Lecture – 12

Date: 11.02.2016

- Electric Field Due to Line Charge (contd.)

Electric Field due to a Line Charge

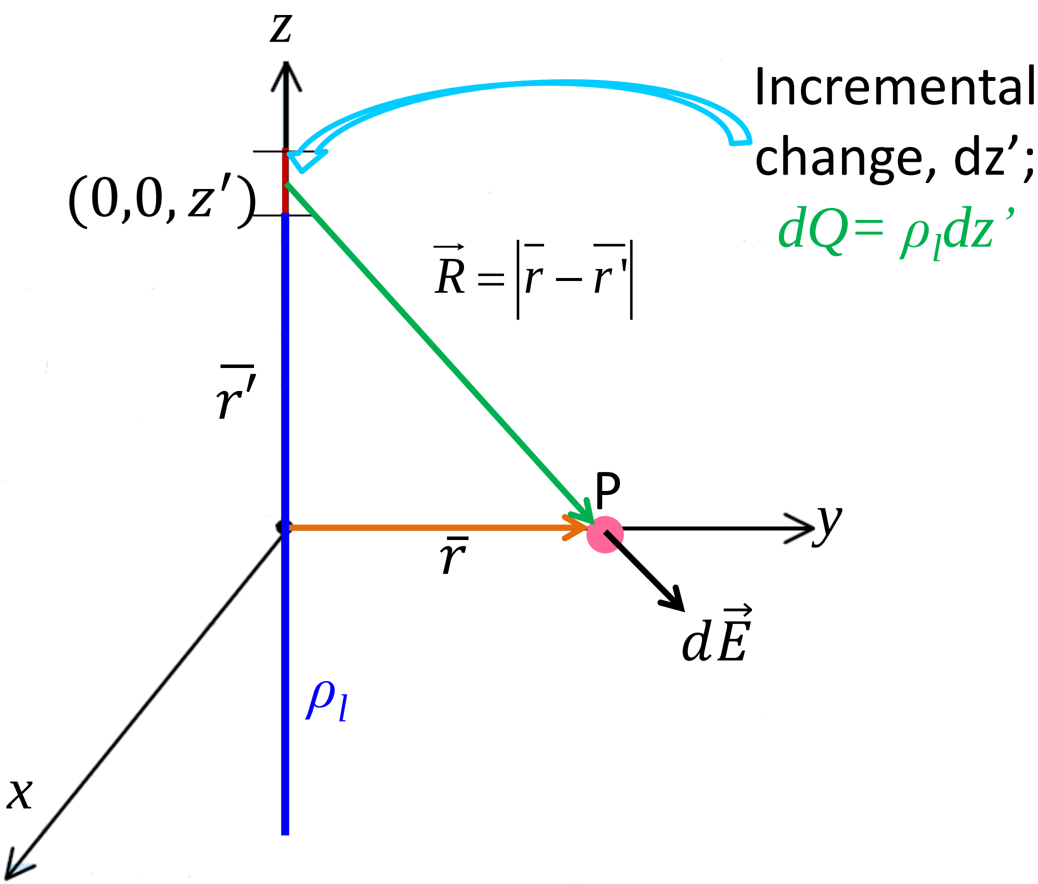
- Filament like distribution of charge density.
- For example, sharp beam in a cathode-ray tube or charged conductor of a very small radius.
- **Let us assume an infinite straight-line charge, with charge density ρ_l C/m, lying along the z-axis.**

Q: What electric field $\vec{E}(\bar{r})$ is produced by this line charge?

A: Apply Coulomb's Law.

Electric Field due to a Line Charge (contd.)

- For the calculation of electric field $\vec{E}$ at $P(0,y,0)$, the first step is to determine the incremental field at P due to the incremental charge $dQ= \rho_l dz'$



We have:

$$\boxed{\overline{dE} = \frac{dQ}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} = \frac{\rho_l dz'}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}}$$

$$\boxed{\bar{r} = y\hat{a}_y = \rho\hat{a}_\rho} \qquad \boxed{\bar{r}' = z'\hat{a}_z}$$

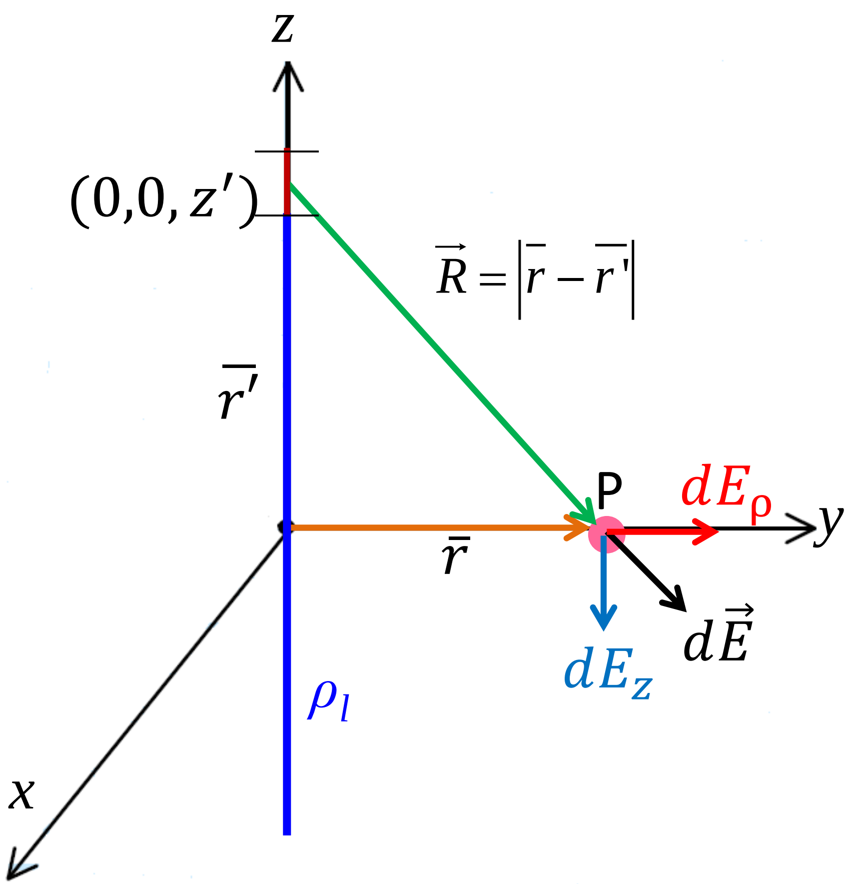
$$\boxed{\therefore \vec{R} = \bar{r} - \bar{r}' = \rho\hat{a}_\rho - z'\hat{a}_z}$$

Electric Field due to a Line Charge (contd.)

$$\Rightarrow \overrightarrow{dE} = \frac{\rho_l dz'}{4\pi\varepsilon_0}\frac{\left(\rho\hat{a}_\rho - z'\hat{a}_z\right)}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\therefore \overrightarrow{dE} = \underbrace{\frac{\rho_l \rho dz'}{4\pi\varepsilon_0\left(\rho^2 + z'^2\right)^{3/2}}}_{d\vec{E}_\rho}\hat{a}_\rho - \underbrace{\frac{\rho_l z' dz'}{4\pi\varepsilon_0\left(\rho^2 + z'^2\right)^{3/2}}}_{d\vec{E}_z}\hat{a}_z$$

$$\therefore \overrightarrow{dE} = \hat{a}_\rho dE_\rho - \hat{a}_z dE_z$$



Electric Field due to a Line Charge (contd.)

Now:

$$E_\rho = \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l \rho}{4\pi\varepsilon_0 \left(\rho^2 + z'^2\right)^{3/2}} dz' = \frac{\rho_l \rho}{4\pi\varepsilon_0} \int_{z'=-\infty}^{z'=\infty} \frac{dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\Rightarrow E_\rho = \frac{\rho_l \rho}{4\pi\varepsilon_0} \left[\frac{1}{\rho^2} \frac{z'}{\left(\rho^2 + z'^2\right)^{1/2}} \right]_{z'=-\infty}^{z'=\infty}$$

$$\boxed{\therefore E_\rho = \frac{\rho_l}{2\pi\varepsilon_0 \rho}}$$

AND:

$$E_z = \frac{\rho_l}{4\pi\varepsilon_0} \int_{z'=-\infty}^{z'=\infty} \frac{z' dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\boxed{\therefore E_z = \frac{\rho_l}{4\pi\varepsilon_0} \times (0) = 0}$$

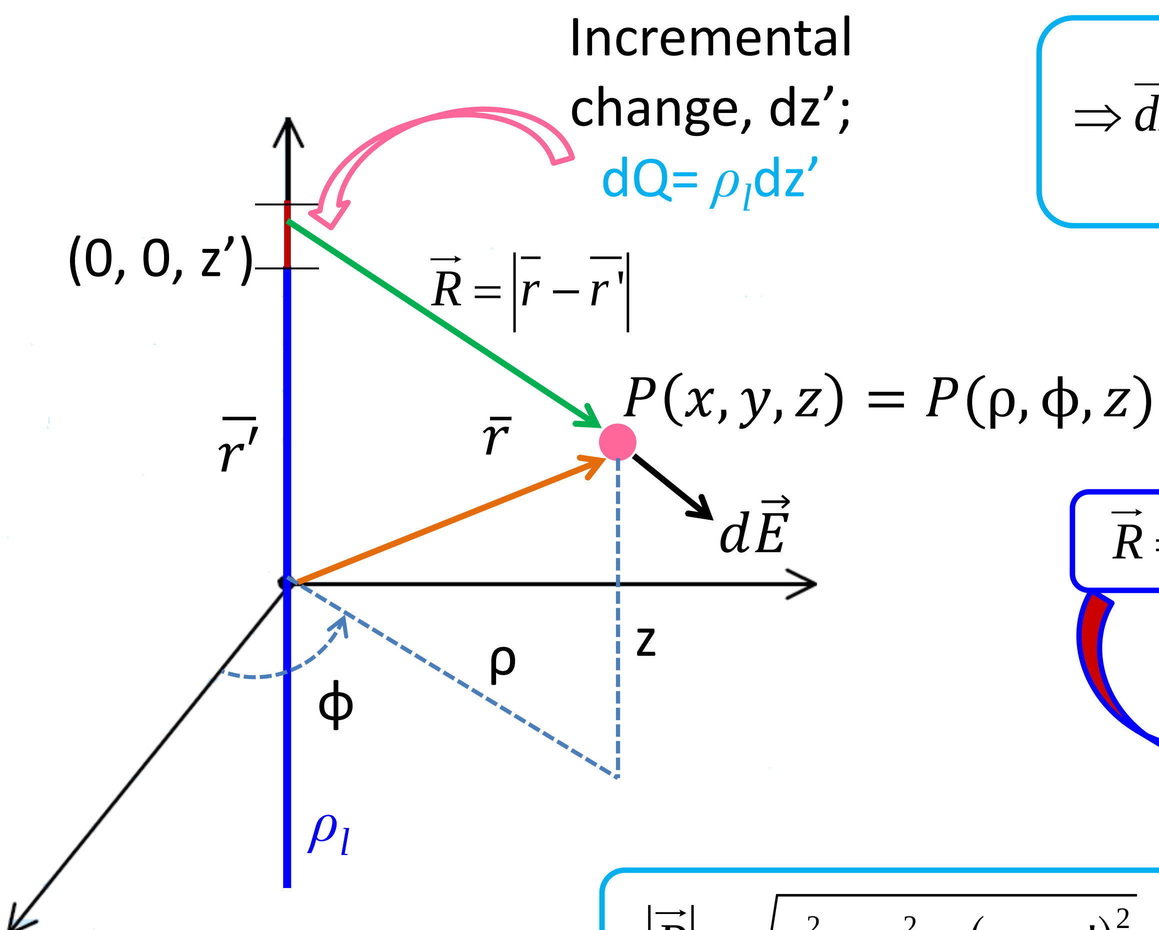
Therefore:

$$\boxed{\vec{E}\left(\bar{r}\right) = E_\rho \hat{a}_\rho - E_z \hat{a}_z = \frac{\rho_l}{2\pi\varepsilon_0 \rho} \hat{a}_\rho}$$

BTW, there are multiple ways of solving this problem.
You can master this art through practice!

Example – 1

- Determine electric field $\vec{E}$ at P(x, y, z)



$$\Rightarrow \overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0}\frac{\left(\bar{r} - \bar{r}'\right)}{\left|\bar{r} - \bar{r}'\right|^3} = \frac{\rho_l dz'}{4\pi\varepsilon_0}\frac{\left(\bar{r} - \bar{r}'\right)}{\left|\bar{r} - \bar{r}'\right|^3}$$

$$\vec{R} = \bar{r} - \bar{r}' = x\hat{a}_x + y\hat{a}_y + \left(z - z'\right)\hat{a}_z$$

$$\vec{R} = \rho\hat{a}_\rho + \left(z - z'\right)\hat{a}_z$$

$$\therefore \left|\vec{R}\right| = \sqrt{x^2 + y^2 + \left(z - z'\right)^2} = \sqrt{\rho^2 + \left(z - z'\right)^2}$$

Example – 1 (contd.)

Now:

$$\vec{E}(\bar{r}) = \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l dz'}{4\pi\varepsilon_0} \frac{\rho \hat{a}_\rho + (z-z')\hat{a}_z}{\left[\rho^2 + (z-z')^2\right]^{3/2}}$$

$$\vec{E}(\bar{r}) = \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l \rho dz'}{4\pi\varepsilon_0\left[\rho^2 + (z-z')^2\right]^{3/2}}\hat{a}_\rho + \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l (z-z') dz'}{4\pi\varepsilon_0\left[\rho^2 + (z-z')^2\right]^{3/2}}\hat{a}_z$$

$$\Rightarrow \vec{E}(\bar{r}) = \frac{\rho_l}{4\pi\varepsilon_0}\left\{\left[\hat{a}_\rho \frac{\rho}{\rho^2}\frac{-(z-z')}{\left[\rho^2 + (z-z')^2\right]^{1/2}}\right]_{z'=-\infty}^{z'=\infty} + \left[\hat{a}_z \frac{-(z-z')}{\left[\rho^2 + (z-z')^2\right]^{1/2}}\right]_{z'=-\infty}^{z'=\infty}\right\}$$

$$\boxed{= \frac{\rho_l}{4\pi\varepsilon_0}\left[\hat{a}_\rho \frac{2}{\rho} + \hat{a}_z \times (0)\right]}$$

$$\boxed{\therefore \vec{E}(\bar{r}) = \frac{\rho_l}{2\pi\varepsilon_0 \rho}\hat{a}_\rho}$$

same result

Electric Field due to a Line Charge (contd.)

$$\vec{E}(\bar{r}) = \frac{\rho_l}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

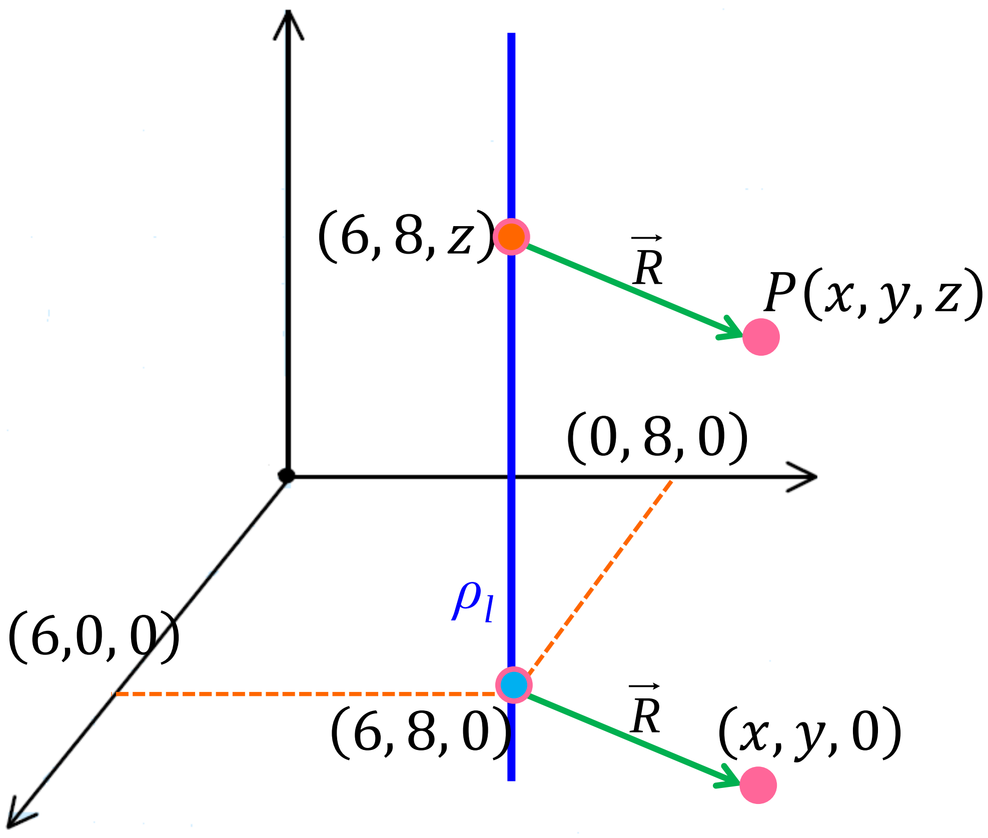
Recall unit vector $\hat{a}_\rho$ is the direction that **points away from** the z-axis.

Thus, the electric field produced by the uniform line charge points away from the line charge, just like the electric field produced by a point charge points away from the charge.

- Note the **magnitude** of the electric field is **proportional** to $1/\rho$, therefore the electric field **diminishes** as we get further from the line charge.
- Note however, the electric field **does not diminish** as **quickly** as that generated by a point charge. Recall in the case of point charge, the magnitude of the electric field diminishes as $1/r^2$.

Example – 2

- **Oh yes!** It is important to note that **not all the line charges** will be located along the z-axis.
- For example, let us consider an infinite line charge parallel to the z-axis at $x = 6, y = 8$. We wish to find $\vec{E}$ at the general field point $P(x, y, z)$.



$$\vec{R} = (x-6)\hat{a}_x + (y-8)\hat{a}_y$$

$$\therefore \left|\vec{R}\right| = \rho = \sqrt{(x-6)^2 + (y-8)^2}$$

$$\Rightarrow \hat{a}_\rho = \frac{(x-6)\hat{a}_x + (y-8)\hat{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

$$\boxed{\therefore \vec{E} = \frac{\rho_l}{2\pi\varepsilon_0} \frac{(x-6)\hat{a}_x + (y-8)\hat{a}_y}{(x-6)^2 + (y-8)^2}}$$

again, not a function of **z**

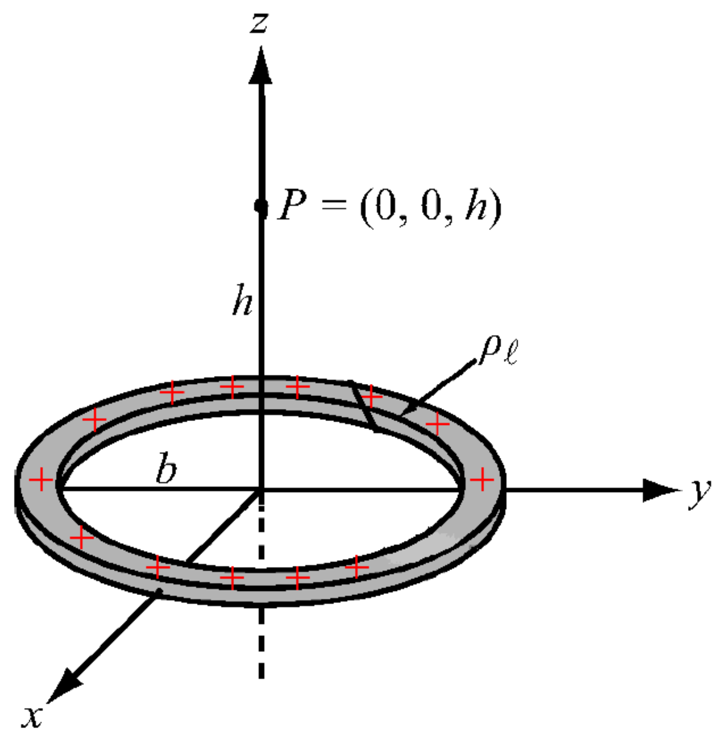
Example – 3

- A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_l. The ring, with its center at (0,0,0), resides in free medium and is positioned in the xy-plane.

i. Determine $\vec{E}$ at point $P = (0, 0, h)$ along the axis of the ring at a distance h from the center.

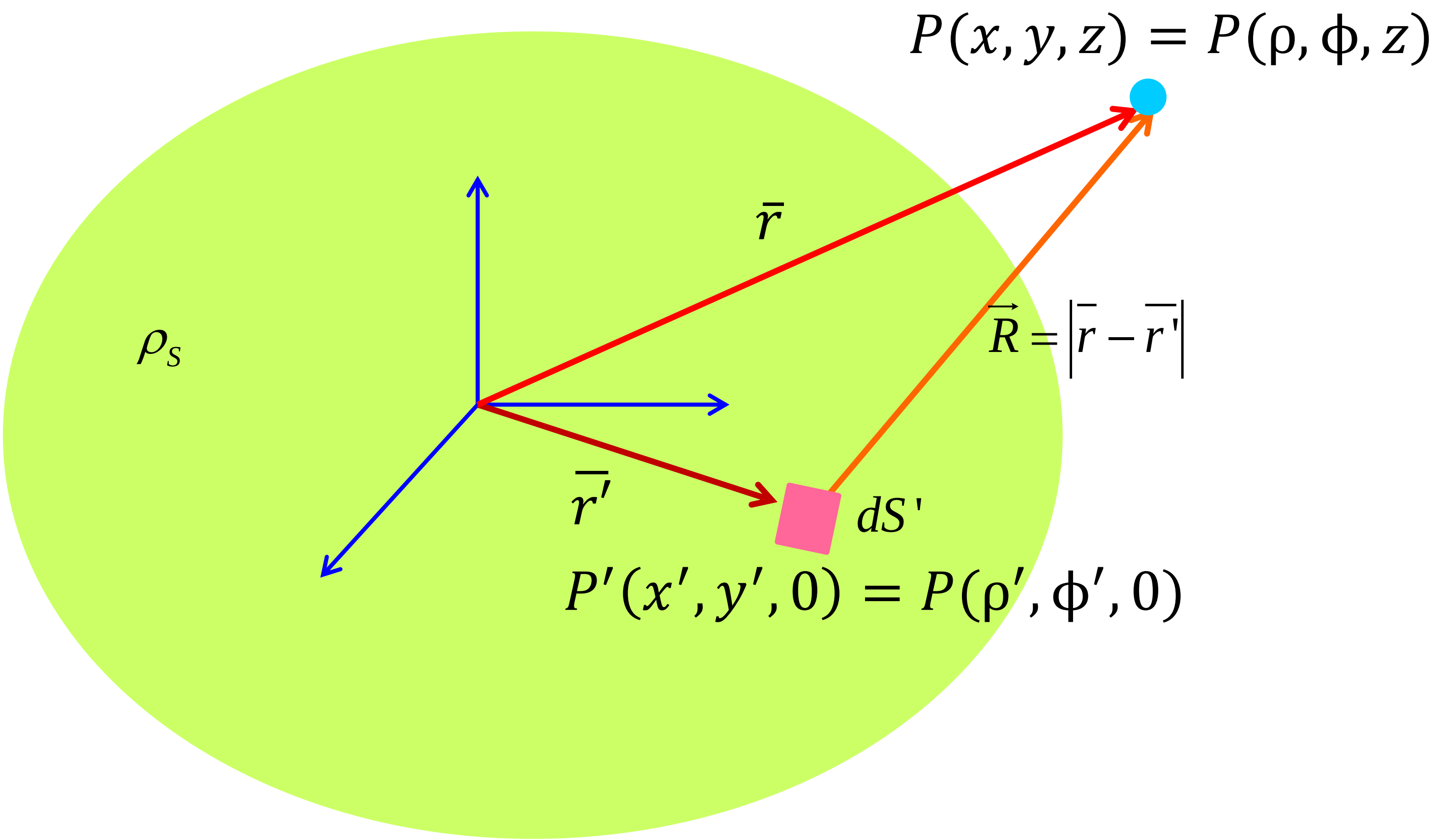
ii. What values of h gives the maximum value of $\vec{E}$.

iii. If the total charge on the ring is Q, find $\vec{E}$ as $b \to 0$.



Electric Field due to a Surface Charge

- Consider a **disk of** **radius** **a**, centered at the origin, and lying entirely on the xy-plane (i.e., z =0 plane). Let us also assume that this disk carries a uniform charge density of ρ_S C/m^2.



Challenge: determine electric field at point P

Electric Field due to a Surface Charge (contd.)

- From Coulomb's Law:

$$\vec{E}(\bar{r}) = \iint_S \frac{\rho_S(r')}{4\pi\varepsilon_0} \frac{\bar{r}-\bar{r}'}{\left|\bar{r}-\bar{r}'\right|^3} dS'$$

$$dS' = \rho' d\rho' d\phi' \qquad 0 < \rho' < a \qquad 0 < \phi' < 2\pi$$

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \qquad \bar{r}' = x'\hat{a}_x + y'\hat{a}_y$$

$$\therefore \vec{R} = \bar{r} - \bar{r}' = (x - x')\hat{a}_x + (y - y')\hat{a}_y + z\hat{a}_z$$

Convert to cylindrical

$$\therefore \vec{R} = \bar{r} - \bar{r}' = (x - \rho'\cos\phi')\hat{a}_x + (y - \rho'\sin\phi')\hat{a}_y + z\hat{a}_z$$

$$\left|\vec{R}\right|^3 = \left|\bar{r} - \bar{r}'\right|^3 = \left[(x - \rho'\cos\phi')^2 + (y - \rho'\sin\phi')^2 + z^2\right]^{3/2}$$

Electric Field due to a Surface Charge (contd.)

$$\boxed{\vec{E}\left(\bar{r}\right)=\iint_S \frac{\rho_S\left(\bar{r}'\right)}{4\pi\varepsilon_0}\frac{\bar{r}-\bar{r}'}{\left|\bar{r}-\bar{r}'\right|^3}dS'}$$

$$=\frac{\rho_S}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{\left(x-\rho'\cos\phi'\right)\hat{a}_x+\left(y-\rho'\sin\phi'\right)\hat{a}_y+z\hat{a}_z}{\left[\left(x-\rho'\cos\phi'\right)^2+\left(y-\rho'\sin\phi'\right)^2+z^2\right]^{3/2}}\rho' d\rho' d\phi'$$

Yikes! What a **mess**!

- To **simplify** our integration let's determine the electric field $\vec{E}(\bar{r})$ along the **z-axis** only. In other words, set x = 0 and y = 0.

$$\Rightarrow \vec{E}(x=0,y=0,z)=\frac{\rho_S}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{\left(0-\rho'\cos\phi'\right)\hat{a}_x+\left(0-\rho'\sin\phi'\right)\hat{a}_y+z\hat{a}_z}{\left[\left(0-\rho'\cos\phi'\right)^2+\left(0-\rho'\sin\phi'\right)^2+z^2\right]^{3/2}}\rho' d\rho' d\phi'$$

Electric Field due to a Surface Charge (contd.)

$$\Rightarrow \vec{E}(\bar{r}) = \frac{\rho_S}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(-\rho'\cos\phi')\hat{a}_x + (-\rho'\sin\phi')\hat{a}_y + z\hat{a}_z}{\left[(\rho'\cos\phi')^2 + (\rho'\sin\phi')^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi'$$

$$= \frac{\rho_S}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(-\rho'\cos\phi')\hat{a}_x + (-\rho'\sin\phi')\hat{a}_y + z\hat{a}_z}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi'$$

$$= \frac{\rho_S}{4\pi\varepsilon_0}\left[\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(-\rho'\cos\phi')\hat{a}_x}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi' + \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(-\rho'\sin\phi')\hat{a}_y}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi' + \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{z\hat{a}_z}{\left[\rho'^2 + z^2\right]^{3/2}}\rho' d\rho' d\phi'\right]$$

=0

We know:

$$\int_0^{2\pi}\sin\phi d\phi = 0 = \int_0^{2\pi}\cos\phi d\phi$$

Electric Field due to a Surface Charge (contd.)

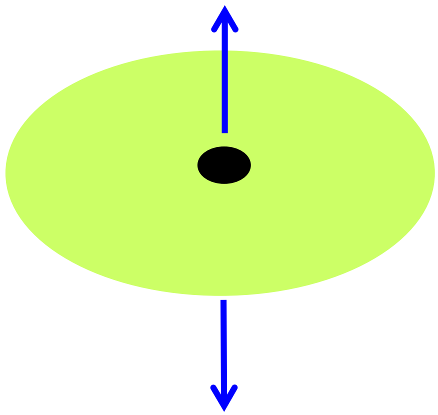
$$\vec{E}(x=0,y=0,z)=\frac{\rho_S}{4\pi\varepsilon_0}\left[\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{z\hat{a}_z}{\left[\rho'^2+z^2\right]^{3/2}}\rho' d\rho' d\phi'\right]$$

$$\vec{E}(x=0,y=0,z)=\begin{cases}\dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z>0\\[2ex] -\dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[1+\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z<0\end{cases}$$

From this expression, we can conclude **two** things. The first is that **above** the disk ($z > 0$), the electric field points in the direction $\hat{a}_z$, and below the disk ($z < 0$), it points in the direction $-\hat{a}_z$.

Electric Field due to a Surface Charge (contd.)

- What a surprise (not)! The electric field **points away** from the charge. It appears to be **diverging** from the charged disk



- Likewise, it is evident that as we move further and **further from** the disk, the electric field will **diminish**. In fact, as distance z goes to **infinity**, the magnitude of the electric field approaches **zero**. This of course is similar to the **point** or **line** charge; as we move an infinite distance away, the electric field diminishes to **nothing**.

Electric Field due to a Surface Charge (contd.)

- Say that we have a **very large** charge disk. So large, in fact, that its radius ***a*** approaches **infinity** !

Q: What electric field is created by this infinite plane?

A: We **already** know! Just evaluate the charge disk solution for the case where the disk **radius** ***a*** is **infinity**.

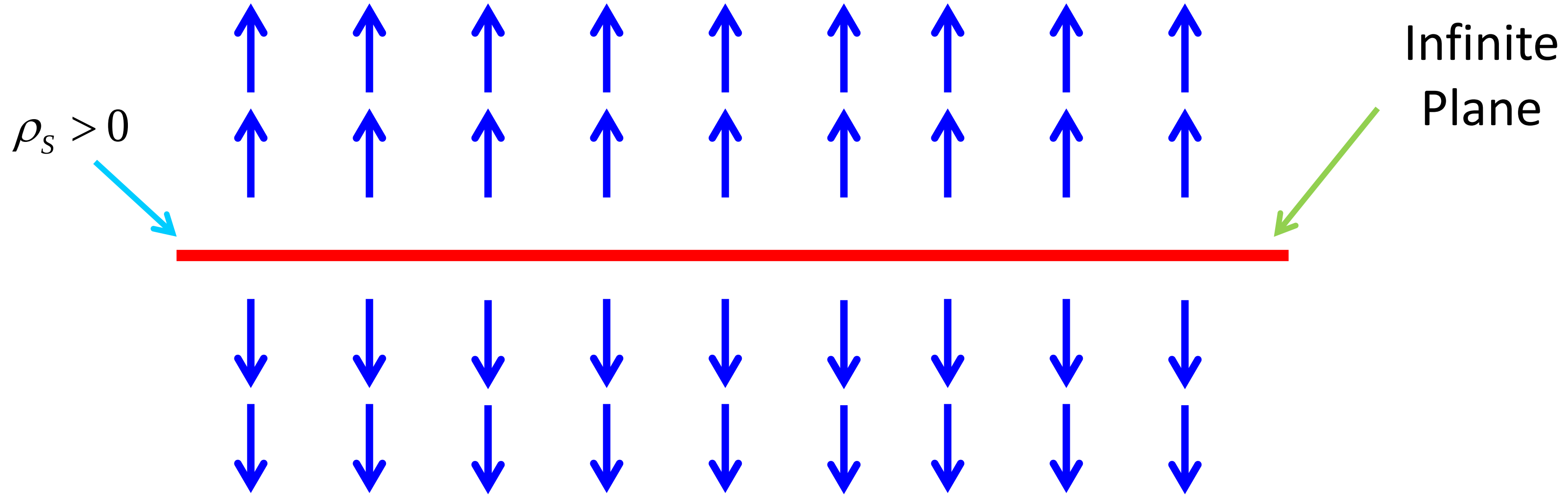
$$\lim_{a\to\infty} \vec{E}(x=0, y=0, z) = \begin{cases} \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[1-\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z>0 \\ -\dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z\left[1+\dfrac{z}{\sqrt{z^2+a^2}}\right] & \text{If } z<0 \end{cases}$$

$$= \begin{cases} \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z & \text{If } z>0 \\ \dfrac{-\rho_S}{2\varepsilon_0}\hat{a}_z & \text{If } z<0 \end{cases}$$

Think about what **this** says!

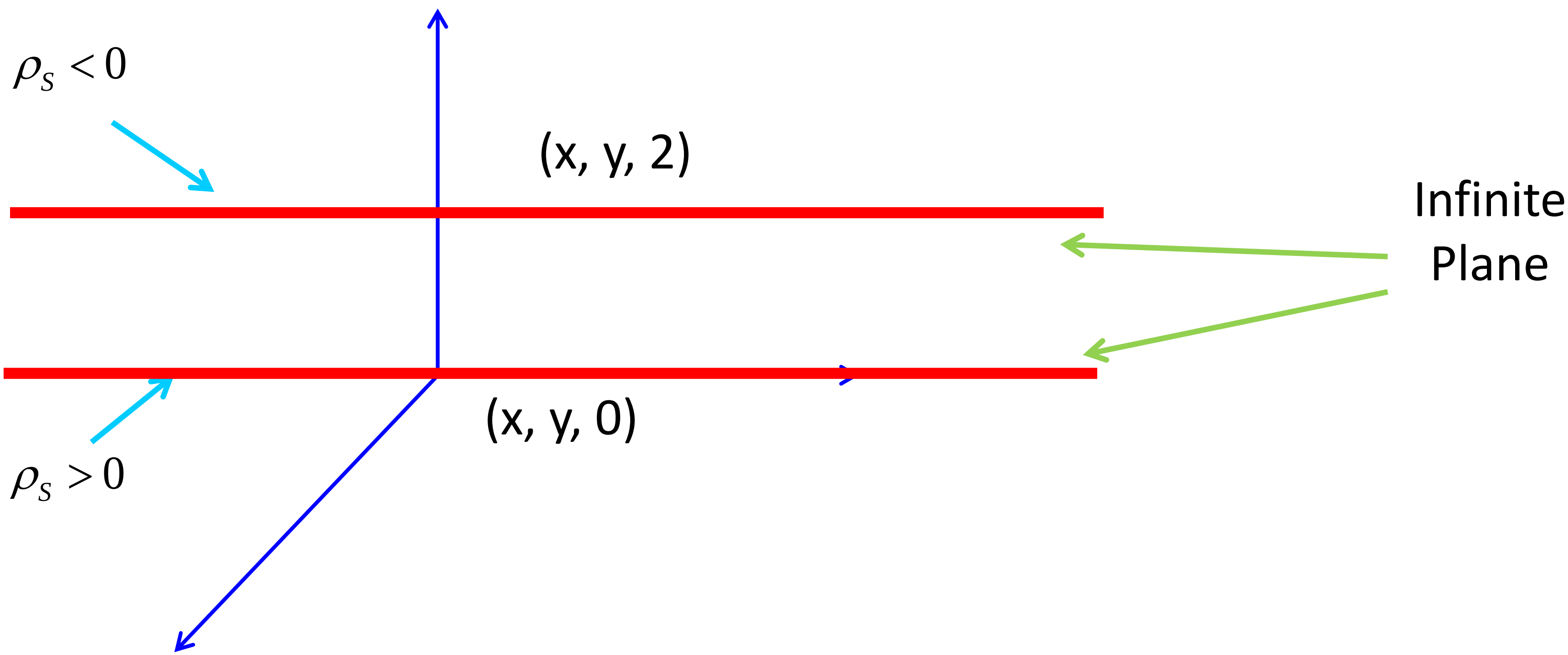
Electric Field due to a Surface Charge (contd.)

- First, we note that the electric field **points away** from the plane if $\boldsymbol{\rho_S}$ is positive, and toward the plane if $\boldsymbol{\rho_S}$ is negative.
- Second, we notice that the magnitude of the electric field is a **constant**—the magnitude is **independent** of the distance from the infinite plane!



Example – 5

- An infinite sheet with uniform surface charge density **ρ_s is located at z=0 (x-y plane)**, and another infinite sheet with **$-\rho_s$ is located at z=2m**, both in free space. Determine $\vec{E}$ everywhere.



Example – 5 (contd.)

For the sheet at z =0

$$\vec{E}_1 = \begin{cases} \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for } z > 0 \\ \dfrac{-\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for } z < 0 \end{cases}$$

For the sheet at z =2m

$$\vec{E}_2 = \begin{cases} \dfrac{-\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for } z > 2\text{m} \\ \dfrac{\rho_S}{2\varepsilon_0}\hat{a}_z & \text{for } z < 2\text{m} \end{cases}$$

Therefore:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \begin{cases} 0 & \text{for } z < 0 \\ \dfrac{\rho_S}{\varepsilon_0}\hat{a}_z & \text{for } 0 < z < 2\text{m} \\ 0 & \text{for } z > 2\text{m} \end{cases}$$

Any thought on this outcome !

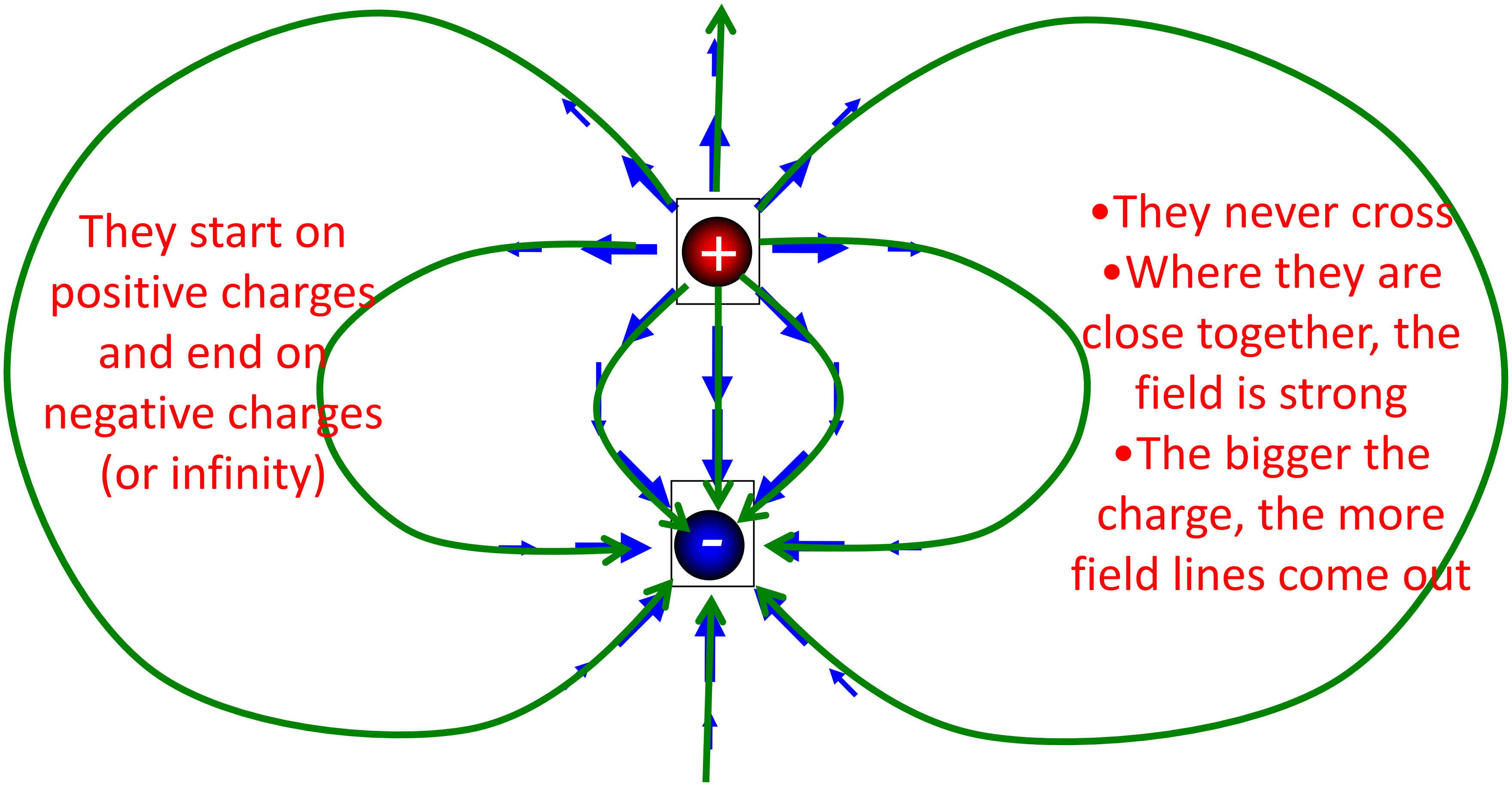
Electric Field of a Charged Sphere

Involves tricky triple integration. Lets first learn Gauss Theorem. It will simplify this problem.

Electric Field Lines

- Electric *field lines* are a pictorial representation of the electric field. These consist of directed lines indicating the direction of electric field at various points in space.
- There is no rule as to how many lines are to be shown. However, it is customary to draw number of lines proportional to the charge. Thus if N number of lines are drawn from or into a charge Q, 2N number of lines would be drawn for charge 2Q.
- Lines are dense close to a source of the electric field and become sparse as one moves away.
- Lines originate from a positive charge and end either on a negative charge or move to infinity.
- Lines of force due to a solitary negative charge is assumed to start at infinity and end at the negative charge.
- Field lines do not cross each other. (if they did, the field at the point of crossing will not be uniquely defined.)

Electric Field Lines (contd.)



They start on positive charges and end on negative charges (or infinity)

- They never cross
- Where they are close together, the field is strong
- The bigger the charge, the more field lines come out

Electric Field Lines (contd.)

Note:

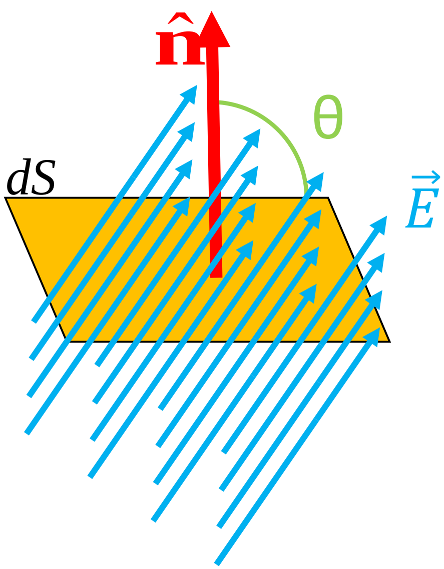
- **Near field:** very close to a charge, the field due to that charge predominates. Therefore, the lines are radial and spherically symmetric.

- **Far field:** far from the system of charges, the pattern should look like that of a single point charge of value $Q = \sum_i Q_i$. Thus, the lines should be radially inward or outward, unless $Q = 0$.

- The direction of the electric field vector $\overrightarrow{E}$ at a point is always tangent to the field lines.

Electric Flux

- Simply speaking, electric flux is the amount of electric field going through a surface. It is defined in terms of a direction, unit vector, perpendicular to the surface.



$$d\psi = \vec{E}.\hat{n}dS \quad \therefore d\psi = \left|\vec{E}\right| dS\cos\theta$$

- For an arbitrary surface S, the flux is obtained by integrating over all the surface elements.

$$\boxed{\psi = \int_S d\psi = \int_S \vec{E}.\hat{n}dS}$$

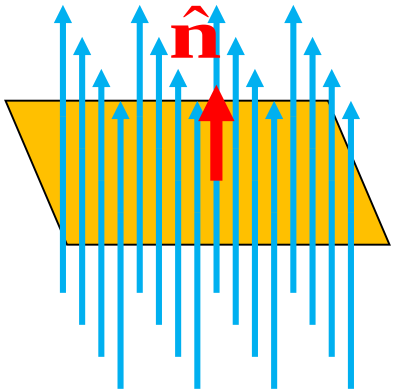
- If the electric field is uniform, the angle θ is constant and we have:

$$\boxed{\psi = E\left(S\cos\theta\right)}$$

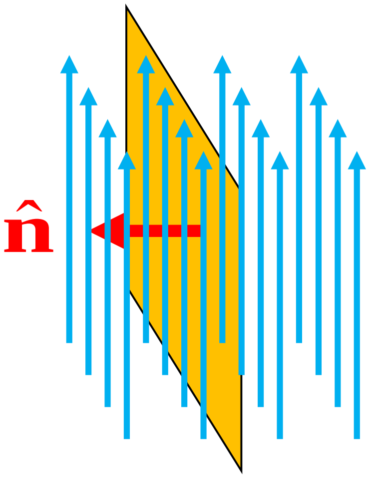
Thus the flux is equal to the product of magnitude of the electric field and the projection of area perpendicular to the field.

Electric Flux (contd.)

- When the surface is flat, and the fields are constant, you can just use multiplication to get the flux.



$$\psi = S\left|\vec{E}\right|$$



$$\psi = 0$$

- When the surface is curved, or the fields are not constant, you have to perform an integration:

$$\boxed{\psi = \int \vec{E} \cdot \hat{\mathbf{n}} dS}$$

Gauss Law

- In practice, electric field intensity is dependent on the medium in which the charge is placed (free space in our discussion).
- Let us define a new vector $\vec{D}$ that relates the medium and the electric field as:

$$\boxed{\vec{D} = \varepsilon_0 \vec{E}}$$

Electric Flux Density

- According to Gauss **[full name: Carl Friedrich Gauss]**, the total electric flux ψ **through any closed surface** is equal to the total charge enclosed by that surface.

$$\psi = Q_{enc}$$

$$Q_{enc} = \int_v \rho_v dv$$

$$\psi = \oint_S d\psi = \oint_S \vec{D}.\vec{dS}$$

$$\Rightarrow \oint_S \vec{D}.\vec{dS} = \int_v \rho_v dv$$

- From Divergence theorem: $\Rightarrow \oint_S \vec{D}.\vec{dS} = \int_v \nabla.\vec{D} dv$

Gauss Law (contd.)

- Therefore: $\boxed{\rho_v = \nabla.\vec{D}}$ ⟵ First of the four Maxwell's equations

Gauss's Law can be used to solve three types of problems:

1. Finding the total charge in a region when you know the electric field outside that region
2. Finding the total flux out of a region when the charge is known
   - It can also be used to find the flux out of one side in symmetrical problems ↔ In such cases, you must first argue from symmetry that the flux is identical through each side
3. Finding the electrical field in highly symmetrical situations
   - One must first use reason to find the direction of the electric field everywhere
   - Then draw a Gaussian surface over which the electric field is constant
   - Use this surface to find the electric field using Gauss's Law
   - Works generally only for spherical, cylindrical, or planar-type problems

Gauss Law (contd.)

1. A continuous charge distribution has rectangular symmetry if it depends only on x $(or\ y\ or\ z)$, cylindrical symmetry if it depends only on ρ, and spherical symmetry if it depends only on r $(independent\ of\ \theta\ and\ \varphi)$.
2. Gauss's Law is also valid for asymmetric charge distribution. However, you can't apply Gauss's Law to determine $\vec{E}$ or $\vec{D}$. In such situations, apply Coulomb's Law.
3. Gaussian surface is chosen **such that $\vec{D}$ is normal or tangential to the surface.** **When $\vec{D}$ is normal to the surface then $\vec{D}.\overline{ds} = \vec{D}$ and for tangential $\vec{D}$ we get $\vec{D}.\overline{ds} = 0$.**

Applications of Gauss's Law

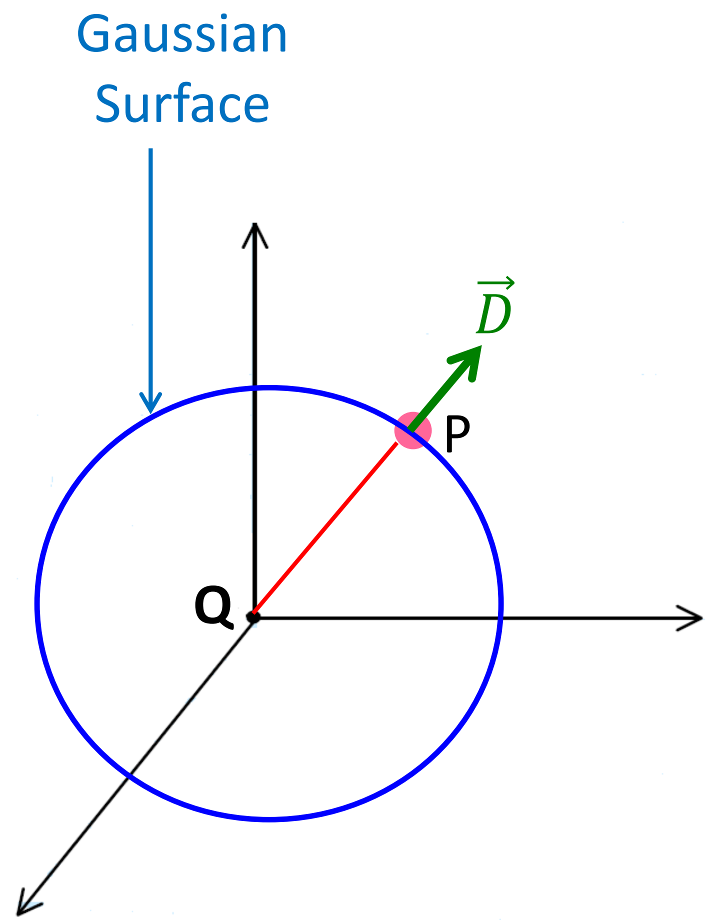
Point Charge

- Suppose a point charge is located at origin.
- Determine $\vec{D}$ at a point P.

- Choose a spherical surface containing P.
- $\vec{D}$ is everywhere normal to the Gaussian surface.

$$Q = \oint_S \vec{D}.\vec{dS} = \oint_S DdS\cos 0^o = \oint_S DdS = D\oint_S dS = D \times 4\pi r^2$$

$$\boxed{\therefore D = \frac{Q}{4\pi r^2}}$$



Applications of Gauss's Law (contd.)

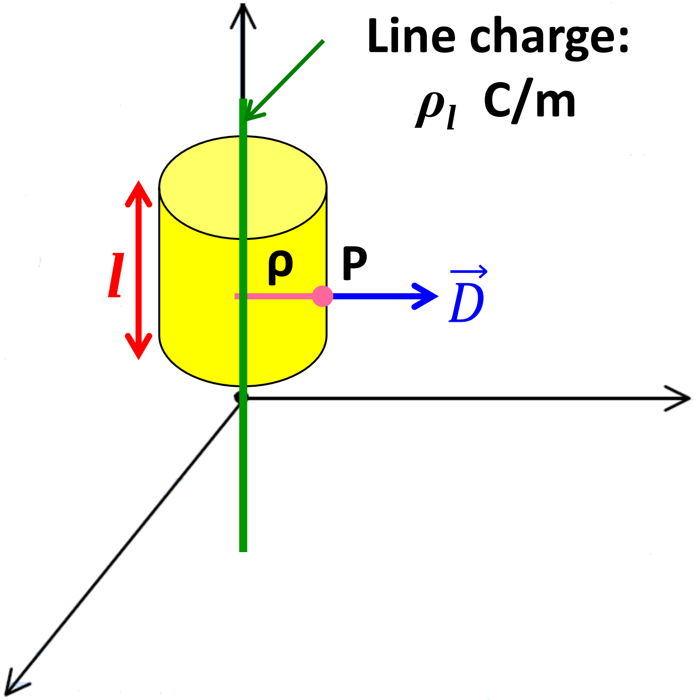
Infinite Line Charge

- To determine $\vec{D}$ at a point P, lets choose a cylindrical surface of arbitrary length ***l***.
- $\vec{D}$ is normal to side surface, doesn't exist on the top and bottom surface (because there is no z-component of $\vec{D}$).
- **Therefore:**

$$Q = \rho_l l = \oint_S \vec{D}.\vec{dS} = D_\rho \oint_S dS = D_\rho \times 2\pi\rho l$$

$$D_\rho = \frac{\rho_l}{2\pi\rho}$$

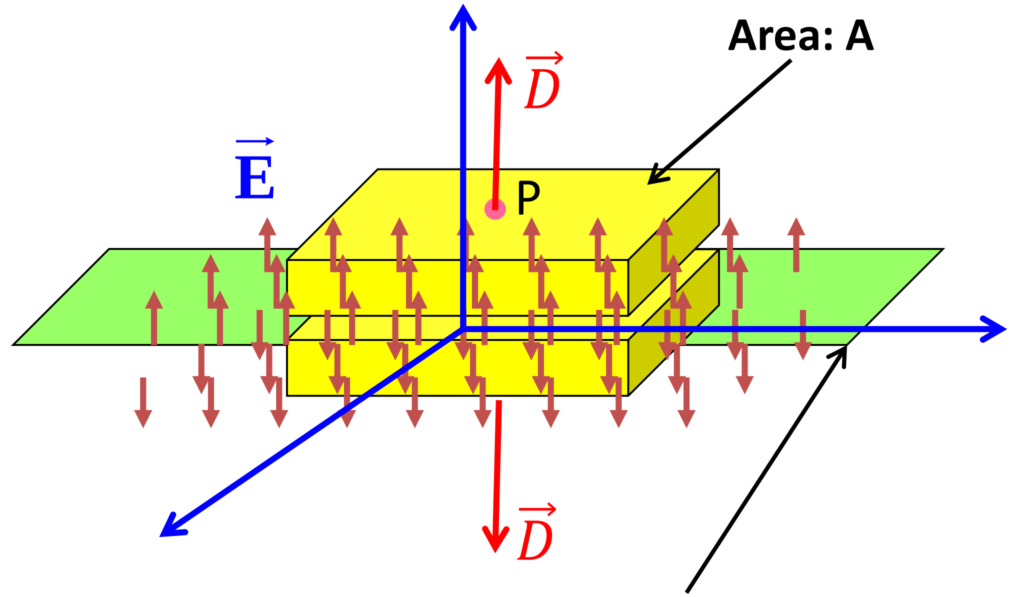
$$\boxed{\therefore D = \frac{\rho_l}{2\pi\rho}\hat{a}_\rho}$$



Applications of Gauss's Law (contd.)

Infinite Sheet Charge

- To determine $\vec{D}$ at a point P, lets choose a rectangular box with top and bottom area A
- $\vec{D}$ is normal to the top and bottom, doesn't exist on the side surface



Infinite sheet of charge: ρ_S C/m^2

- **Therefore:**

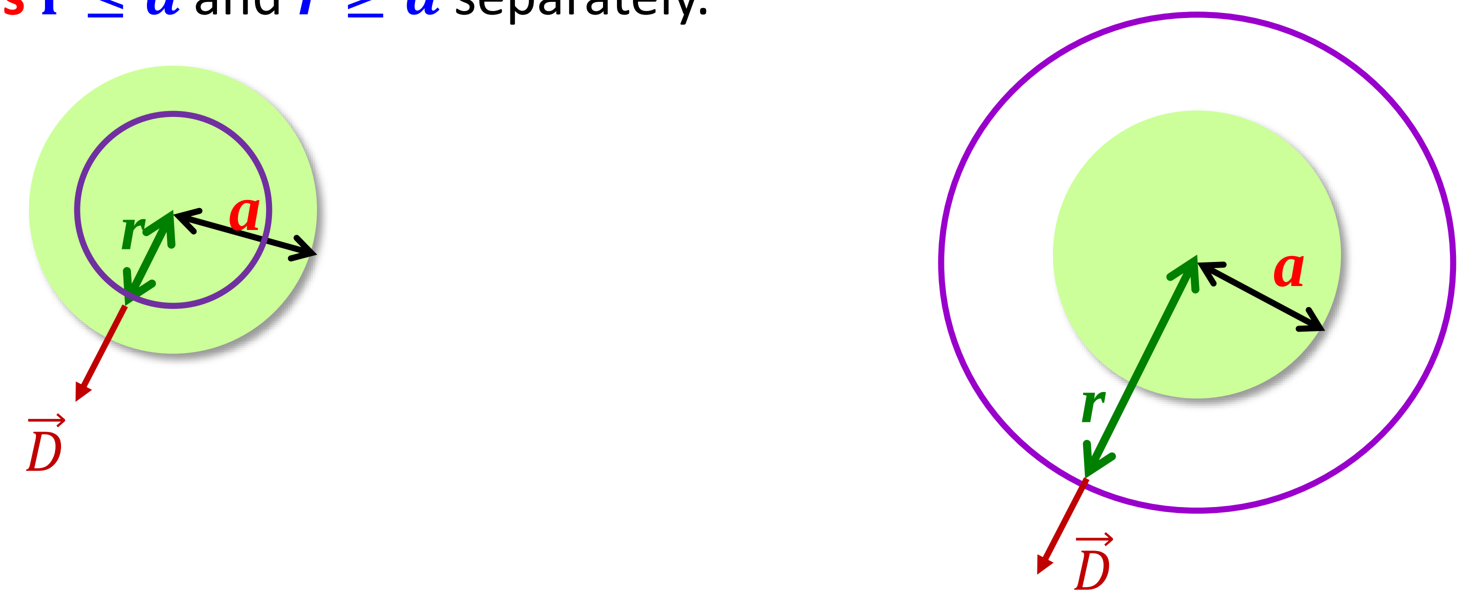
$$Q = \rho_S \int dS = \oint_S \vec{D}.\vec{dS} = D_z \left[\int_{top} dS + \int_{bottom} dS \right]$$

$$\Rightarrow \rho_S A = D_z [A + A] \quad \Rightarrow D_z = \frac{\rho_S}{2} \quad \boxed{\therefore \vec{D} = \frac{\rho_S}{2} \hat{a}_z} \Rightarrow \boxed{\therefore \vec{E} = \frac{\rho_S}{2\varepsilon_0} \hat{a}_z}$$

Applications of Gauss's Law (contd.)

Uniformly Charged Sphere

- A sphere of radius a has uniform charge density ρ_v **throughout**. What is the direction and magnitude of the electric field everywhere?
  - To determine $\vec{D}$ **everywhere**, **let us construct Gaussian surfaces for cases** $\mathbf{r} \leq \boldsymbol{a}$ and $\boldsymbol{r} \geq \boldsymbol{a}$ separately.

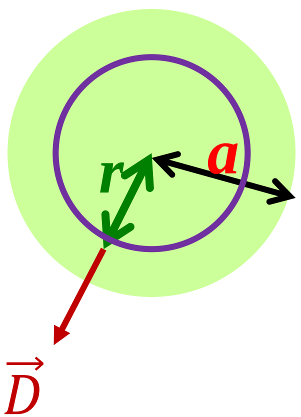


- Clearly, all directions are created equal in this problem
- Certainly the electric field will point away from the sphere at all points
- The electric field must depend *only* on the distance

Applications of Gauss's Law (contd.)

Uniformly Charged Sphere

Case-I: $r \le a$



- When computing the flux for a Gaussian surface, only include the electric charges *inside* the surface. Here, the enclosed charge is:

$$Q_{enc} = \int_v \rho_v dv = \rho_v \int_v dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta dr d\theta d\phi \qquad \boxed{\therefore Q_{enc} = \rho_v \frac{4}{3}\pi r^3}$$

- **The total flux:**

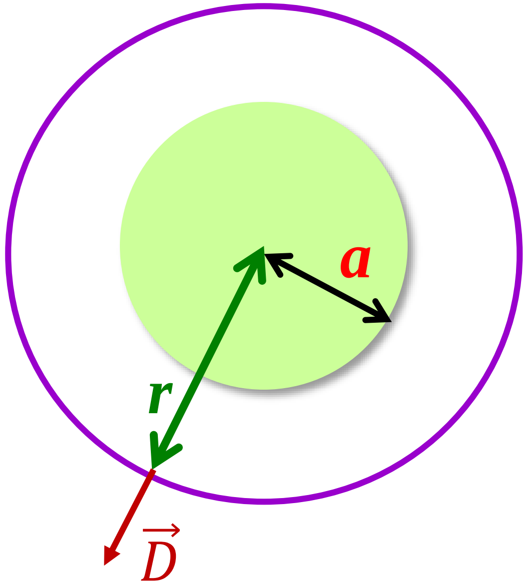
$$\psi = \oint_S \vec{D}.\vec{dS} = D_r \oint_S dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\phi \qquad \boxed{\therefore \psi = D_r 4\pi r^2}$$

- **From Gauss's Law:** $\psi = Q_{enc}$ $\quad \Rightarrow D_r 4\pi r^2 = \rho_v \frac{4}{3}\pi r^3 \quad \Rightarrow D_r = \frac{\rho_v}{3} r$

$$\boxed{\therefore \vec{D} = \frac{r}{3}\rho_v \hat{a}_r}$$

Applications of Gauss's Law (contd.)

Uniformly Charged Sphere



Case-II: $r \geq a$

- The charge enclosed in this case is the entire charge:

$$Q_{enc} = \int_v \rho_v dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta dr d\theta d\phi \qquad \boxed{\therefore Q_{enc} = \rho_v \frac{4}{3}\pi a^3}$$

- **While:**

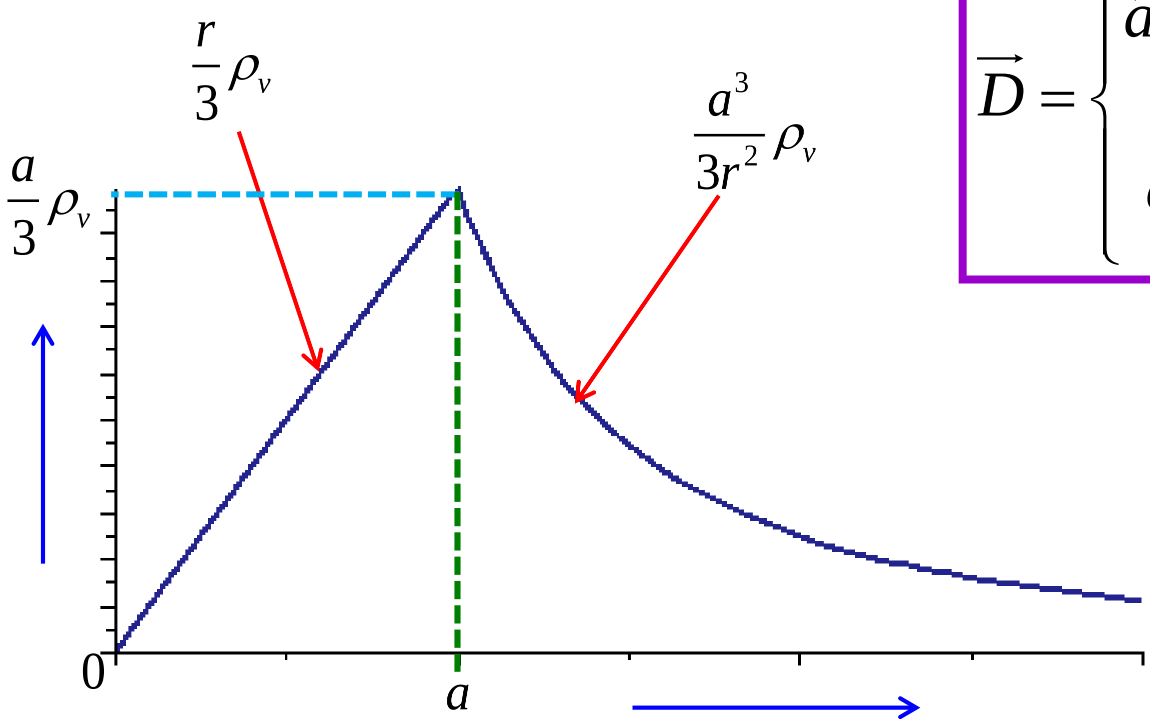
$$\psi = \oint_S \vec{D}.\vec{dS} = D_r \oint_S dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\phi \qquad \boxed{\therefore \psi = D_r 4\pi r^2}$$

- **From Gauss's Law:** $\psi = Q_{enc} \quad \Rightarrow D_r 4\pi r^2 = \rho_v \frac{4}{3}\pi a^3 \quad \Rightarrow D_r = \frac{a^3}{3r^2}\rho_v$

$$\boxed{\therefore \vec{D} = \frac{a^3}{3r^2}\rho_v \hat{a}_r}$$

Applications of Gauss's Law (contd.)

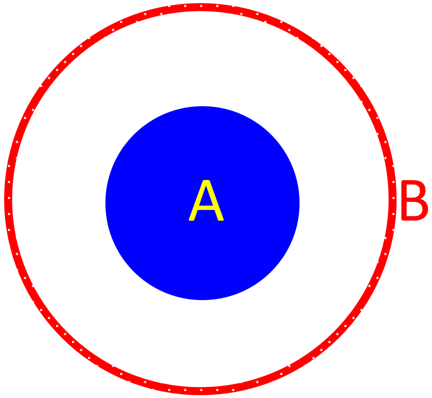
Uniformly Charged Sphere



$$\vec{D} = \begin{cases} \hat{a}_r \rho_v \dfrac{a^3}{3r^2} & \text{for } r > a, \\ \hat{a}_r \dfrac{r}{3}\rho_v & \text{for } r < a. \end{cases}$$

Example – 6

A blue sphere A is contained within a red spherical shell B. There is a charge Q_A on the blue sphere and charge Q_B on the red spherical shell.



- The electric field in the region between the spheres is completely independent of Q_B the charge on the red spherical shell.

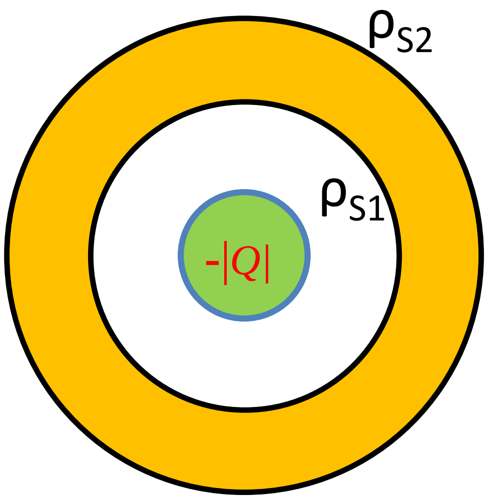
True

False

Example – 7

Consider the following topology:

A) A solid non-conducting sphere carries a total charge Q = -3 mC distributed evenly throughout. It is surrounded by an *uncharged* conducting spherical shell.

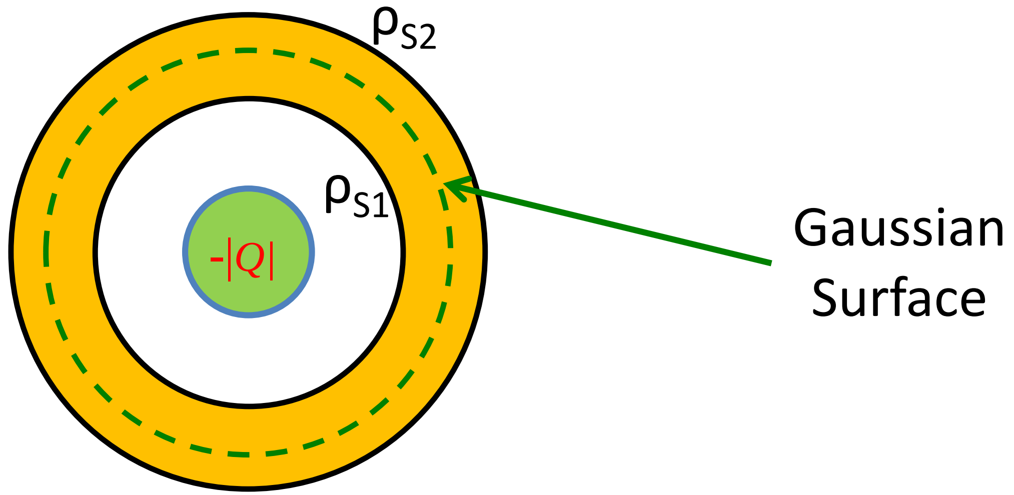


- What is the surface charge density ρ_{S1} on the inner surface of the conducting shell?

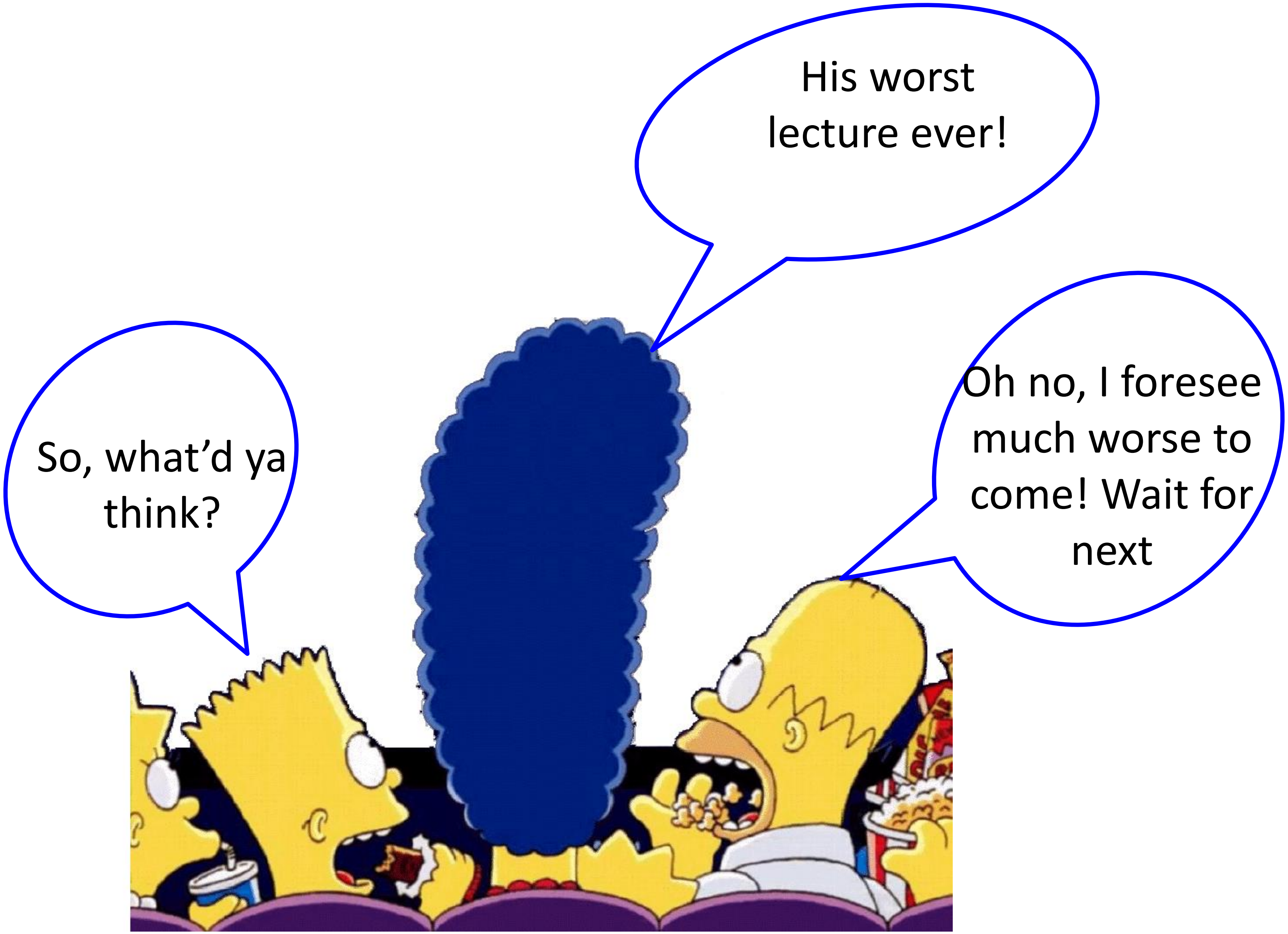
(c) $\rho_{S1} < 0$ (c) $\rho_{S1} = 0$ (c) $\rho_{S1} > 0$

Example – 7 (contd.)

- Inside the conductor, we know the field $\vec{E} = 0$
- Select a Gaussian surface inside the conductor
  - Since $\vec{E} = 0$ on this surface, the total enclosed charge must be 0.
  - Therefore, the surface charge density on the inner surface of the conducting shell must be positive, to cancel the charge -|Q|.



(a) $\rho_{s1} < 0$ (b) $\rho_{s1} = 0$ (c) $\rho_{S1} > 0$


So, what'd ya think?
His worst lecture ever!
Oh no, I foresee much worse to come! Wait for next