 1 of the loop.





Here: $\hat{a}_l = \hat{a}_x$ $\hat{a}_R = \hat{a}_z$

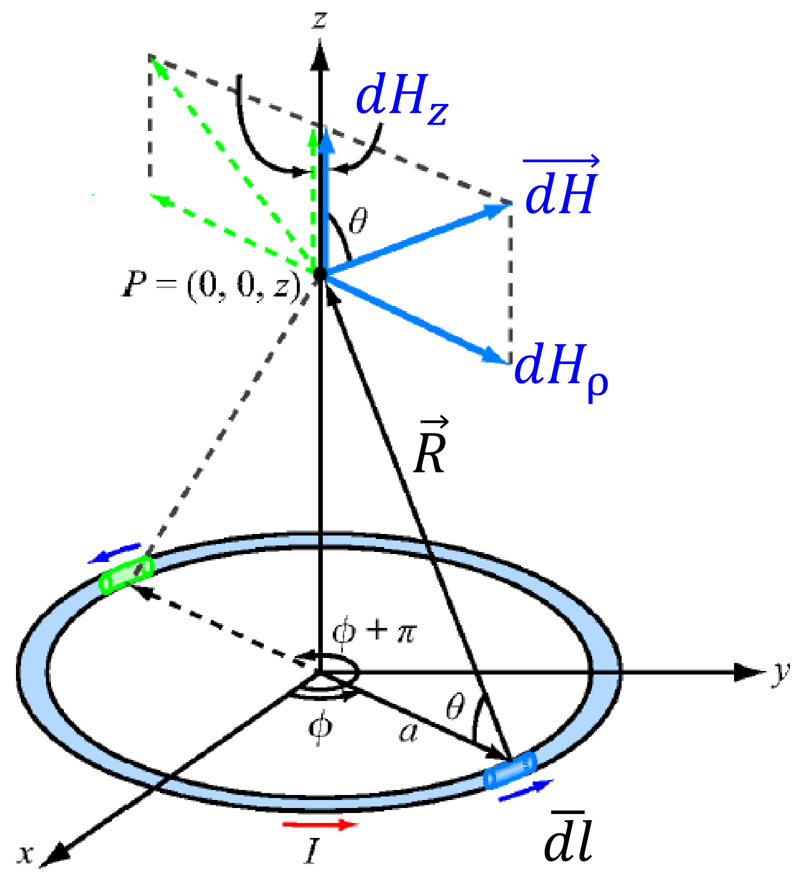
$$\therefore \hat{a}_\phi = \hat{a}_x \times \hat{a}_z = -\hat{a}_y$$

$$\cos\alpha_1 = 0 \qquad \cos\alpha_2 = -\frac{2}{\sqrt{29}} \qquad r = 5$$

$$\boxed{\Rightarrow \vec{H}(\bar{r}) = \hat{a}_\phi \frac{I}{4\pi r}\left(\cos\alpha_1 - \cos\alpha_2\right) = -59.1\hat{a}_y mA/m}$$

Example – 3

- A circular loop of radius a carries a steady current I. Determine the magnetic field $\vec{H}$ at a point on the axis of the loop.



- Let us place the loop in the xy-plane as shown.
- We want to obtain expression for $\vec{H}$ at (0, 0, z).

- Let us take an element $\overline{dl}$ at (x, y, 0)
- The magnetic field $\overrightarrow{dH}$ due to this element is:

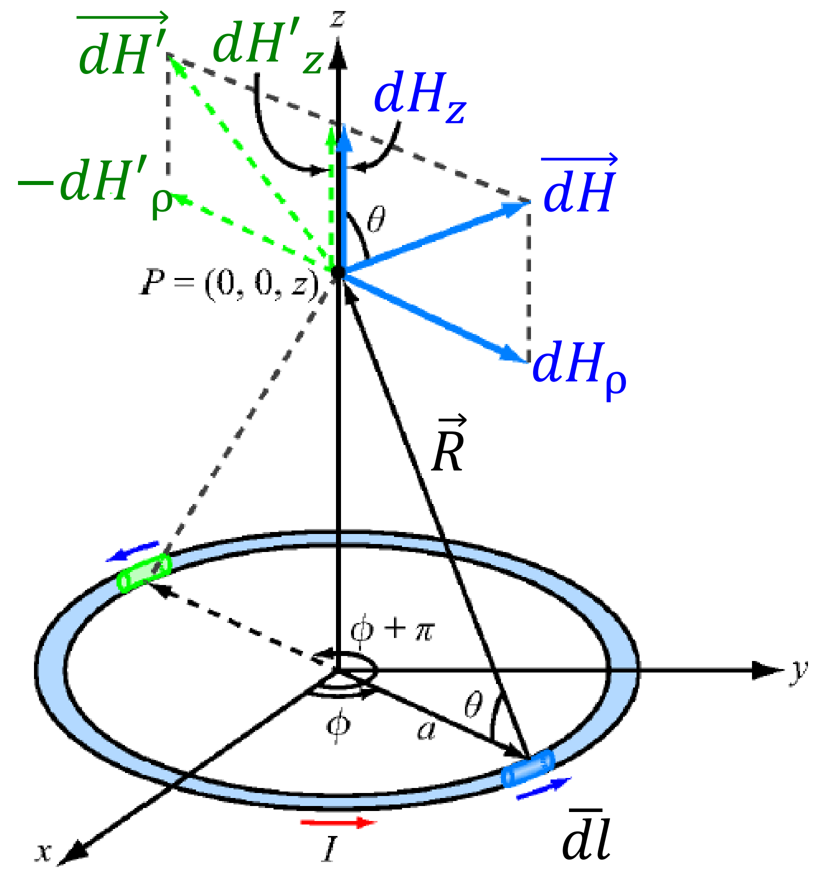
$$\overrightarrow{dH} = \frac{I\,\overline{dl} \times \hat{a}_R}{4\pi R^2}$$

$$\vec{R} = (0,0,z) - (x,y,0) = -a\hat{a}_\rho + z\hat{a}_z$$

Clearly indicates two components of $\vec{H}$

Example – 3 (contd.)

- If we consider element $\overline{dl}'$ located diametrically opposite to $\overline{dl}$ then we observe that the z-components of the magnetic fields due to $\overline{dl}'$ and $\overline{dl}$ add because they are in the same direction, but their ρ-components cancel because they are in opposite directions.
- Hence the net magnetic field is along z-axis only.



- We have: $$\left|\overrightarrow{dH}\right| = \left|\frac{I\,\overline{dl} \times \hat{a}_R}{4\pi R^2}\right| = \frac{Idl}{4\pi(a^2 + z^2)}$$

- Therefore: $$\overrightarrow{dH} = \hat{a}_z dH_z = \hat{a}_z dH\cos\theta = \hat{a}_z \frac{I(\cos\theta)}{4\pi(a^2 + z^2)} dl$$

Example – 3 (contd.)

- For a fixed point $P(0,0,z)$ on the axis of the loop, all quantities in the above expression are constant except for $\overline{dl}$, therefore:

$$\vec{H} = \hat{a}_z \frac{I(\cos\theta)}{4\pi(a^2+z^2)} \oint dl$$

- Thus:

$$\vec{H} = \hat{a}_z \frac{I(\cos\theta)}{4\pi(a^2+z^2)} (2\pi a)$$

- We can also derive:

$$\cos\theta = \frac{a}{\sqrt{a^2+z^2}}$$

$$\therefore \vec{H} = \hat{a}_z \frac{Ia^2}{2(a^2+z^2)^{3/2}}$$

- At the center of the loop $(z=0)$:

$$\therefore \vec{H} = \hat{a}_z \frac{I}{2a}$$

- At a point far away from the loop $(|z| \gg a)$:

$$\therefore \vec{H} = \hat{a}_z \frac{Ia^2}{2|z|^3}$$

Example – 4

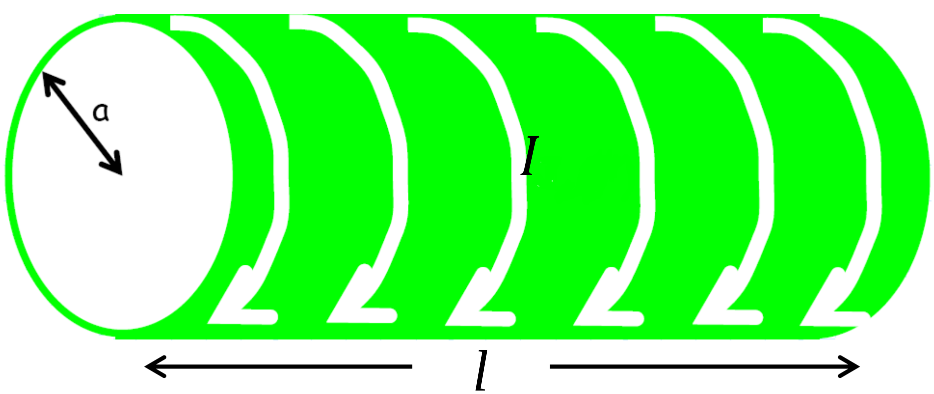
- A solenoid, lying along z-axis, of length l and radius a consists of N turns of wire carrying current I. show that at point P along its axis:

$$\vec{H} = \hat{a}_z \frac{NI}{2l}\left(\cos\theta_2 - \cos\theta_1\right)$$

where, θ_1 and θ_2 are the angle subtended at P by the end turns.

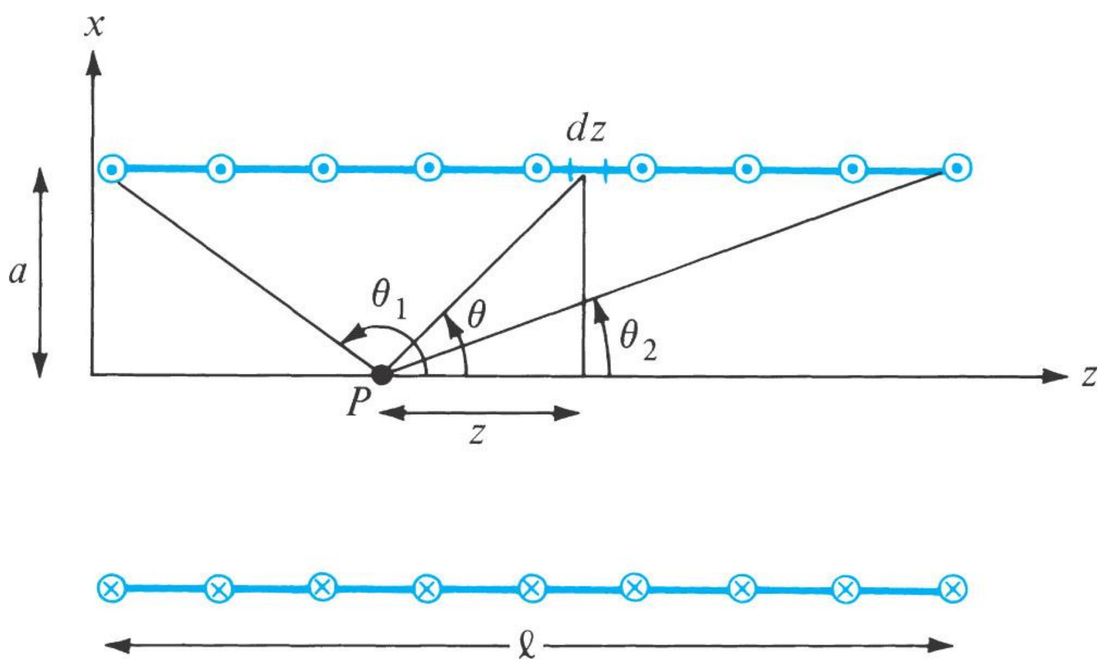
- Alo show that if $l \gg a$, at the center of the solenoid: $\vec{H} = \hat{a}_z \dfrac{NI}{l}$

- An important structure in electrical and computer engineering is the **solenoid**.
- A solenoid is a **tube of current**. However, it is different from the hollow cylinder, in that the current flows **around** the tube, rather than down the tube:



Example – 4 (contd.)

- Let us consider the cross section of solenoid as shown below.



Make use of example-3

- The magnetic field at P due to length dz is:

$$\boxed{dH_z = \frac{NIa^2}{2l(a^2+z^2)^{3/2}}dz}$$

where

$$dH_z = \frac{NIa^2}{2l(a^2+z^2)^{3/2}}dz$$

- **From figure:** $\tan\theta = \dfrac{a}{z}$ ⇒ $dz = -a\,\mathrm{cosec}^2\theta\, d\theta$ ⇒ $dz = -a\dfrac{\left(z^2+a^2\right)^{3/2}}{a^2}\sin\theta\, d\theta$

Example – 4 (contd.)

- Therefore:

$$dH_z = -\frac{NI}{2l}\sin\theta d\theta \quad \Rightarrow \quad H_z = -\frac{NI}{2l}\int_{\theta_1}^{\theta_2}\sin\theta d\theta$$

$$\therefore \vec{H} = \frac{NI}{2l}\left(\cos\theta_2 - \cos\theta_1\right)\hat{a}_z$$

- At the center of the Solenoid:

$$\cos\theta_2 = \frac{l/2}{\left[a^2 + \frac{l^2}{4}\right]^{1/2}} = -\cos\theta_1$$

- Thus:

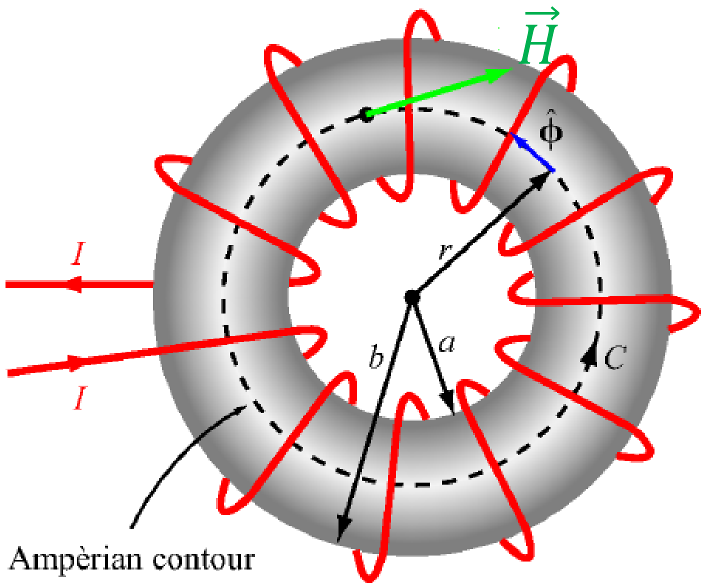
$$\therefore \vec{H} = \frac{NI}{2\left[a^2 + \frac{l^2}{4}\right]^{1/2}}\hat{a}_z$$

- If $l \gg a$, then:

$$\therefore \vec{H} = \frac{NI}{l}\hat{a}_z$$

Example – 5

- A toroidal coil is a doughnut-shaped structure (called the core) wrapped in a closely spaced turns of wire (as shown in figure). For clarity, the turns have been shown as spaced far apart, but in practice they are wound in a closely spaced arrangement. The toroid is used to magnetically couple multiple circuits and to measure the magnetic properties of materials. For a toroid with N turns carrying a current I, determine the magnetic field $\vec{H}$ in each of the following three regions: $r < a$, $a < r < b$, and $r > b$, all in the azimuthal plane symmetry of the toroid.



Ampere Circuital Law

- Earlier we learnt that the electrostatic field is conservative, meaning its line integral along a closed contour always vanishes.
- This property was expressed as: $\nabla \times \vec{E} = 0 \Leftrightarrow \oint_C \vec{E}.\overline{dl} = 0$
- The magnetostatic counterpart known as Ampere's Law is: $\nabla \times \vec{H} = \vec{J} \Leftrightarrow \oint_C \vec{H}.\overline{dl} = I_{encl}$
- The sign convention for the direction of contour path C in Ampere's law is taken so that I and $\vec{H}$ satisfy the right-hand rule defined earlier in connection with Biot-Savart law→ If the direction of I is aligned with the direction of the thumb then the direction of the contour C should be chosen along that of the other four fingers.
- In words, Ampere's circuital law states that the line integral of $\vec{H}$ around a closed path is equal to the current traversing the surface bounded by that path.

$$\boxed{\oint_C \vec{H}.\overline{dl} = I_{encl}}$$

We know: $\boxed{I_{encl} = \int_S \vec{J}.\overline{ds}}$

Apply Stoke's Theorem

$$\boxed{\oint_C \vec{H}.\overline{dl} = \int_S (\nabla \times \vec{H}).\overline{ds}}$$

Ampere Circuital Law (contd.)

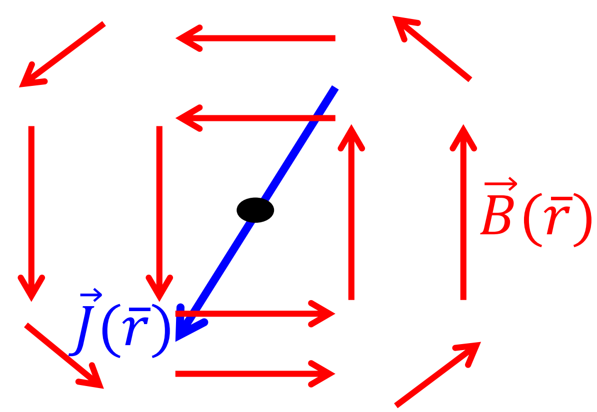
- Therefore: $\nabla \times \vec{H} = \vec{J}$

Maxwell's Equation for Magnetostatics

Magnetostatic field is not conservative

- This Maxwell's equation for magnetostatic equation is referred to as **Ampere's Circuital Law**: $\nabla \times \vec{H}(\bar{r}) = \vec{J}(\bar{r})$

This equation indicates that the magnetic flux density $\vec{H}(\bar{r})$ **rotates around** current density $\vec{J}(\bar{r})$ --the **source** of magnetic field intensity is current!.



Applications of Ampere's Law

$$\oint_C \vec{H}.\overline{dl} = I_{encl}$$

This equation holds regardless of whether the current distribution is symmetrical or otherwise

But $\vec{H}$ can be determined using this expression only if the symmetrical current distribution exists

Examples include: an infinite line current, an infinite sheet of current, and an infinitely long coaxial transmission line

In each case, we apply $\oint_C \vec{H}.\overline{dl} = I_{enc}$. For symmetrical current distribution, $\vec{H}$ is either parallel or perpendicular to $\overline{dl}$. When $\vec{H}$ is parallel to $\overline{dl}$, $|\vec{H}| = constant$.

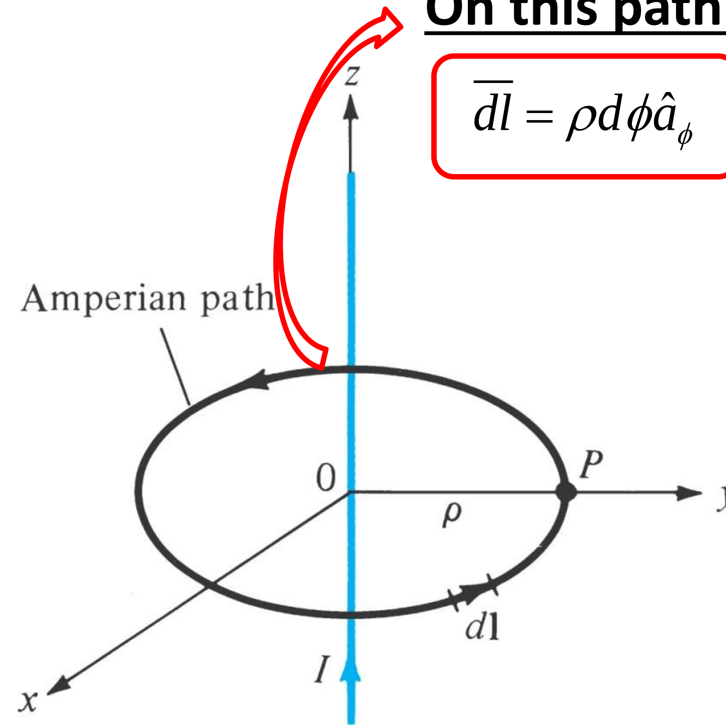
Applications of Ampere's Law (contd.)

Infinite Line Current

- Let us consider an infinitely long filamentary current along the z-axis.
- To determine $\vec{H}$ at point P, let us form a closed path to pass through P.
- This path is called Amperian path (analogous to Gaussian surface).
- From Ampere's law we can write:

On this path:

$$\overline{dl} = \rho d\phi \hat{a}_\phi$$



$$\oint_C \vec{H}.\overline{dl} = I = \int H_\phi \hat{a}_\phi . \rho d\phi \hat{a}_\phi$$

As $\vec{H}$ is parallel to $\overline{dl}$

$$\Rightarrow I = H_\phi \int \rho d\phi$$

For fixed ρ

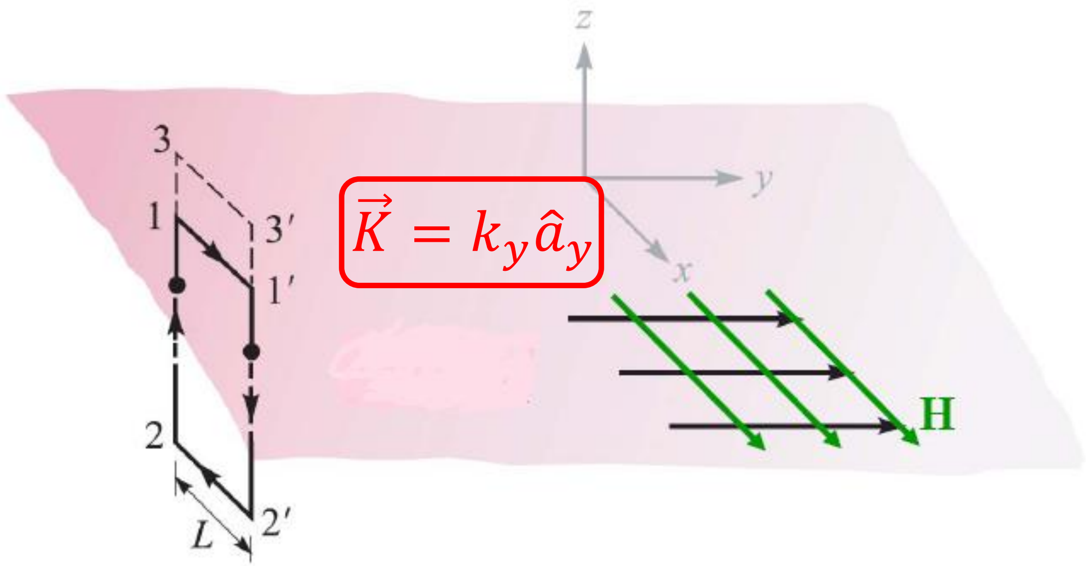
$$\Rightarrow I = H_\phi (2\pi\rho)$$

$$\therefore \vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi$$

Applications of Ampere's Law (contd.)

Infinite Sheet of Current

- Let us consider an infinite current sheet in the $z = 0$ plane.
- The sheet has a uniform current density $\vec{K} = k_y \hat{a}_y$ A/m as shown.

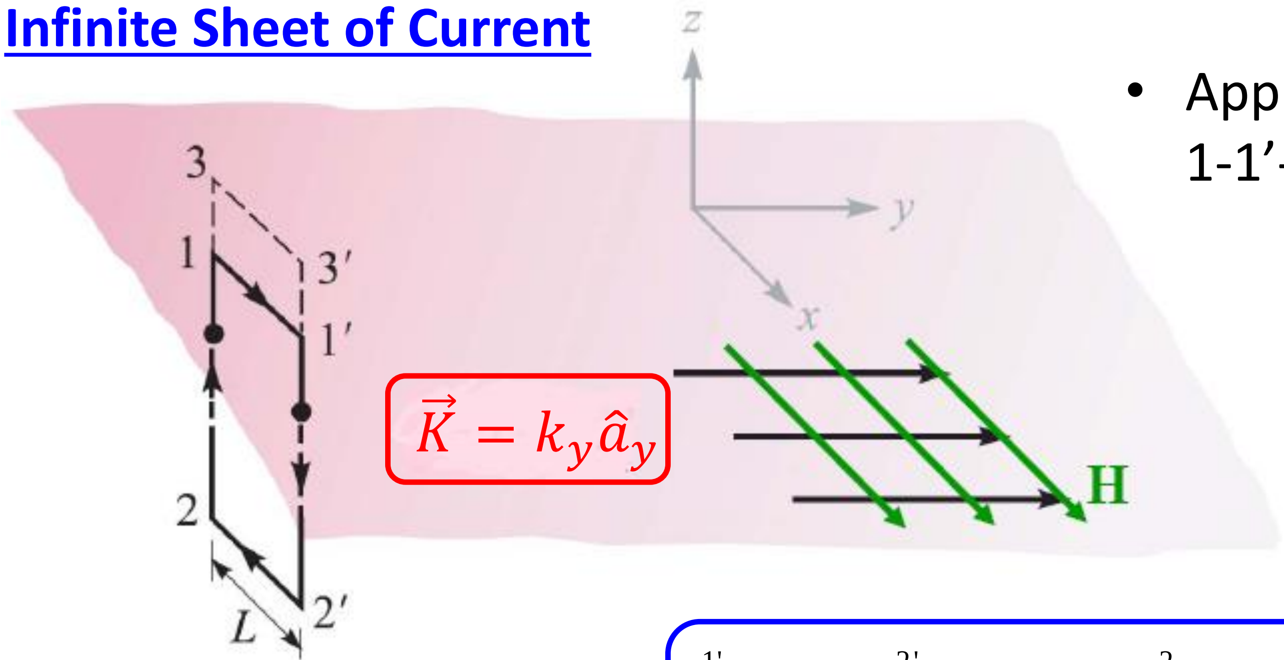


- Consider the sheet as a finite number of filaments cascaded together
- Field doesn't vary with x and y as the source doesn't vary with x and y
- $H_y = 0$, since current is along $y - axis$ [field is perpendicular to current]

- $H_z = 0$, as two symmetric filamentary elements along $x - axis$ will cancel the $z - components$.
- Resultant fields will be along $x - axis$ and doesn't vary with x and y.

Applications of Ampere's Law (contd.)

Infinite Sheet of Current



$\vec{K} = k_y \hat{a}_y$

- Apply Ampere's law along 1-1'-2'-2-1

$$\oint_C \vec{H}.\overline{dl} = I$$

$$\int_{1}^{1'} H_{x1}dx + \int_{1'}^{2'} (-H_{z1})dz + \int_{2'}^{2} (-H_{x2})dx + \int_{2}^{1} H_{z1}dz = K_y L$$

Doesn't vary with x

Zero contribution from segments 1'-2' and 1-2 ($H_z = 0$)

$$\Rightarrow H_{x1}L - H_{x2}L = K_y L$$

$$\therefore H_{x1} - H_{x2} = K_y$$

Applications of Ampere's Law (contd.)

- Similarly application of Ampere's law along 3-3'-2'-2-3 results into

$$\therefore H_{x3} - H_{x2} = K_y$$

$$\therefore H_{x1} - H_{x2} = K_y$$

$$\therefore H_{x3} - H_{x2} = K_y$$

- Simplification gives:

$$H_{x1} = H_{x3} = \frac{K_y}{2}$$

$$H_{x2} = -\frac{K_y}{2}$$

Therefore, it can be said that the field is same for all positive z and similarly the same for all negative z

- Because of symmetry, the magnetic field intensity on one side of the current sheet is negative of that on the other.

$$H_x = \frac{K_y}{2} \qquad (z > 0)$$

$$H_x = -\frac{K_y}{2} \qquad (z < 0)$$

Applications of Ampere's Law (contd.)

Infinite Sheet of Current

- If $\hat{a}_N$ is the unit vector normal (outward) to the current sheet, the result may be expressed as:

$$\boxed{\vec{H} = \frac{1}{2}\vec{K} \times \hat{a}_N}$$

- Magnetic field doesn't depend on the distance from the infinite current sheet → analogous to $\vec{D}$ $field$ of an infinite charge sheet.

$$\boxed{\vec{H} = \frac{1}{2}\vec{K} \times \hat{a}_N} \qquad \boxed{\vec{D} = \frac{1}{2}\rho_s \hat{a}_N}$$

- If a second sheet of current flowing in the opposite direction, $\vec{K} = -k_y \hat{a}_y$, is placed at $z = h$, then the field in the region between the sheets is:

$$\boxed{\vec{H} = \vec{K} \times \hat{a}_N \qquad \mathbf{(0 < z < h)}}$$

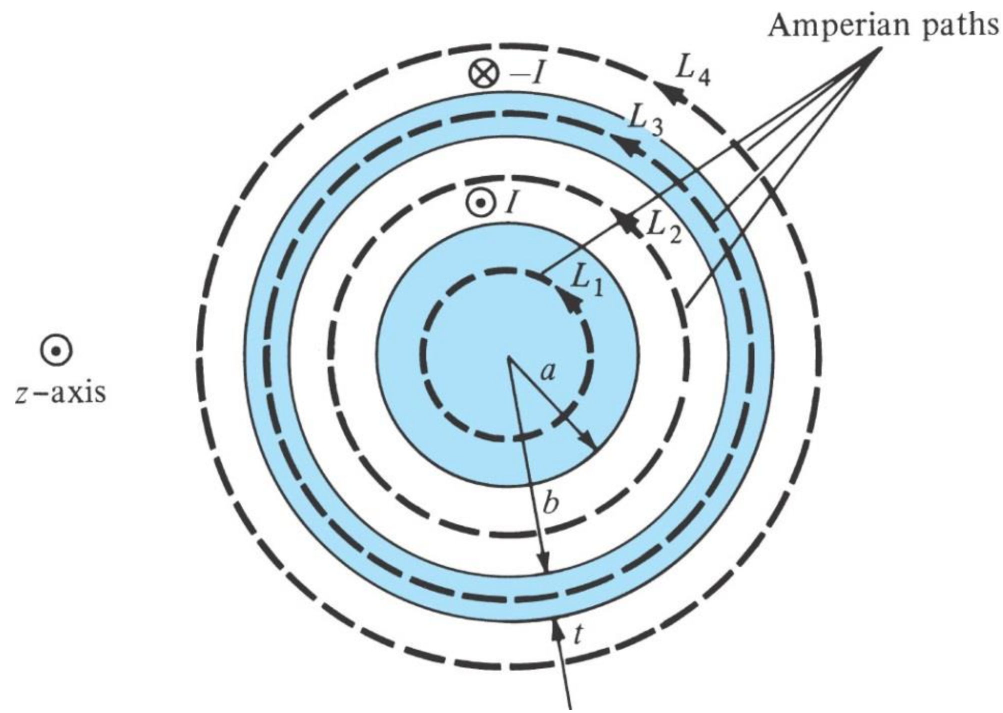
- and is zero elsewhere:

$$\boxed{\vec{H} = 0 \qquad \mathbf{(z < 0, \qquad z > h)}}$$

Applications of Ampere's Law (contd.)

Infinitely Long Coaxial Transmission Line

- Let us consider coaxial transmission line with two concentric cylinders having their axes along the z-axis, where the z-axis is out of page.
- The inner conductor has radius a and carries current I, while the outer conductor has inner radius b and thickness t and carries return current $-I$.
- Determine field $\vec{H}$ everywhere.



Since the current distribution is symmetric, we apply Ampere's law along the Amperian path for each of the four possible regions: $0 \leq \rho \leq a$, $a \leq \rho \leq b$, $b \leq \rho \leq b + t$, $\rho \geq b + t$

Applications of Ampere's Law (contd.)

Infinitely Long Coaxial Transmission Line

- For region $0 \le \rho \le a$, we have:

$$\vec{J} = \frac{I}{\pi a^2}\hat{a}_z \qquad \overline{dS} = \rho d\phi d\rho \hat{a}_z \Rightarrow I_{enc} = \int \vec{J}.\overline{dS} = \frac{I}{\pi a^2}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \rho d\phi d\rho \qquad \therefore I_{enc} = \frac{I\rho^2}{a^2}$$

Therefore application of Ampere's law over path L_1 gives:

$$H_\phi \int_{L_1} dl = H_\phi(2\pi\rho) = \frac{I\rho^2}{a^2} \Rightarrow \therefore H_\phi = \frac{I\rho}{2\pi a^2}$$

- For region $a \le \rho \le b$, we have: $I_{enc} = I$

Therefore application of Ampere's law over path L_2 gives:

$$H_\phi \int_{L_2} dl = H_\phi(2\pi\rho) = I \Rightarrow \therefore H_\phi = \frac{I}{2\pi\rho}$$

Applications of Ampere's Law (contd.)

Infinitely Long Coaxial Transmission Line

- For region $b \leq \rho \leq b + t$, we get:

$$I_{enc} = I + \int \vec{J}.\overline{dS}$$

Here, $\vec{J}$ is the current density of the outer conductor and is along $-\hat{a}_z$

$$\vec{J} = -\frac{I}{\pi\left[(b+t)^2 - b^2\right]}\hat{a}_z$$

$$I_{enc} = I - \frac{I}{\pi\left[(b+t)^2 - b^2\right]}\int_{\phi=0}^{2\pi}\int_{\rho=b}^{\rho}\rho d\phi d\rho$$

$$I_{enc} = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

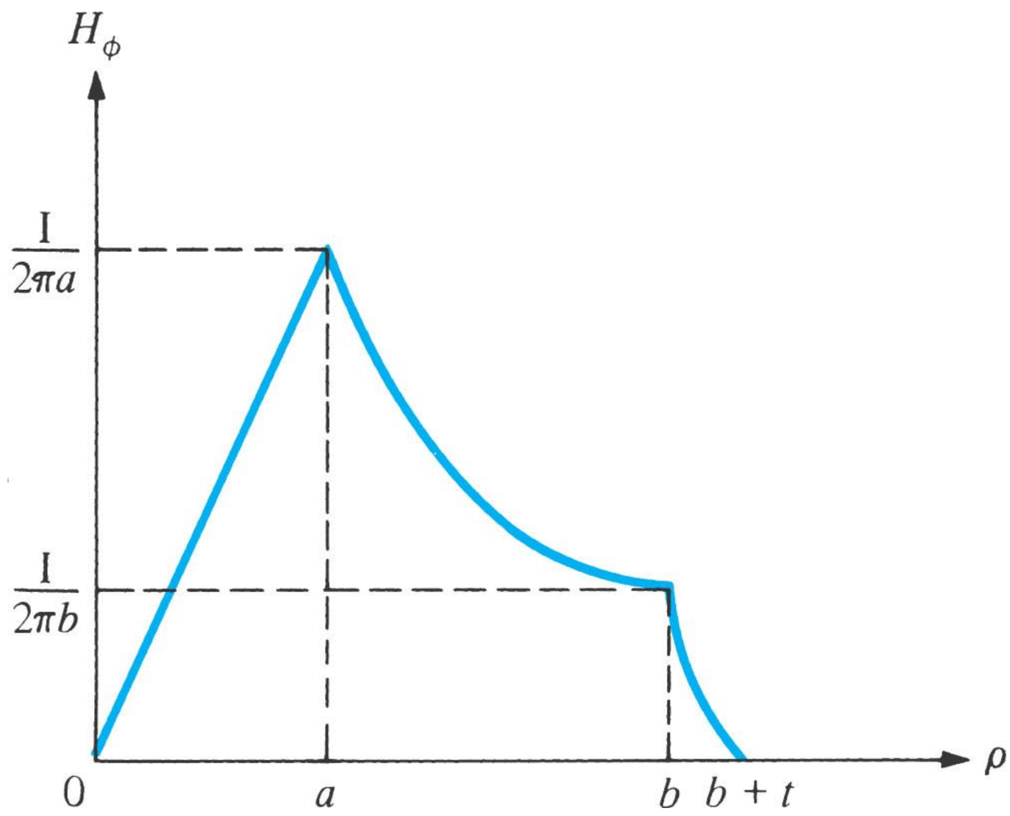
$$\Rightarrow H_\phi = \frac{I_{enc}}{2\pi\rho}$$

$$\therefore H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

Applications of Ampere's Law (contd.)

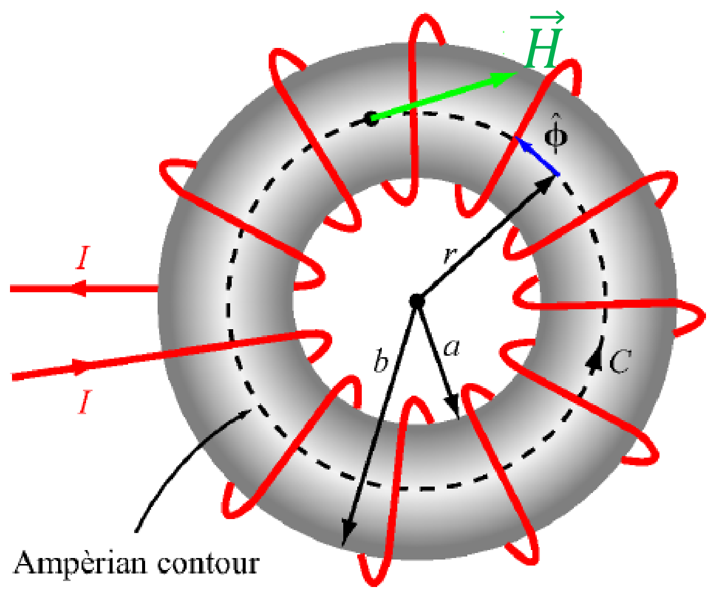
Infinitely Long Coaxial Transmission Line

- For region $\rho \geq b + t$, we get: $\boxed{I_{enc} = I - I = 0}$ $\boxed{\therefore H_\phi = 0}$



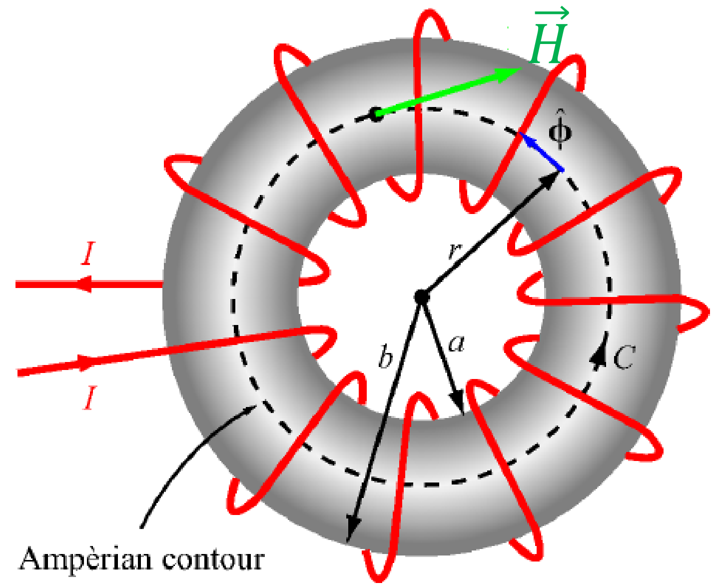
Example – 6

- A toroidal coil is a doughnut-shaped structure (called the core) wrapped in a closely spaced turns of wire (as shown in figure). For clarity, the turns have been shown as spaced far apart, but in practice they are wound in a closely spaced arrangement. The toroid is used to magnetically couple multiple circuits and to measure the magnetic properties of materials. For a toroid with N turns carrying a current I, determine the magnetic field $\vec{H}$ in each of the following three regions: $r < a$, $a < r < b$, and $r > b$, all in the azimuthal plane symmetry of the toroid.



Example – 6 (contd.)

- From Symmetry: It is apparent that $\vec{H}$ is uniform in the azimuthal direction.
- For circular Amperian path $r < a$, there will be no current through the surface of the contour.
- Similarly, for circular Amperian path $r > b$, there will be no current through the surface of the contour.



- Therefore, $\vec{H} = 0$ in the region external to the core.
- For region inside the core: Let us construct path of radius r.
- For each loop of radius r, we know that the field $\vec{H}$ at the center of the loop points along the axis of the loop, which in this case is the $\varphi - direction$.
- Now solve using Ampere's Circuital Law!!!

Magnetic Flux Density

- The magnetic flux density is similar to electric flux density $\vec{D}$.
- $\vec{E} = \varepsilon_0 \vec{E}$ in free space → similarly, the magnetic flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ as: $\boxed{\vec{B} = \mu_0 \vec{H}}$

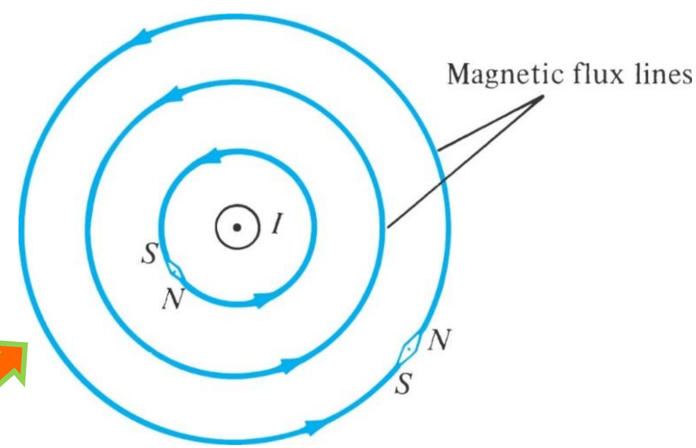
  Where, μ_0 is a constant known as permeability of free space. The constant is in henrys per meter (H/m) and has the value:

$$\boxed{\mu_0 = 4\pi \times 10^{-7} H/m}$$

- The magnetic flux through a surface S is given by: $\boxed{\psi = \int_S \vec{B}.\overline{ds}}$ ← **Webers (Wb)**

- Magnetic flux line is a path to which $\vec{B}$ is tangential at every point on the line.
- It is the line along which the needle of a magnetic compass will orient itself if placed in the presence of a magnetic field.

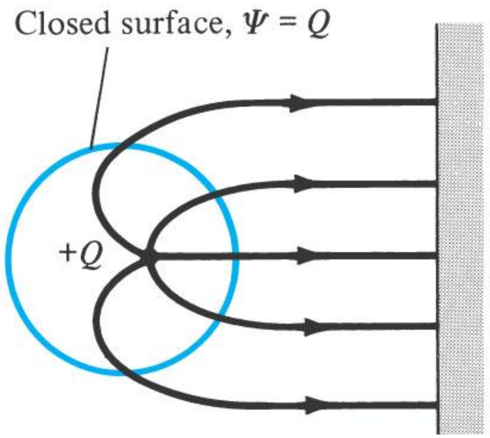
For example, the magnetic flux lines due to a straight long wire is



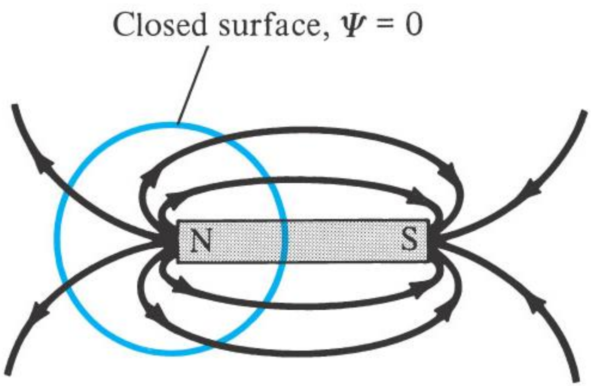
Magnetic Flux Density (contd.)

Note that each flux lines is closed and has no beginning or end. It is generally true that magnetic flux lines are closed and do not cross each other regardless of the current distribution.

- In an electrostatic field, the flux passing through a closed surface is the same as charge enclosed ($\psi = \oint \vec{D}.\overline{ds} = Q$) → thus it is possible to have an isolated electric charge such that flux lines are not necessarily closed.

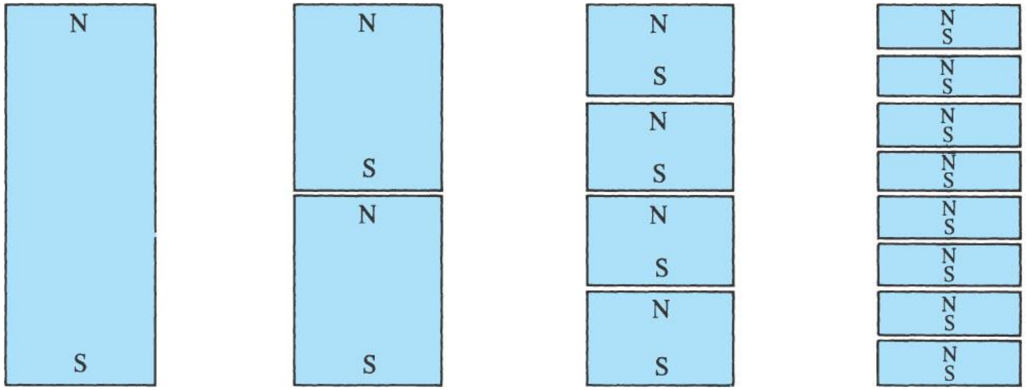


- Unlike electric flux lines, magnetic flux lines always close upon themselves → therefore, the total flux through a closed surface in a magnetic field must be zero ($\psi = \oint \vec{B}.\overline{ds} = 0$) → not possible to have isolated magnetic poles or magnetic charges.



Magnetic Flux Density (contd.)

- If we want an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having north and south poles → we find it impossible to separate the north pole from the south pole.



$$\oint \vec{B}.\overline{ds} = 0$$

Law of conservation of magnetic flux or Gauss's law for magnetostatic fields

Divergence Theorem

$$\oint \vec{B}.\overline{ds} = \int_v \nabla.\vec{B}dv = 0$$

$$\nabla.\vec{B} = 0$$

Maxwell Equation

Magnetic fields have no source or sinks ↔ Magnetic field lines are always continuous

Maxwell's Equations for Static Fields

Differential Form	Integral Form	Remarks
$\nabla.\vec{D} = \rho_v$	$\oint_S \vec{D}.\overline{ds} = \int_v \rho_v dv$	Gauss's Law
$\nabla.\vec{B} = 0$	$\oint_S \vec{B}.\overline{ds} = 0$	None existence of magnetic monopole
$\nabla \times \vec{E} = 0$	$\oint_C \vec{E}.\overline{dl} = 0$	Conservative Nature of $\vec{E}$
$\nabla \times \vec{H} = \vec{J}$	$\oint_C \vec{H}.\overline{dl} = \int_S \vec{J}.\overline{ds}$	Ampere's Law

Magnetic Scalar and Vector Potentials

- Some electrostatic problems became simpler by relating electric field intensity $\vec{E}$ $(\vec{E} = -\nabla V)$.
- Similarly, one can define potential associated with $\vec{H}$ or $\vec{B}$.
- The idea is that $\vec{B}$ should be defined in such a way that divergence of $\vec{B}$ should be always zero.
- Actually, magnetic potential could be scalar denoted as V_m or vector denoted as $\vec{A}$.

Magnetic Scalar and Vector Potentials

- Let us use these two identities:

$$\nabla \times (\nabla V) = 0$$

This holds for any scalar V

$$\nabla . (\nabla \times \vec{A}) = 0$$

This holds for any vector $\vec{A}$

- We define the magnetic scalar potential as: $\vec{H} = -\nabla V_m$

$$\Rightarrow \vec{J} = \nabla \times \vec{H} = \nabla \times \left(-\nabla V_m\right)$$

This Form $\nabla \times (\nabla V) = 0$

$$\vec{J} = \nabla \times \vec{H} = \nabla \times \left(-\nabla V_m\right) = 0$$

Thus magnetic scalar potential is valid only in a region where $\vec{J} = 0$

Very useful term for defining parameters of a permanent magnet

V_m satisfies Laplace's equation $\Rightarrow$ $\nabla^2 V_m = 0$

- Furthermore, $\nabla . \vec{B} = 0$ $\quad$ $\nabla . (\nabla \times \vec{A}) = 0$ $\Rightarrow$ $\vec{B} = \nabla \times \vec{A}$

Gives definition of vector magnetic potential

Magnetic Scalar and Vector Potentials (contd.)

- We defined: $V = \int \frac{dQ}{4\pi\varepsilon_0 r}$
- Similarly we can define: $\vec{A} = \int_C \frac{\mu_0 I \overline{dl}}{4\pi R}$ For line current

$\vec{A} = \int_C \frac{\mu_0 \vec{K} ds}{4\pi R}$ For surface current

$\vec{A} = \int_v \frac{\mu_0 \vec{J} dv}{4\pi R}$ For volume current

- We can express flux alternatively as:

$$\psi = \int_S \vec{B}.\overline{ds} \Rightarrow \psi = \int_S \left(\nabla \times \vec{A}\right).\overline{ds} \xrightarrow{\text{Stoke's Theorem}} \psi = \int_C \vec{A}.\overline{dl}$$

Thus the magnetic flux through a given area can be found using the magnetic vector potential

The use of magnetic vector potential provides a powerful approach to solving EM problems, particularly those relating to antennas → For antennas, its more convenient to find $\vec{A}$ than finding $\vec{B}$

Example – 7

- Given the magnetic vector potential $\vec{A} = -\frac{\rho^2}{4}\hat{a}_z \frac{\mathrm{Wb}}{\mathrm{m}}$, calculate the total magnetic flux crossing the surface $\phi = \frac{\pi}{2}$, $1 \le \rho \le 2\, m$, $0 \le z \le 5\, m$.

Method-1: $\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_z}{\partial \rho}\hat{a}_\phi$ $\quad \overline{dS} = d\rho dz \hat{a}_\phi$

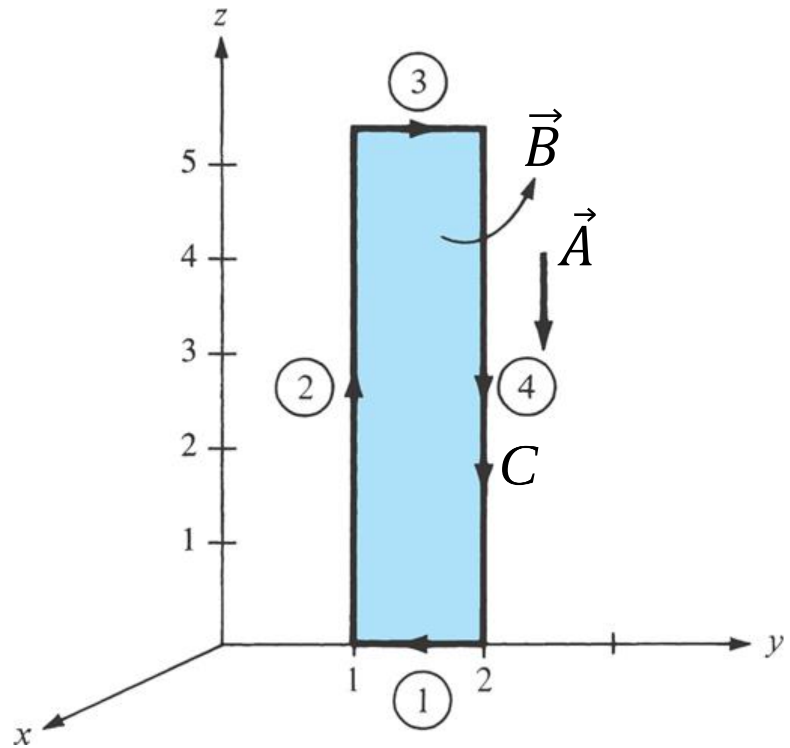
- Therefore: $\boxed{\psi = \int_S \vec{B}.\overline{ds}} \Rightarrow \boxed{\psi = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho d\rho dz} \Rightarrow \boxed{\psi = 3.75Wb}$

Example – 7 (contd.)

Method-2:

- We use: $\psi = \int_C \vec{A}.\overline{dl} = \psi_1 + \psi_2 + \psi_3 + \psi_4$

where, C is the path bounding surface S; ψ_1, ψ_2, ψ_3, and ψ_4 are respectively the evaluations of $\int \vec{A}.\overline{dl}$ along segments of C labeled 1 to 4.



- Since $\vec{A}$ has only z-component: $\psi_1 = \psi_3 = 0$

- Therefore: $\psi = \psi_2 + \psi_4 = -\frac{1}{4}\left[(1)^2 \int_0^5 dz + (2)^2 \int_5^0 dz\right]$ $\Rightarrow$ $\psi = 3.75Wb$

Example – 8

- A current distribution gives rise to the vector magnetic potential $\vec{A} = x^2 y\hat{a}_x + y^2 x\hat{a}_y - 4xyz\hat{a}_z \frac{Wb}{m}$. Calculate the following:
  (a) $\vec{B}$ at $(-1,\ 2,\ 5)$
  (b) The flux through the surface defined by $z = 1, 0 \le x \le 1, -1 \le y \le 4$

(a) $\vec{B} = \nabla \times \vec{A} \quad = (-4xz - 0)\hat{a}_x + (0 + 4yz)\hat{a}_y + (y^2 - x^2)\hat{a}_z$

$$\boxed{\therefore \vec{B}(-1,2,5) = 20\hat{a}_x + 40\hat{a}_y + 3\hat{a}_z}$$

(b) The flux through the given surface:

$$\boxed{\psi = \int_S \overline{B}.\overline{ds}} \Rightarrow \boxed{\psi = \int_{y=-1}^{4}\int_{x=0}^{1}\left(y^2 - x^2\right)\partial x \partial y} \Rightarrow \boxed{\psi = 20Wb}$$

<u>Alternatively:</u>

$$\boxed{\psi = \int_C \overline{A}.\overline{dl}} \Rightarrow \boxed{\psi = \int_0^1 x^2(-1)\partial x + \int_{-1}^{4} y^2(1)\partial y + \int_1^0 x^2(4)\partial x + 0} \Rightarrow \boxed{\psi = 20Wb}$$