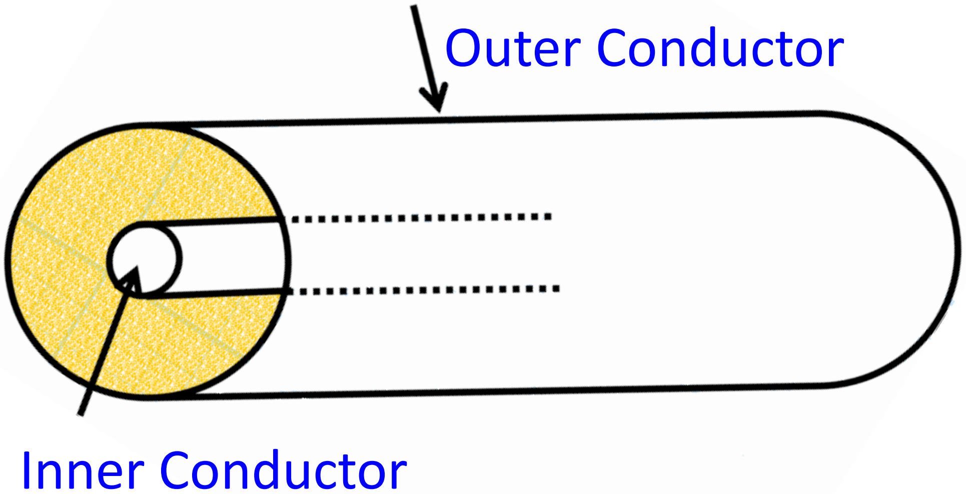
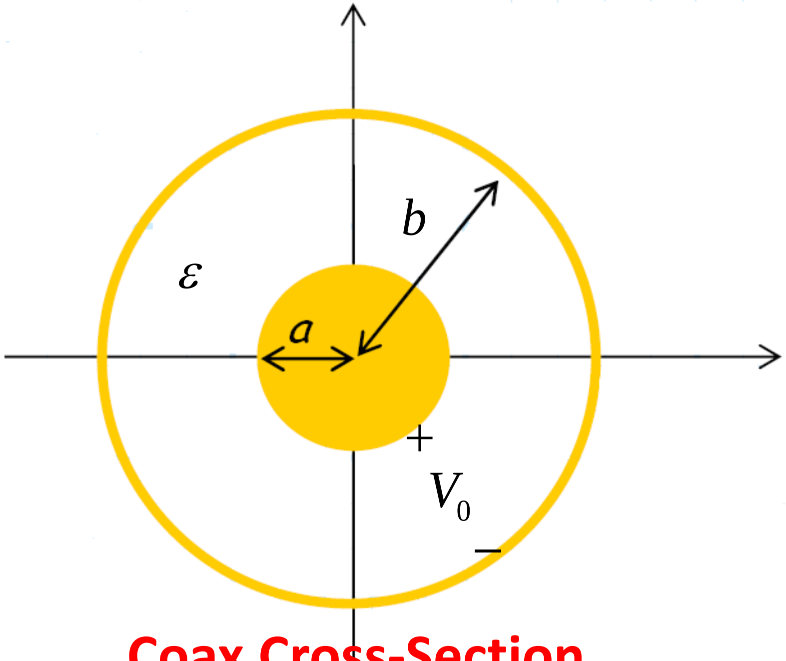
Lecture – 17 **Date: 07.03.2016**

- Electrostatic Boundary Value Problems (contd.)
- Energy Storage in a Capacitor
- Magnetostatics
- Biot-Savart Law

Example – 1: The Electrostatic Fields of a Coaxial Line

- A common form of a transmission line is the **coaxial** cable.





Coax Cross-Section

The coax has an **outer** radius b, and an **inner** radius a. The space between the conductors is filled with **dielectric** material of permittivity ε .

Say a voltage V_0 is placed across the conductors, such that the electric potential of the **outer** conductor is **zero**, and the electric potential of the **inner** conductor is V_0.

Example – 1 (contd.)

- The potential **difference** between the inner and outer conductor is therefore $V_0 - 0 = V_0$ volts.

Q: What electric potential field $V(\bar{r})$, electric field $\vec{E}(\bar{r})$, and charge density $\rho_s(\bar{r})$ is produced by this situation?

A: We must solve a **boundary-value** problem! We must find solutions that:

a) Satisfy the **differential equations** of electrostatics (e.g., Poisson's, Laplace's, Gauss's).

b) Satisfy the electrostatic **boundary conditions**.

Yikes! Where do we start ?

Example – 1 (contd.)

We might start with the electric potential field $V(\bar{r})$, since it is a **scalar** field.

a) The electric potential function must satisfy **Poisson's** equation:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0}$$

b) It must also satisfy the **boundary conditions**: $V(\rho = a) = V_0$ $\quad V(\rho = b) = 0$

- Consider first the **dielectric** region $(a < \rho < b)$. Since the region is a dielectric, there is **no** free charge, and:

$$\rho_v(\bar{r}) = 0$$

- Therefore, Poisson's equation reduces to **Laplace's** equation:

$$\rho_v(\bar{r}) = 0$$

- This particular problem (i.e., coaxial line) is directly solvable because the structure is **cylindrically symmetric**. Rotating the coax around the z-axis (i.e., in the $\hat{a}_\phi$ direction) does not change the geometry at all. As a result, we know that the electric potential field is a function of ρ **only**! i.e.,:

$$V(\bar{r}) = V(\rho)$$

Example – 1 (contd.)

- This make the problem much **easier**. Laplace's equation becomes:

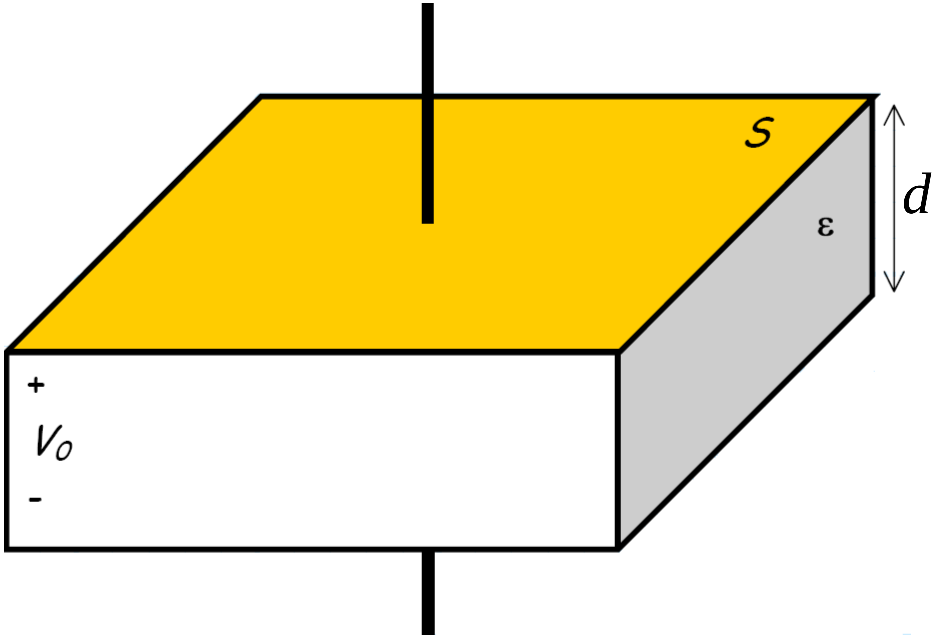
$$\nabla^2 V(\bar{r}) = 0 \quad \Rightarrow \quad \nabla^2 V(\rho) = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V(\rho)}{\partial \rho}\right) + 0 + 0 = 0 \quad \Rightarrow \quad \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V(\rho)}{\partial \rho}\right) = 0$$

Be **very** careful during **this** step! Make sure you implement the **Laplacian** operator correctly.

The Parallel Plate Capacitor

- Consider the geometry of a **parallel plate capacitor**:



Recall that we determined the fields and surface charge density of an **infinite** pair of parallel plates. We can use those results to approximate the fields and charge densities of this **finite** structure, where the **area** of each plate is S.

Where:

V_0 = the **potential difference** between the plates

S = **surface area** of each conducting plate

d = **distance** between plates

ε = **permittivity** of the dielectric between the plates

The Parallel Plate Capacitor (contd.)

- For example, we determined that the **surface charge density** on the upper plate is:

$$\rho_{s+}(\bar{r}) = \frac{\varepsilon V_0}{d}$$

- The **total charge** on the upper plate is therefore:

$$Q = \iint_{S_+} \rho_{s+}(\bar{r})\, ds \Rightarrow Q = \iint_{S_+} \frac{\varepsilon V_0}{d}\, ds \Rightarrow Q = \frac{\varepsilon V_0}{d} \iint_{S_+} ds \Rightarrow Q = \frac{\varepsilon V_0 S}{d}$$

- The **capacitance** of this structure is therefore:

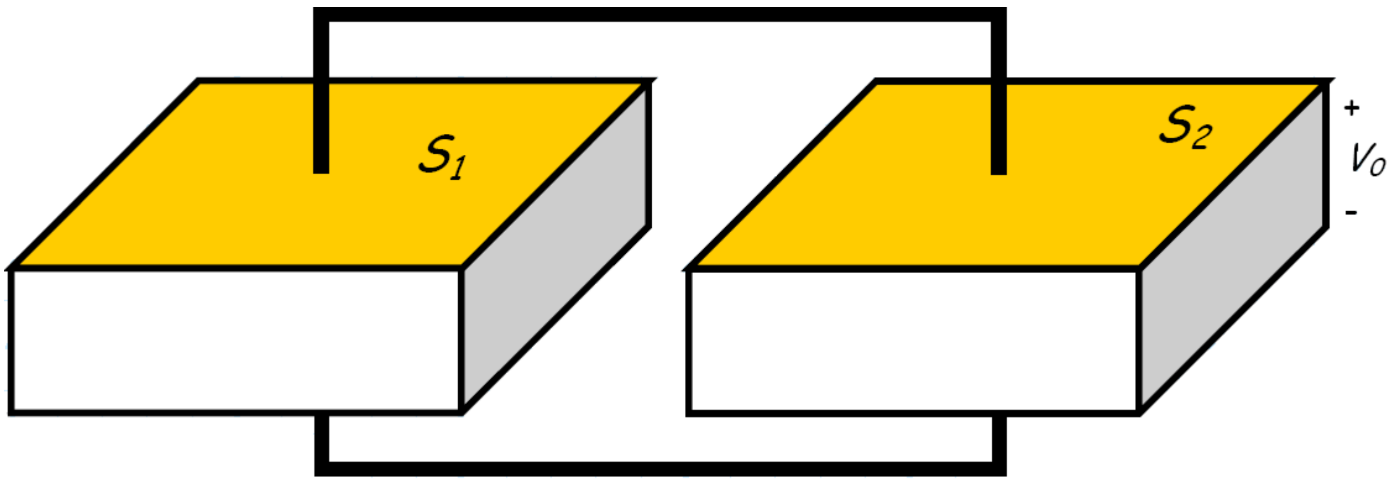
$$C = \frac{Q}{V} \Rightarrow C = \left(\frac{\varepsilon V_0 S}{d}\right)\left(\frac{1}{V_0}\right) \Rightarrow \therefore C = \frac{\varepsilon S}{d}$$

Therefore, we can **increase** the capacitance of a parallel plate capacitor by:

1) **Increasing** surface area S.
2) Decreasing separation distance d.
3) **Increasing** the dielectric permittivity ε.

The Parallel Plate Capacitor (contd.)

- Consider now the structure:



Note the **two** upper plates form **one** conducting structure, and the **two** bottom plates form **another**.

Q: What is the **capacitance** between these two conducting structures?

A: The potential difference between them is V_0. The **total charge** on one conducting structure is simply the **sum** of the charges on **each plate**:

$$Q = Q_1 + Q_2 = \frac{\varepsilon V_0 S_1}{d} + \frac{\varepsilon V_0 S_2}{d}$$

The Parallel Plate Capacitor (contd.)

- Therefore, the **capacitance** of this structure is:

$$C = \frac{Q}{V} = \left(\frac{\varepsilon V_0 \left(S_1 + S_2\right)}{d}\right)\left(\frac{1}{V_0}\right)$$

$$C = \frac{\varepsilon\left(S_1 + S_2\right)}{d}$$

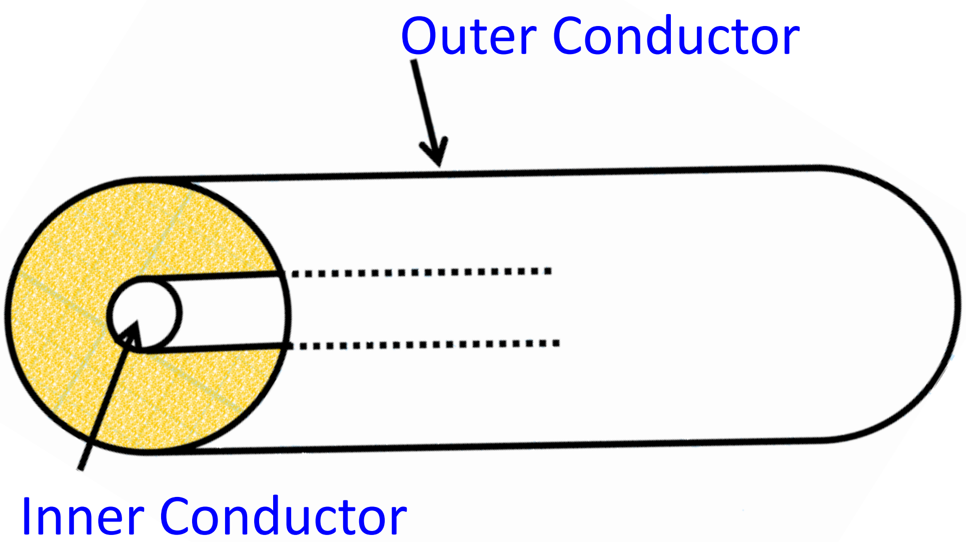
$$C = \frac{\varepsilon S_1}{d} + \frac{\varepsilon S_2}{d}$$

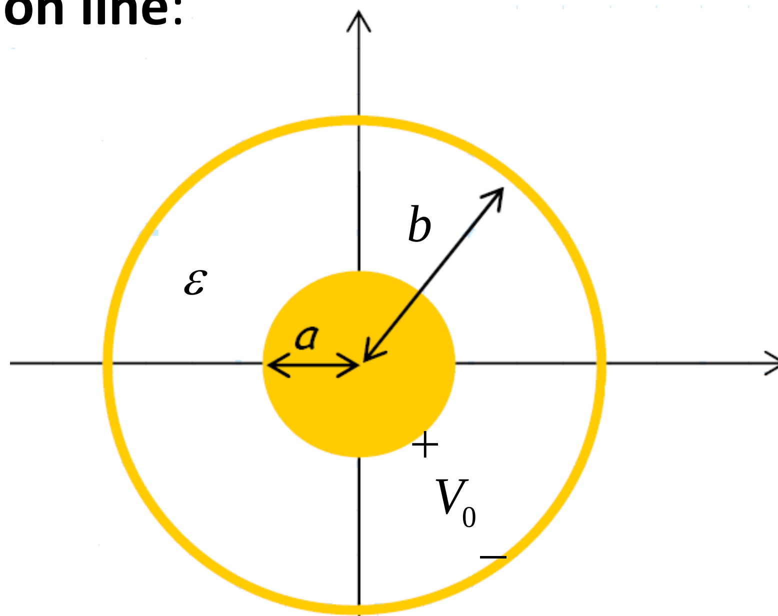
$$C = C_1 + C_2$$

But **you** knew this! The total capacitance of two capacitors in **parallel** is equal to the **sum** of **each** capacitance.

Capacitance of a Coaxial Transmission Line

- Recall the geometry of a **coaxial transmission line**:





<u>Coax Cross-Section</u>

- In earlier problem, you can determine, the **surface charge density** on the **inner** conductor is:

$$\rho_{sa}(\bar{r}) = \frac{\varepsilon V_0}{\ln[b/a]}\frac{1}{a} \qquad \rho = a$$

Capacitance of a Coaxial Transmission Line (contd.)

- The **total charge** Q on the **inner** conductor of a coax of length l is determined by **integrating** the surface charge density across the **conductor surface**:

$$Q = \iint_{S_+} \rho_{s+}(\bar{r})ds$$

$$Q = \int_0^l \int_0^{2\pi} \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \rho d\phi dz$$

$$Q = \frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \rho \int_0^l \int_0^{2\pi} d\phi dz$$

$$Q = \left[\frac{\varepsilon V_0}{\ln[b/a]} \frac{1}{a} \rho \right]_{\rho=a} \int_0^l \int_0^{2\pi} d\phi dz$$

$$\therefore Q = \frac{\varepsilon V_0}{\ln[b/a]} 2\pi l$$

Capacitance of a Coaxial Transmission Line (contd.)

- We can now determine the **capacitance** of this coaxial line!
- Since $C = Q/V$, and since the **potential difference** between the conductors is $V = V_0$, we find:

$$C = \frac{Q}{V} = \left(\frac{\varepsilon V_0}{\ln[b/a]} 2\pi l \right)\left(\frac{1}{V_0} \right) \Rightarrow C = \frac{2\pi\varepsilon}{\ln[b/a]} l$$

- This value represents the capacitance of a coaxial line of length l. A more useful expression is the capacitance of a coaxial line **per unit length** (e.g. farads/meter). We find this simply by **dividing** it by length l:

$$\frac{C}{l} = \frac{2\pi\varepsilon}{\ln[b/a]} \qquad \left[\frac{Farads}{metre}\right]$$

Capacitance of a Coaxial Transmission Line (contd.)

$$C = \frac{2\pi\varepsilon}{\ln\left[b/a\right]} l$$

Note the **longer** the transmission line, the **greater** the capacitance!

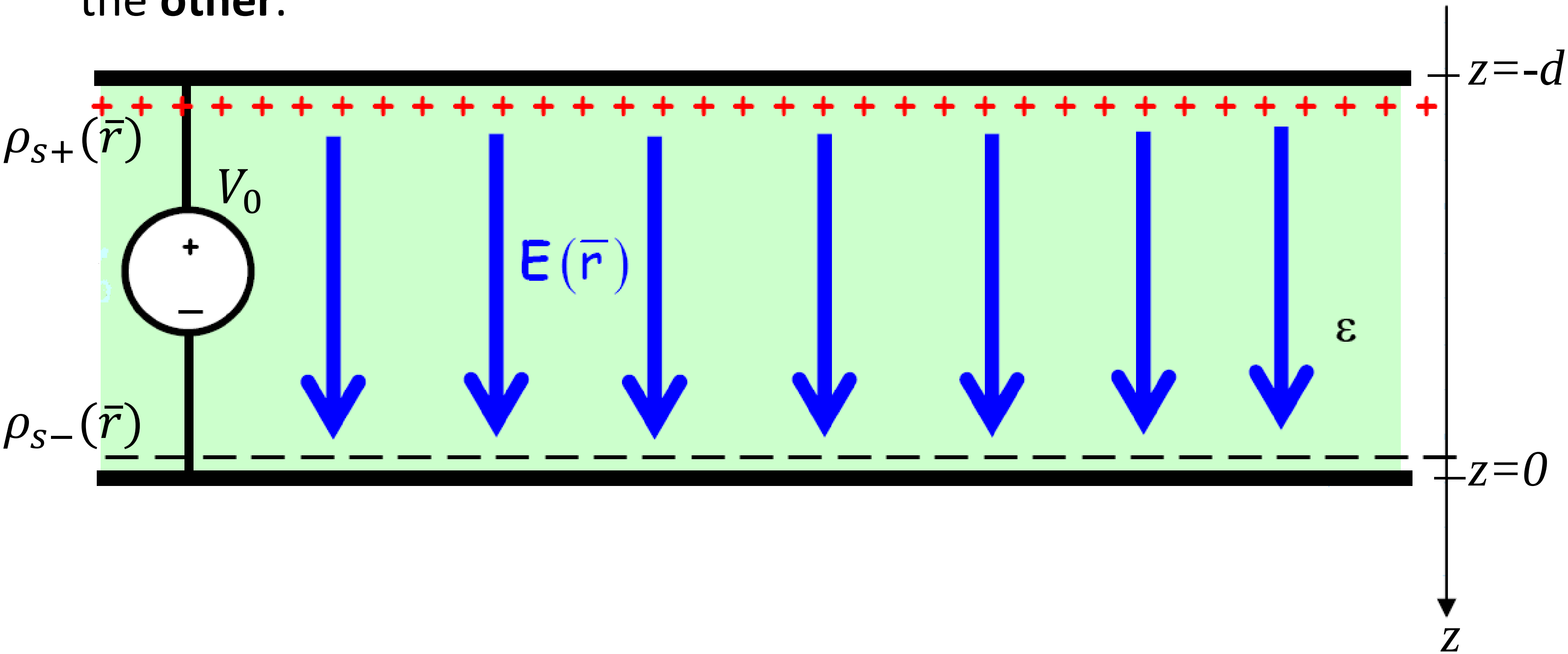
This can cause **great difficulty** if the voltage across the transmission line conductors is **time varying** (as it almost certainly will be!).

For **long** transmission lines, engineers cannot consider a transmission line simply as a "**wire**" conductor that connects circuit elements together. Instead, capacitance (and inductance) make the transmission line **itself** a **circuit element**!

In this case, engineers must use **transmission line theory** to analyze circuits!

Energy Storage in Capacitors

- Recall in a **parallel plate capacitor**, a surface charge distribution $\rho_{s+}(\bar{r})$ is created on **one** conductor, while charge distribution $\rho_{s-}(\bar{r})$ is created on the **other**.



Q: How much **energy** is stored by these charges?

Energy Storage in Capacitors (contd.)

- We learnt that the energy **stored** by a **charge distribution** is:

$$W_e = \frac{1}{2}\iiint_v \rho_v(\bar{r})V(\bar{r})dv$$

- The **equivalent** equation for **surface** charge distributions is:

$$W_e = \frac{1}{2}\iint_S \rho_s(\bar{r})V(\bar{r})dS$$

- For the parallel plate capacitor, we must integrate over **both** plates:

$$W_e = \frac{1}{2}\iint_{S_+} \rho_{s+}(\bar{r})V(\bar{r})dS + \frac{1}{2}\iint_{S_-} \rho_{s-}(\bar{r})V(\bar{r})dS$$

- But on the **top** plate (i.e., S_+), we know that: $V(z=-d) = V_0$

- While on the **bottom** plate (i.e., S_-): $V(z=0) = 0$

- Therefore: $W_e = \frac{V_0}{2}\iint_{S_+} \rho_{s+}(\bar{r})dS + \frac{0}{2}\iint_{S_-} \rho_{s-}(\bar{r})dS$ → $W_e = \frac{V_0}{2}\iint_{S_+} \rho_{s+}(\bar{r})dS$

$$\therefore W_e = \frac{1}{2}QV_0 \rightarrow W_e = \frac{1}{2}CV_0^2$$

Energy Storage in Capacitors (contd.)

$$W_e = \frac{1}{2} C V^2$$

It shows that the **energy stored** within a capacitor is proportional to the product of its capacitance and the squared value of the voltage across the capacitor.

- Recall that we also can determine the stored energy from the **fields** within the dielectric:

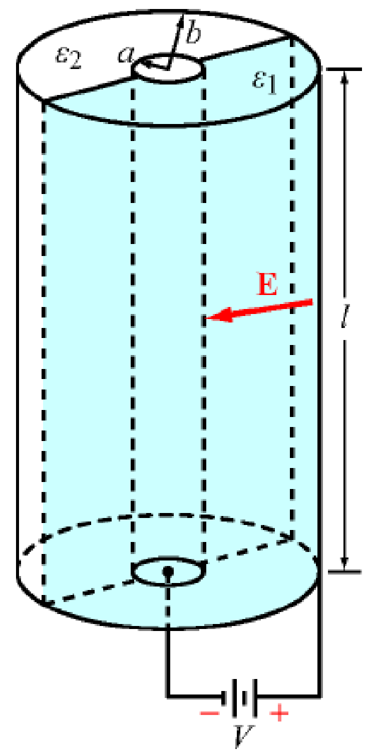
$$W_e = \frac{1}{2} \iiint_v \vec{D}(\bar{r}).\vec{E}(\bar{r})dv \quad \Rightarrow \quad W_e = \frac{1}{2} \frac{\varepsilon V^2}{d^2} (volume)$$

- Here $volume = Sd$, therefore:

$$W_e = \frac{1}{2} \frac{\varepsilon S}{d} V^2 \quad \Rightarrow \quad W_e = \frac{1}{2} \frac{\varepsilon S}{d} V^2$$

Example – 2

- A coaxial capacitor consists of two concentric, conducting, cylindrical surfaces, one of radius a and another of radius b. The insulating layer separating the two conducting surfaces is divided equally into two semi-cylindrical sections, one filled with dielectric ε_1 and the other filled with dielectric ε_2.



(a) Develop an expression for C in terms of the length l and the given quantities.

(b) Evaluate the value of C for a = 2 mm, b = 6 mm, ε_{r1} = 2, ε_{r2} = 4, and l = 4 cm.

Example – 2 (contd.)

(a) For the indicated voltage polarity, the **electric field** inside the capacitor exists in only the dielectric materials and **points radially inward**. Let $\vec{E}_1$ be the field in dielectric ε_1 and $\vec{E}_2$ be the field in dielectric ε_2.

(b) At the interface between the two dielectric sections, $\vec{E}_1$ is parallel to $\vec{E}_2$ and both are tangential to the interface.

(a) Since boundary conditions require that the tangential components of $\vec{E}_1$ and $\vec{E}_2$ be the same, it follows that:

$$\vec{D}_1 = \varepsilon_1 \vec{E}_1 = \rho_{s1}\hat{a}_n$$

- At r = a (surface of inner conductor), in medium 1, the boundary condition on $\vec{D}$, leads to:

$$\vec{D}_1 = \varepsilon_1 \vec{E}_1 = \rho_{s1}\hat{a}_n \Rightarrow -\varepsilon_1 E\hat{a}_\rho = \rho_{s1}\hat{a}_\rho \Rightarrow \vec{D}_1 = \varepsilon_1 \vec{E}_1 = \rho_{s1}\hat{a}_n$$

Example – 2 (contd.)

- Similarly, in medium 2: $\rho_{s2} = -\varepsilon_2 E$
- Thus, the electric fields will be the same in the two dielectrics, but the charge densities will be different along the two sides of the inner conducting cylinder.
- Since the same voltage applies for the two sections of the capacitor, we can treat them as two capacitors in parallel. For first half of the cylinder that includes dielectric ε_1, we can express:

$$C_1 = \frac{\pi\varepsilon_1}{\ln[b/a]} l$$

Only half cylinder

- Similarly:

$$C_2 = \frac{\pi\varepsilon_2}{\ln[b/a]} l$$

Therefore:

$$C = C_1 + C_2 = \frac{\pi l(\varepsilon_1 + \varepsilon_2)}{\ln[b/a]}$$

Magnetostatics

- **Magnetostatics** is the branch of electromagnetics dealing with the effects of electric charges in steady motion (i.e, steady current or DC).
- The fundamental law of magnetostatics is **Ampere's law of force**.
- Ampere's law of force is analogous to Coulomb's law in electrostatics.
- In magnetostatics, the magnetic field is produced by steady currents.
- The magnetostatic field does not allow for
  - inductive coupling between circuits
  - coupling between electric and magnetic fields

Magnetostatic Fields

- Static magnetic fields are characterized by $\vec{H}$ or $\vec{B}$.
- These are **analogous** to $\vec{E}$ or $\vec{D}$
- A magnetostatic field is produced by a constant current flow (or direct current).
- These currents could be due to **magnetization currents** as in permanent magnets, electron beam currents as in vacuum tubes, or conduction currents as in current-carrying wires.
- **Foremost**, study of magnetostatics is not a dispensable luxury.
- Its indispensable necessity.
- Motors, Transformers, Microphones, Compasses, Telephone Bell Ringers, Television Focusing Controls, Advertising Displays, Magnetically Levitated High Speed Trains, Volatile and Non-Volatile Memories, Magnetic Separators etc could not have been developed without an understanding of magnetostatic phenomena.

Maxwell's Equations for Magnetostatics

- From the **point form** of Maxwell's equations, we find that the **static** case reduces to another (in addition to electrostatics) pair of **decoupled differential equations** involving magnetic flux density $\vec{B}(\bar{r})$ and current density $\vec{J}(\bar{r})$:

$$\nabla.\vec{B}(\bar{r}) = 0$$

$$\nabla.\vec{B}(\bar{r}) = 0$$

- We know from the **Lorentz force equation** that the magnetic flux density $\vec{B}(\bar{r})$ will apply a **force** on current density $\vec{J}(\bar{r})$ flowing in volume dv equal to:

$$d\vec{F} = \left(\vec{J}(\bar{r}) \times \vec{B}(\bar{r})\right) dv$$

- Current density $\vec{J}(\bar{r})$ is of course expressed in units of **Amps/meter²**. The units of magnetic flux density $\vec{B}(\bar{r})$ are:

$$\frac{Newton.seconds}{Coulomb.meter} \equiv \frac{Weber}{meter^2} \equiv Tesla$$

The concept of **magnetic flux** is much more important and useful than the concept of electric flux, as there is **no** such thing as **magnetic charge**.

Maxwell's Equations for Magnetostatics (contd.)

First, we note that equations specify both the **divergence** and **curl** of magnetic flux density $\vec{B}(\bar{r})$, thus **completely** specifying this vector field.

Second, it is apparent that the magnetic flux density $\vec{B}(\bar{r})$ is **not conservative** (i.e, $\nabla \times \vec{B}(\bar{r}) = \mu_0 \vec{J}(\bar{r}) \neq 0$).

Finally, we note that the magnetic flux density **is** a **solenoidal** vector field (i.e, $\nabla . \vec{B}(\bar{r}) = 0$).

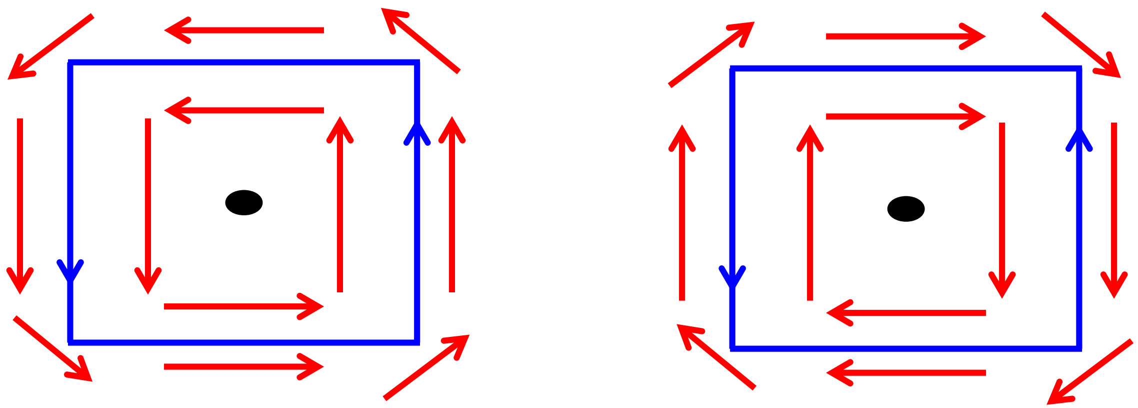
- Consider the **first** of the magnetostatic equations: $\nabla . \vec{B}(\bar{r}) = 0$

This equation is sometimes referred to as **Gauss's Law for magnetics**, for its obvious **similarity** to Gauss's Law of electrostatics.

This equation essentially states that the magnetic flux density does **not diverge** nor converge from any point. In other words, it states that there is no such thing as **magnetic charge** !

Maxwell's Equations for Magnetostatics (contd.)

- This of course is **consistent** with our understanding of **solenoidal** vector fields. The vector field will **rotate** about a point, but not diverge from it.



Q: Just what **does** the magnetic flux density $\vec{B}(\bar{r})$ rotate around ?

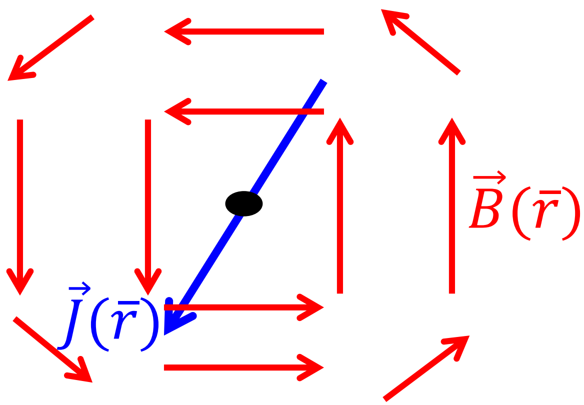
A: Look at the **second** magnetostatic equation!

- The **second** magnetostatic equation is referred to as **Ampere's Circuital Law**:

$$\nabla \times \vec{B}(\bar{r}) = \mu_0 \vec{J}(\bar{r})$$

This equation indicates that the magnetic flux density $\vec{B}(\bar{r})$ **rotates around** current density $\vec{J}(\bar{r})$ --the **source** of magnetic flux density is current!.

Maxwell's Equations for Magnetostatics (contd.)



The Integral Form of Magnetostatics

- Say, we evaluate the **surface integral** of the point form of **Ampere's Law** over some arbitrary surface S.

$$\iint_S \nabla \times \vec{B}(\bar{r}).\overline{ds} = \mu_0 \iint_S \vec{J}(\bar{r}).\overline{ds}$$

- Using **Stoke's Theorem**, we can write the **left** side of the above equation as:

$$\iint_S \nabla \times \vec{B}(\bar{r}).\overline{ds} = \oint_C \vec{B}(\bar{r}).\overline{dl}$$

- We also recognize that the **right** side of the equation is:

$$\mu_0 \iint_S \vec{J}(\bar{r}).\overline{ds} = \mu_0 I$$

- where I is the **current** flowing through surface S.

The Integral Form of Magnetostatics (contd.)

- Therefore, we find the integral form of **Ampere's Law** (Note the **direction** of I is defined by the **right-hand rule**): $\oint_C \vec{B}(\bar{r}).\overline{dl} = \mu_0 I$

Ampere's law states that the **line integral** of $\vec{B}(\bar{r})$ around a **closed contour** C is proportional to the **total current** I flowing **through** this closed contour ($\vec{B}(\bar{r})$ is **not** conservative!).

- Likewise, we can take a **volume integral** over both sides of the magnetostatic equation $\nabla.\vec{B}(\bar{r}) = 0$: $\iiint_v \nabla.\vec{B}(\bar{r})dv = 0$
- But wait! The left side can be rewritten using the **Divergence Theorem**

$$\iiint_v \nabla.\vec{B}(\bar{r})dv = \oiint_S \vec{B}(\bar{r}).\overline{ds}$$

where S is the **closed surface** that **surrounds** volume V.

- Therefore, we can write the integral form of $\nabla.\vec{B}(\bar{r}) = 0$ as: $\oiint_S \vec{B}(\bar{r}).\overline{ds} = 0$
- Summarizing, the **integral form** of the magnetostatic equations are:

$$\oiint_S \vec{B}(\bar{r}).\overline{ds} = 0 \qquad \oint_C \vec{B}(\bar{r}).\overline{dl} = \mu_0 I$$

Magnetic Vector Potential

Q: Given some **field** $\vec{B}(\bar{r})$, how can we determine the **source** $\vec{J}(\bar{r})$ that created it?

A: Easy! $\vec{J}(\vec{r}) = \dfrac{\nabla\times\vec{B}(\vec{r})}{\mu_0}$

Q: OK, given some **source** $\vec{J}(\bar{r})$, how can we determine what **field** $\vec{B}(\bar{r})$ it creates?

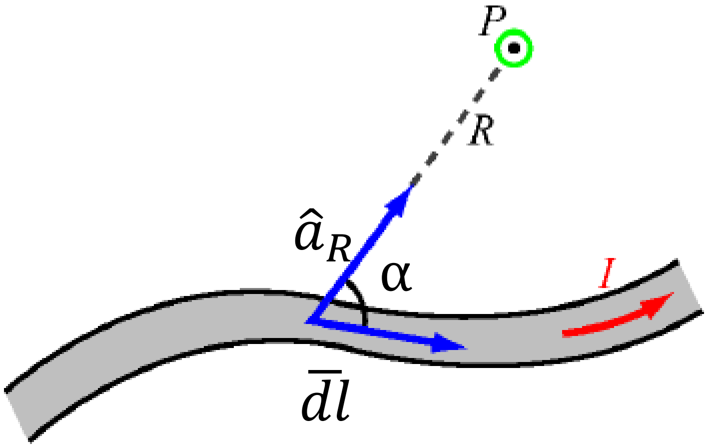
A: Magnetic Vector Potential **!**

Biot-Savart's Law

- It states that: differential magnetic field intensity $\overrightarrow{dH}(\bar{r})$ produced at point P, shown in figure, by the differential current element $Id\bar{l}$ is related as:

$$dH(\bar{r}) \propto Idl\sin\alpha$$

$$dH(\bar{r}) \propto \frac{1}{R^2}$$



Biot-Savart's Law (contd.)

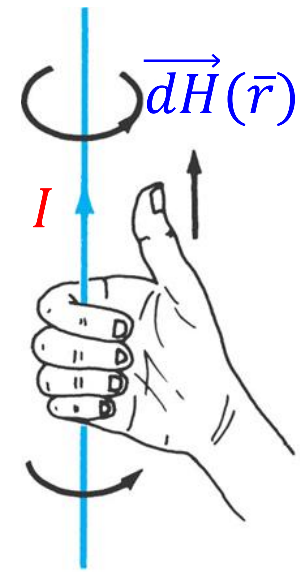
- Combining them together results into:

$$dH(\bar{r}) = \frac{k}{R^2} Idl \sin\alpha \xrightarrow{\text{In SI units}} dH(\bar{r}) = \frac{1}{4\pi R^2} Idl \sin\alpha$$

- From the definition of cross product, we can transform the equation in vector form as:

$$\overrightarrow{dH}(\bar{r}) = \frac{I\,\overline{dl} \times \hat{a}_R}{4\pi R^2} \Rightarrow \overrightarrow{dH}(\bar{r}) = \frac{I\,\overline{dl} \times \vec{R}}{4\pi R^3}$$

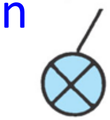
- This direction of $\overrightarrow{dH}(\bar{r})$ can be obtained from right-hand rule: **right-hand thumb points in the direction of current** **and** **the right hand fingers encircle the wire in the direction of $\overrightarrow{dH}(\bar{r})$.**



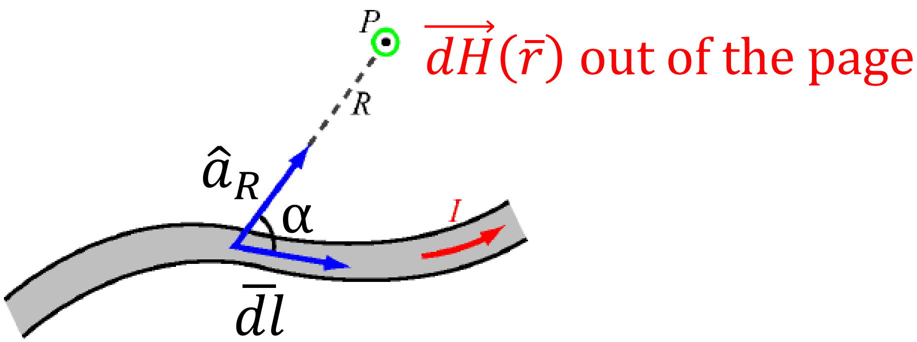
Biot-Savart's Law (contd.)

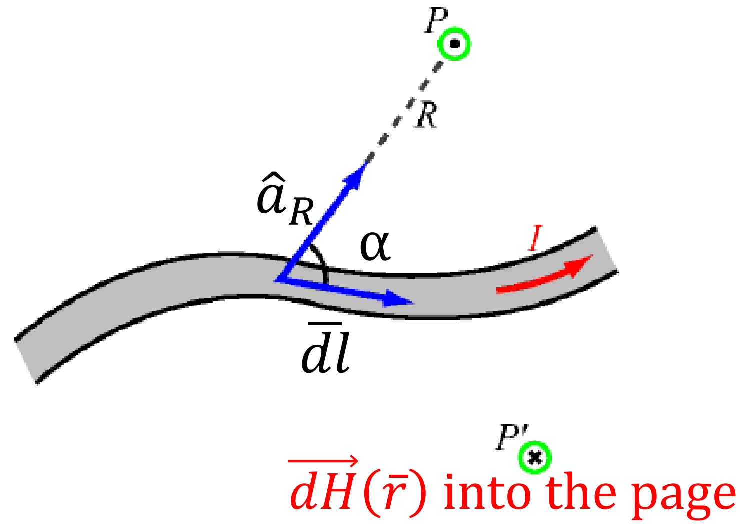
$\vec{H}(\bar{r})$ (or I) is out

$\vec{H}(\bar{r})$ (or I) is in



- It is a standard practice to represent the direction of $\vec{H}(\bar{r})$ (or current I) as:





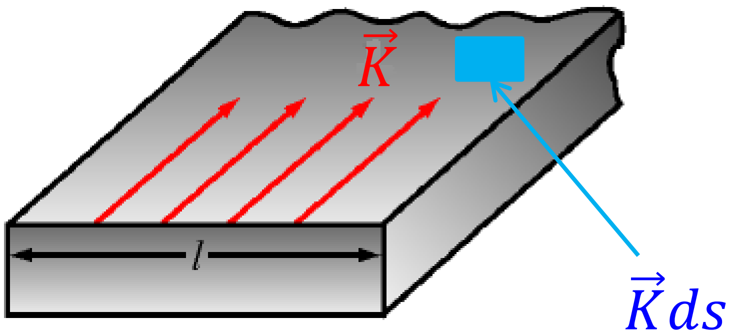
- For total $\vec{H}(\bar{r})$ due to a finite sized conductor, need to sum up the contributions due to all the current elements making up the conductor.
- Therefore the Biot-Savart law becomes:

$$\vec{H}(\bar{r}) = \frac{I}{4\pi}\int_L \frac{\bar{d}l \times \hat{a}_R}{R^2}$$

Magnetic field due to line current

Biot-Savart's Law (contd.)

- If we define $\vec{K}$ as the surface current density then the total magnetic field $\vec{H}(\bar{r})$ can be expressed as:



$$\vec{H}(\bar{r}) = \frac{1}{4\pi}\int_L \frac{\vec{K}\times\hat{a}_R}{R^2}ds$$

$$\vec{H}(\bar{r}) = \frac{1}{4\pi}\int_L \frac{K\,\overline{ds}\times\hat{a}_R}{R^2}$$

- Similarly, we can express the magnetic field $\vec{H}(\bar{r})$ due to volume current $\vec{J}$ as:

$$\vec{H}(\bar{r}) = \frac{1}{4\pi}\int_L \frac{\vec{J}\times\hat{a}_R}{R^2}dv$$