Lecture – 11

Date: 08.02.2016

- Charge, Charge Density, Total Charge
- Coulomb’s Law
- Electric Field Due to Point Charge and Line Charge

Maxwell's Equations

$$\nabla . \vec{D} = \rho_v$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- Under the static conditions the Maxwell's equations become:

$$\nabla . \vec{D} = \rho_v$$

$$\nabla \times \vec{E} = 0$$

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

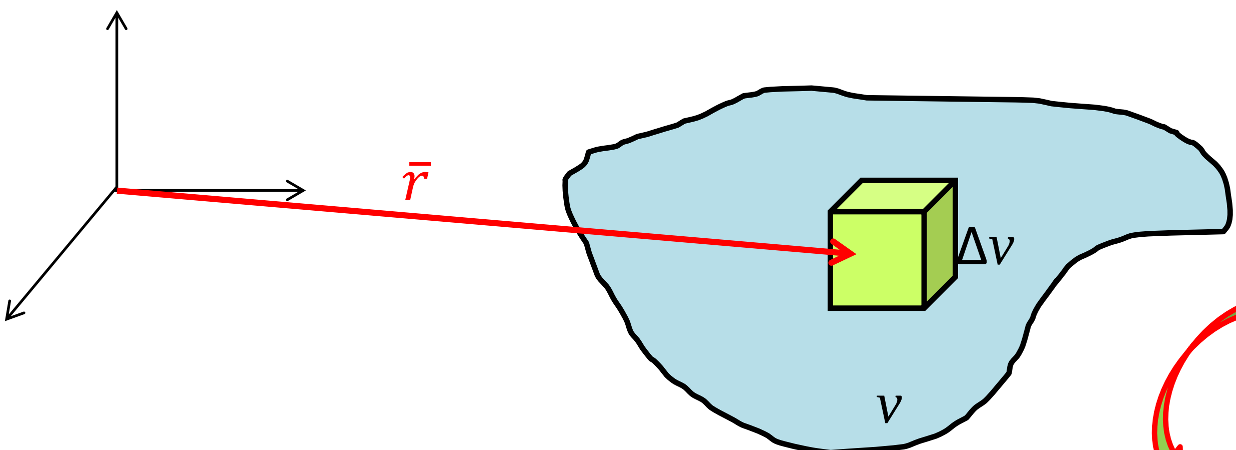
Electric and Magnetic fields become decoupled under static conditions

Enables us to study electricity and magnetism as distinct separate phenomena

We refer the study of electric and magnetic phenomena under static conditions as ***electrostatics*** and ***magnetostatics***

Charge Density

- In many cases, charged particles (e.g., electrons, protons, positive ions) are **unevenly distributed** throughout some volume v.
- We define **volume charge density** at a specific point $\bar{r}$ by evaluating the total net charge ΔQ in a small volume Δv surrounding the point.



Volume Charge Density

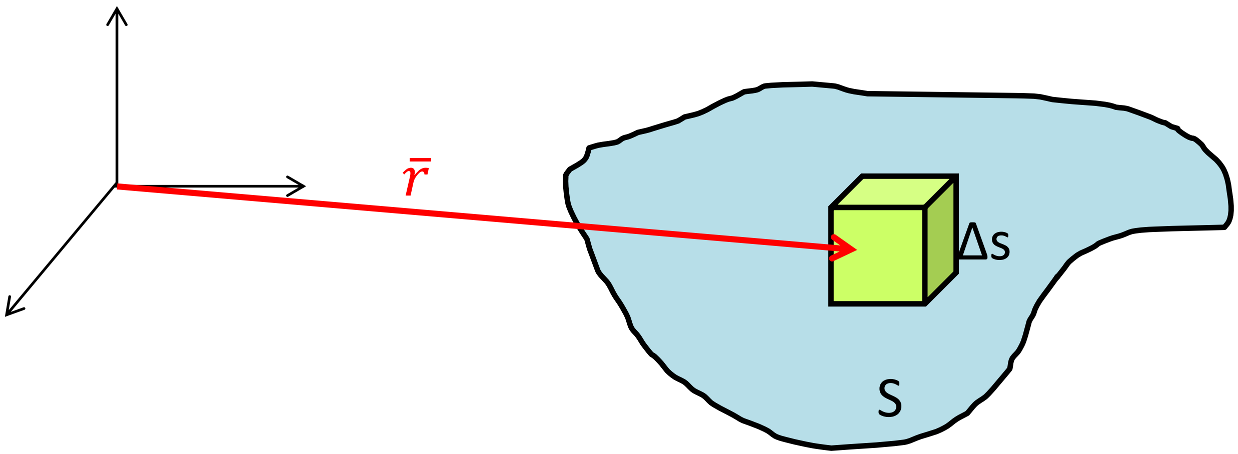
$$\rho_v(\bar{r}) = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

IMPORTANT NOTE: Volume charge density indicates the **net** charge density at each point $\bar{r}$ within volume v.

Volume charge density is a **scalar field**, and is expressed with units such as **coulombs/m³**.

Surface Charge Density

- Another possibility is that charge is unevenly distributed across some surface S. In this case, we can define a **surface charge density** as by evaluating the total charge ΔQ on a small patch of surface Δs, located at point $\bar{r}$ on surface S:



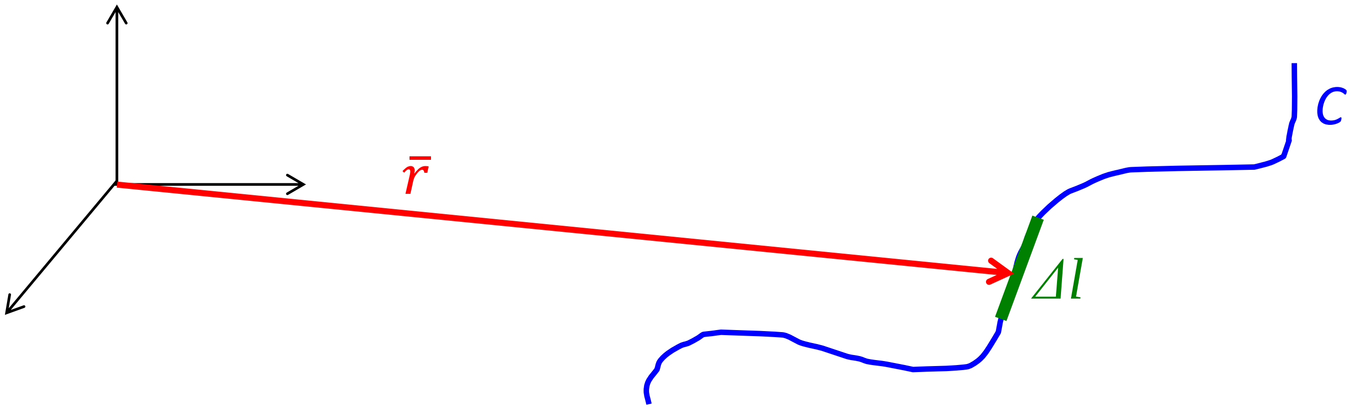
- **Surface** charge density $\rho_s(\bar{r})$ is defined as:

$$\rho_s(\bar{r}) \doteq \lim_{\Delta s \to 0} \frac{\Delta Q}{\Delta s}$$

Note the **units** for surface charge density will be **charge/area** (e.g. **C/m^2**).

Line Charge Density

- Finally, let us consider the case where charge is unevenly distributed across some **contour** C. We can therefore define a **line charge density** as the charge ΔQ along a small distance Δl, located at point $\bar{r}$ of contour C.



- **Line** charge density $\rho_l(\bar{r})$ is defined as: $$\rho_l(\bar{r}) \doteq \lim_{\Delta l \to 0} \frac{\Delta Q}{\Delta l}$$

As you might expect, the units of a line charge density is charge per length (e.g., **C/m**).

Example – 1

Find the total charge on a circular disc defined by $\rho \leq a$ and $z = 0$ if: $\rho_S = \rho_{S0} e^{-\rho}$ (C/m²).

Example – 2

A circular beam of charge of **radius $\boldsymbol{a}$** consists of electrons moving with a **constant speed $\boldsymbol{u}$** along the $+z$ **direction**. The beam's axis is coincident with the z-axis and the electron **charge density is given by: $\boldsymbol{\rho_v = -c\rho^2}$ (C/m³),** where c is a constant and ρ is the radial distance from the axis of the beam. **Determine the charge density per unit length.**

Example – 3

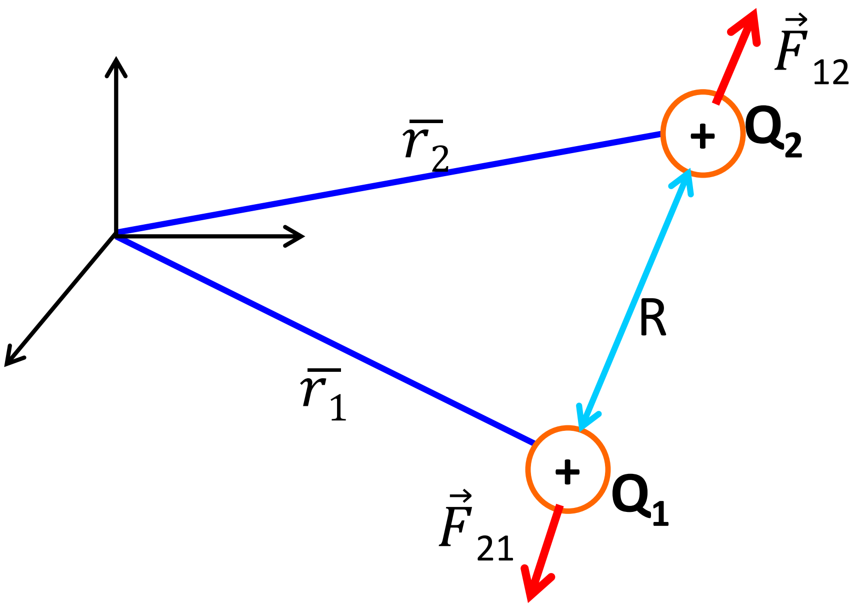
A square plate in the x–y plane is situated in the space defined by $-3m \leq x \leq 3m$ and $-3m \leq y \leq 3m$. Find the total charge on the plate if the surface charge density is given by $\rho_s = 4y^2 \left({}^{\mu C}/_{m^2}\right)$.

Goal of next few lectures

- Develop dexterity in applying the expressions for the electric field intensity $\vec{E}$ induced by specified distribution of charge.
- For now, our discussion will be limited to electrostatic fields generated by stationary charges.
- We will begin by considering the expression for the electric field developed by **Coulomb**.

Coulomb's Law

- Let us Consider **two positive** point charges, Q_1 and Q_2, in free space located at positions $\bar{r}_1$ and $\bar{r}_2$, respectively.
- Clarification: by point charge it is assumed that the charge is located on a body whose dimensions are much smaller than other relevant dimensions.



- **Each** charge exert a **force** $\vec{F}$ (with magnitude and direction) on the other.
- This force is **dependent** on both the **sign** (+ or −) and the **magnitude** of charges Q_1 and Q_2, as well as the **distance** R between the charges.

- **Charles Coulomb** determined this relationship in the 18th century! We call his result **Coulomb's Law**:

$$\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}\hat{a}_{21}$$

This force $\vec{F}_{21}$ is the force exerted on charge Q_1 by Q_2.

Coulomb's Law (contd.)

- Likewise, the force exerted **by** charge Q_1 **on** charge Q_2 is equal to:

$$\overrightarrow{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{a}_{12}$$

- In these formula, the value ε_0 is a **constant** that describes the **permittivity of free space** (i.e., a vacuum) given by:

$$\varepsilon_0 = 8.854\times10^{-12} \text{ C}^2/\text{Nm}^2 \text{ or F/m} \Rightarrow = \frac{1}{36\pi}\times10^{-9} \text{ C}^2/\text{Nm}^2 \text{ or F/m}$$

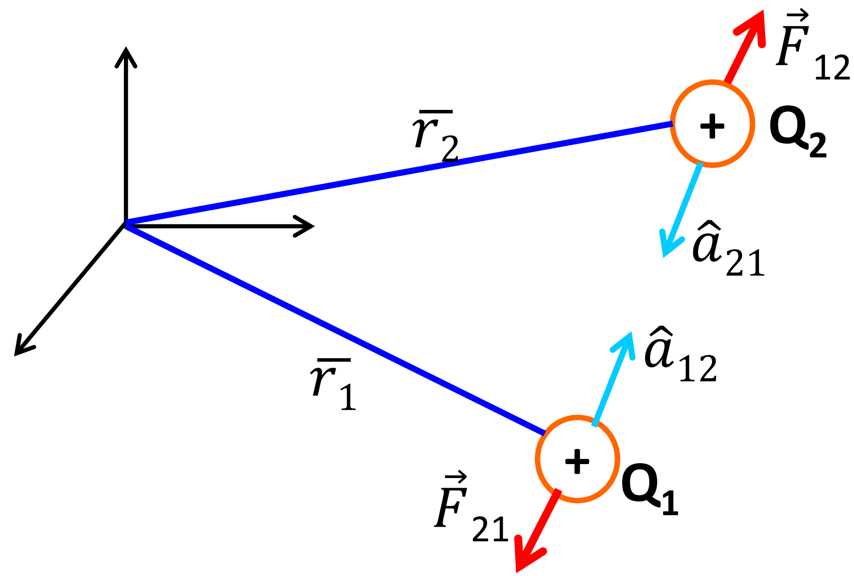
- Note the **only difference** between the equations for forces $\vec{F}_{21}$ and $\vec{F}_{12}$ are the **unit vectors** $\hat{a}_{21}$ and $\hat{a}_{12}$.
- Unit vector $\hat{a}_{21}$ points **from** the location of Q_2 (i.e., $\bar{r}_2$) **to** the location of charge Q_1 (i.e., $\bar{r}_1$).
- Likewise, unit vector $\hat{a}_{12}$ points **from** the location of Q_1 (i.e., $\bar{r}_1$) **to** the location of charge Q_2 (i.e., $\bar{r}_2$).
- Note therefore, that these unit vectors point in **opposite** directions, a result we express mathematically as $\hat{a}_{21} = -\hat{a}_{12}$.

Coulomb's Law (contd.)

- Therefore :

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}\hat{a}_{21} \Rightarrow = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}(-\hat{a}_{12}) \Rightarrow = -\left(\frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}\hat{a}_{12}\right) \Rightarrow = -\vec{F}_{12}$$

- **Look!** Forces $\vec{F}_{21}$ and $\vec{F}_{12}$ have **equal magnitude**, but point in **opposite directions**!



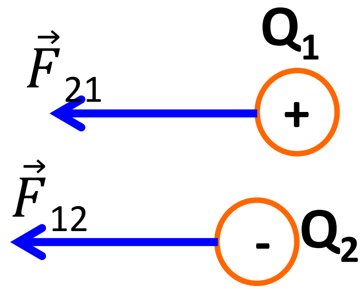
Note in this case **both** charges were **positive**.

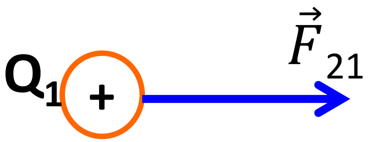
Q: What happens when **one** of the charges is **negative**?

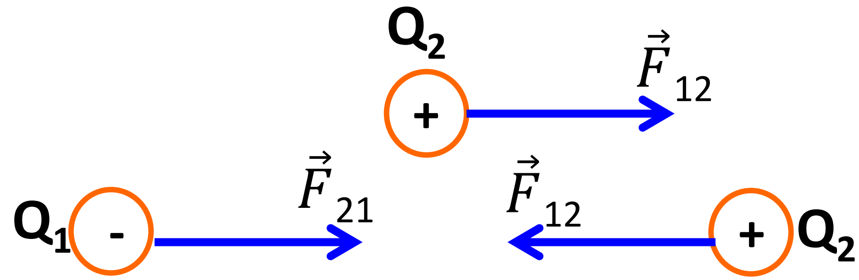
A: Look at Coulomb's Law! If one charge is positive, and the other is negative, then the **product** Q_1Q_2 is **negative**. The resulting force vectors are therefore negative—they point in the **opposite** direction of the previous (i.e., both positive) case!

Coulomb's Law (contd.)

- Therefore:







- What about this case?

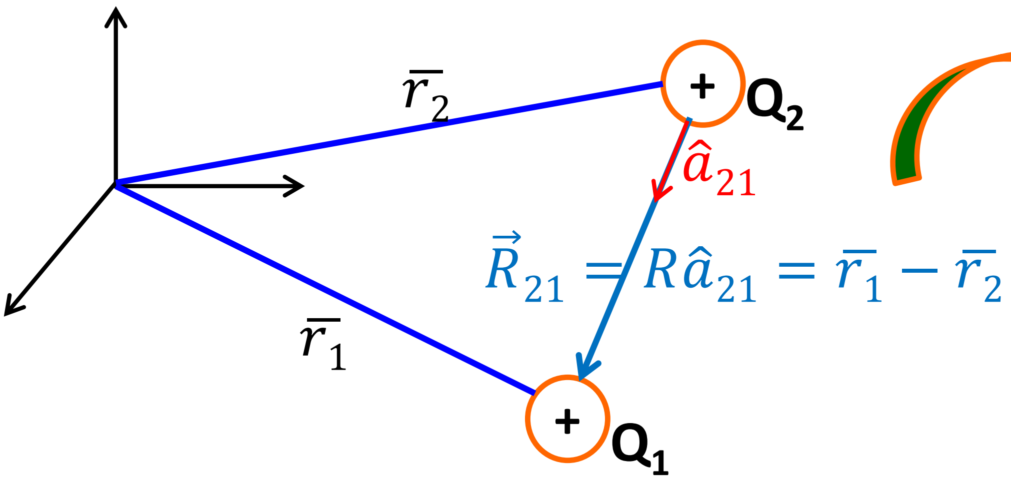
The Vector Form of Coulomb's Law of Force

- Recall: $$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{a}_{21}$$

Specifically, we ask ourselves the question: **how** do we determine the **unit vector** $\hat{a}_{21}$ and **distance** R ??

- Recall the **unit vector** $\hat{a}_{21}$ is a unit vector directed **from** Q_2 **toward** Q_1, and R is the **distance** between the two charges.
- The **directed distance** vector $\vec{R}_{21} = R\hat{a}_{21}$ can be determined from the **difference** of position vectors $\overline{r}_1$ and $\overline{r}_2$.

The Vector Form of Coulomb's Law of Force (contd.)



This directed distance $\vec{R}_{21} = \bar{r}_1 - \bar{r}_2$ is **all** we need to determine **both** unit vector $\hat{a}_{21}$ and distance R (i.e., $\vec{R}_{21} = R\hat{a}_{21}$)!

- For example, since the **direction** of directed distance $\vec{R}_{21}$ is $\hat{a}_{21}$, we can **explicitly** find this unit vector by **dividing** $\vec{R}_{21}$ by its **magnitude**:

$$\hat{a}_{21} = \frac{\vec{R}_{21}}{|\bar{r}_1 - \bar{r}_2|}$$

- Likewise, the **distance** R between the two charges is simply the magnitude of directed distance $\vec{R}_{21}$!

$$R = |\vec{R}_{21}| = |\bar{r}_1 - \bar{r}_2|$$

- We can therefore express **Coulomb's Law** entirely in terms of $\vec{R}_{21}$:

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}\hat{a}_{21} \;\Rightarrow\; \vec{F}_{21} = \frac{Q_1Q_2}{4\pi\varepsilon_0}\frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|^3} \;\Rightarrow\; = \frac{Q_1Q_2}{4\pi\varepsilon_0}\frac{\vec{R}_{21}}{|\vec{R}_{21}|^3}$$

The Vector Form of Coulomb's Law of Force (contd.)

- Explicitly using the relation $\vec{R}_{21} = \bar{r}_1 - \bar{r}_2$, we can express:

$$\overrightarrow{F_{21}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\bar{r}_1 - \bar{r}_2}{\left|\bar{r}_1 - \bar{r}_2\right|^3}$$

- We could likewise define a directed distance:

$$\overrightarrow{R_{12}} = \left|\bar{r}_2 - \bar{r}_1\right|$$

relates the location of Q_2 with respect to Q_1.

- We can then describe the force on charge Q_2 as:

$$\overrightarrow{F_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \frac{\bar{r}_2 - \bar{r}_1}{\left|\bar{r}_2 - \bar{r}_1\right|^3}$$

- Note since $\vec{R}_{12} = -\vec{R}_{21}$, (thus $|\vec{R}_{12}| = |\vec{R}_{21}|$), it is apparent that:

$$\vec{F}_{12} = -\vec{F}_{21}$$

The forces on each charge have **equal** magnitude but **opposite** direction.

Confirmation of Newton's 3rd Law

Example – 4

- Point charge 5nC is located at (2, 0, 4). Determine the force on a 1nC point charge located at (1, –3, 7)

Electric Field

- An electric field is an *invisible entity* which exists in the region around a charged particle. It is caused to exist by the charged particle.
- **The effect of an electric field** is to **exert a force on any charged particle (other than the charged particle causing the electric field to exist)** that finds itself at a point in space at which the electric field exists.
- **The electric field at an empty point in space** is the **force-per-unit-charge-of-would-be-victim at that empty point in space.**
- The charged particle that is causing the electric field to exist is called a **source charge.**
- The electric field exists in the region around the **source charge** whether or not **there is a victim** charged particle for the electric field to exert a force upon.
- Where electric field exists, it has both magnitude and direction. The electric field is a vector at each point in space at which it exists.
- We call the force-per-unit-charge-of-would-be-victim **vector** at a particular point in space the **"electric field"** at that point.
- We also call the infinite set of all such vectors, **in the region around the source charge**, the electric field of the source charge.

Electric Field (contd.)

- **We use the symbol** $\vec{E}$ to represent the electric field. I am using the word "victim" for any particle upon which an electric field is exerting a force.
- The electric field will **only** exert a force on a particle if that particle has charge. So all "victims" of an electric field have charge.

- If there does happen to be a charged particle in an electric field, then that charged particle (the victim) will experience a force:

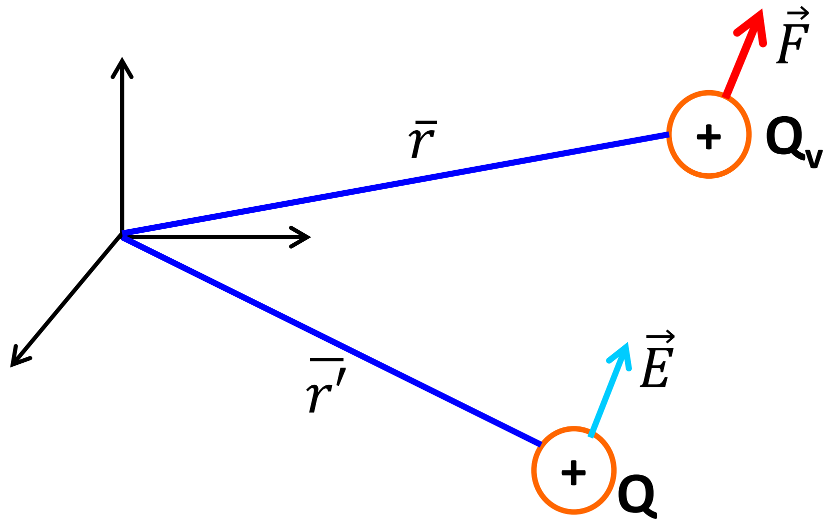
$$\vec{F} = Q_v \vec{E}$$

where Q_v is the charge of the victim and $\vec{E}$ is the electric field vector at the location of the victim.

We can think of the electric field as a characteristic of space. The force experienced by the victim charged particle is the product of a characteristic of the victim (its charge) and a characteristic of the point in space (the electric field) at which the victim happens to be.

Electric Field due to a Point Charge

- **For Q > 0**, the electric field $\vec{E}$ is in the direction of $\vec{F}$.



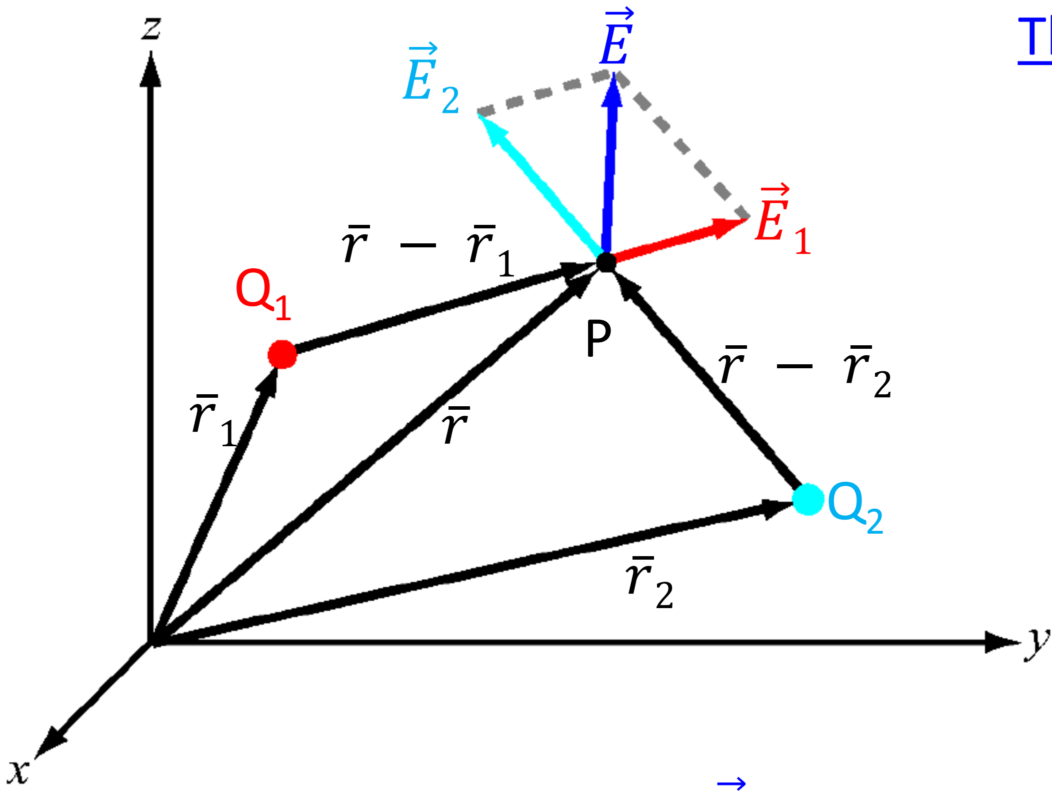
- Therefore, the electric field at point $\bar{r}$ due to a point charge located at $\bar{r}'$ is:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \frac{Q}{4\pi\varepsilon_0}\frac{\left(\bar{r}-\bar{r}'\right)}{\left|\bar{r}-\bar{r}'\right|^3}$$

- The electric field $\vec{E}$ expression due to a single point charge can be extended to multiple point charges.

Electric Field due to a Point Charge (contd.)

- Let us consider two **positive point charges** Q_1 and Q_2 located at position vectors $\bar{r}_1$ and $\bar{r}_2$. Then evaluate the electric field $\vec{E}$ at point P located at $\bar{r}$.



Therefore:

$$\vec{E}_1 = \frac{Q_1}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_1)}{|\bar{r}-\bar{r}_1|^3}$$ Due to Q_1

$$\vec{E}_2 = \frac{Q_2}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_2)}{|\bar{r}-\bar{r}_2|^3}$$ Due to Q_2

- The total electric field $\vec{E}$ due to both point charges is the vector sum of the individual electric fields $\vec{E}_1$ and $\vec{E}_2$.

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

Electric Field of Point Charge (contd.)

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_1)}{|\bar{r}-\bar{r}_1|^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_2)}{|\bar{r}-\bar{r}_2|^3}$$

$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1(\bar{r}-\bar{r}_1)}{|\bar{r}-\bar{r}_1|^3} + \frac{Q_2(\bar{r}-\bar{r}_2)}{|\bar{r}-\bar{r}_2|^3}\right]$$

Obeys principles of superposition

- For *N* point charges Q_1, Q_2, …., Q_N located at $\bar{r}_1, \bar{r}_2, \ldots \bar{r}_N$, the electric field at point $\bar{r}$ is given by:

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_1)}{|\bar{r}-\bar{r}_1|^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_2)}{|\bar{r}-\bar{r}_2|^3} + \ldots + \frac{Q_N}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_N)}{|\bar{r}-\bar{r}_N|^3}$$

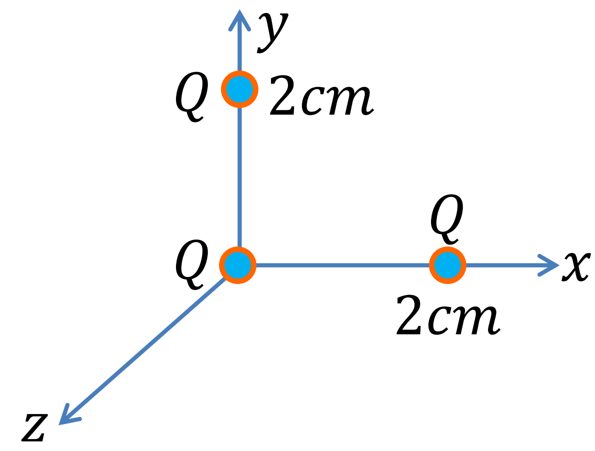
$$= \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\bar{r}-\bar{r}_k)}{|\bar{r}-\bar{r}_k|^3}$$

Example – 5

- Two point charges with $Q_1 = 2 \times 10^{-5}\ C$ and $Q_2 = -4 \times 10^{-5}\ C$ are located in free space at points with Cartesian coordinates (1, 3, -1) and (-3, 1, -2) respectively. Find (a) the electric field $\vec{E}$ at (3, 1, -2) and (b) the force on $8 \times 10^{-5}\ C$ charge located at that point. All distances are in meters.

Example – 6

Three point charges, each with $Q = 3\,nC$ are located at the corners of a triangle in the xy-plane, with one corner at the origin, another at $(2cm, 0, 0)$, and the third at $(0, 2cm, 0)$. Find the force acting on the charge located at the origin.



Example – 7

- Two identical charges are located on the x-axis at $x = 3$ and $x = 7$. At what point in space is the **net electric field** zero.

Example – 8

- Using Coulomb's law, determine the units of permittivity of free space.

Example – 9

- Find the magnitude of the Coulomb force that exists between an electron and proton in a hydrogen atom. Compare the Coulomb force and the gravitational force between the two particles. The two particles are separated approximately by 1×10^{-10} m.

$$F_{Coulomb} = \frac{Q^2}{4\pi\varepsilon_0 R^2} \approx \frac{\left(1.602 \times 10^{-19}\right)^2}{4\pi\left(\frac{10^{-9}}{36\pi}\right)\left(10^{-10}\right)^2} \approx 2.3 \times 10^{-8} N$$

$$F_{gravitational} = G\frac{M_{electron} M_{proton}}{R^2} = \left(6.67 \times 10^{-11}\right)\left(\frac{\left(9.11 \times 10^{-31}\right)\left(1836 \times 9.11 \times 10^{-31}\right)}{\left(1 \times 10^{-10}\right)^2}\right) = 1.02 \times 10^{-47} N$$

$$\therefore \frac{F_{Coulomb}}{F_{gravitational}} = 2.27 \times 10^{39}$$

Such a large Coulomb force helps explain why chemical bonds that hold atoms, molecules, and compounds together can be very strong.

Electric Field due to a Line Charge

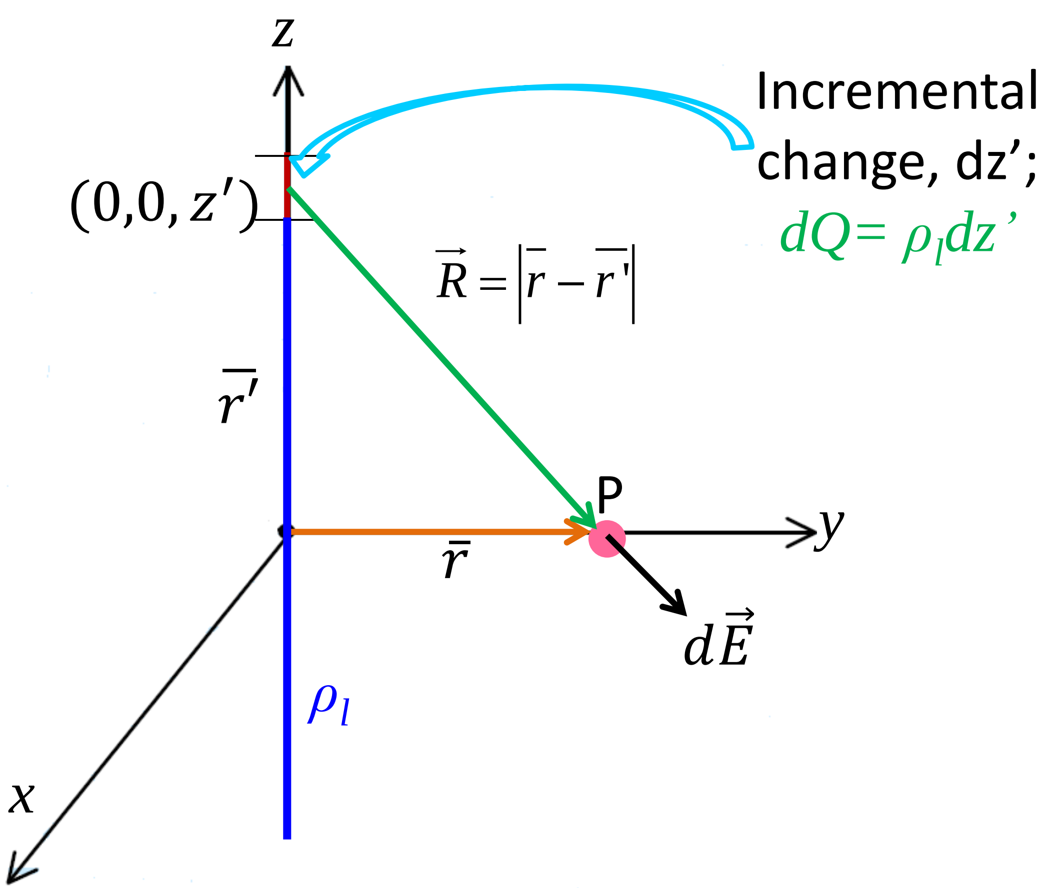
- Filament like distribution of charge density.
- For example, sharp beam in a cathode-ray tube or charged conductor of a very small radius.
- Let us assume an infinite straight-line charge, with charge density ρ_l C/m, lying along the z-axis.

Q: What electric field $\vec{E}(\bar{r})$ is produced by this line charge?

A: Apply Coulomb's Law.

Electric Field due to a Line Charge (contd.)

- For the calculation of electric field $\vec{E}$ at $P(0,y,0)$, the first step is to determine the incremental field at P due to the incremental charge $dQ = \rho_l dz'$



We have:

$$\overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} = \frac{\rho_l dz'}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$$

$$\bar{r} = y\hat{a}_y = \rho\hat{a}_\rho$$

$$\bar{r}' = z'\hat{a}_z$$

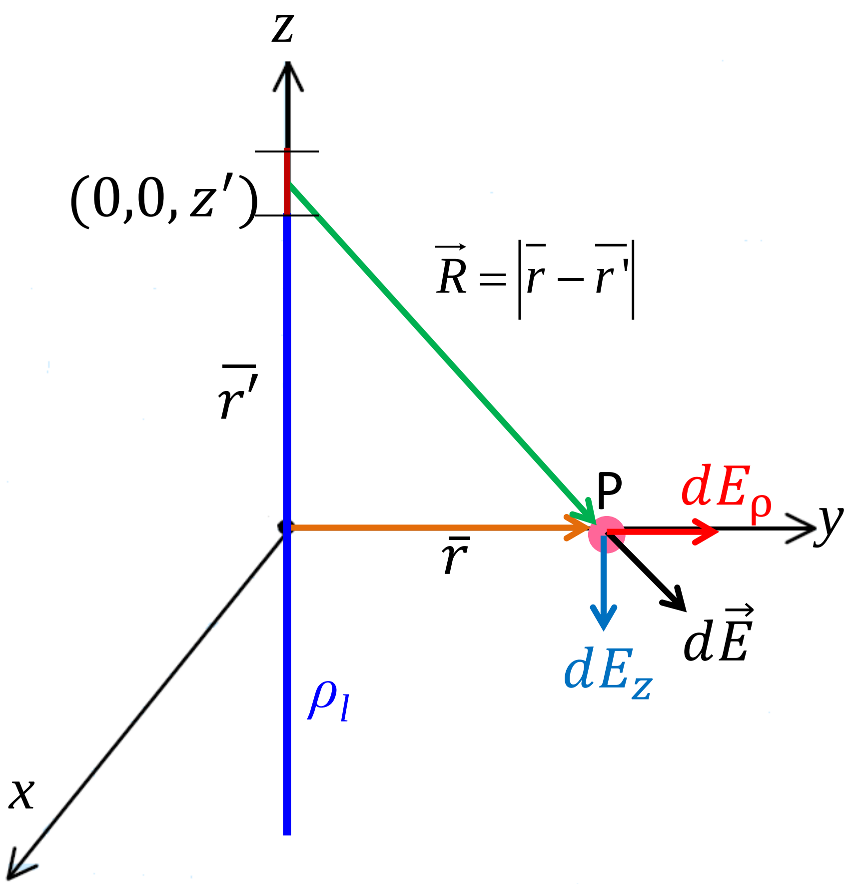
$$\therefore \vec{R} = \bar{r} - \bar{r}' = \rho\hat{a}_\rho - z'\hat{a}_z$$

Electric Field due to a Line Charge (contd.)

$$\Rightarrow \overrightarrow{dE} = \frac{\rho_l dz'}{4\pi\varepsilon_0} \frac{\left(\rho \hat{a}_\rho - z'\hat{a}_z\right)}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\therefore \overrightarrow{dE} = \frac{\rho_l \rho dz'}{4\pi\varepsilon_0 \left(\rho^2 + z'^2\right)^{3/2}} \hat{a}_\rho - \frac{\rho_l z' dz'}{4\pi\varepsilon_0 \left(\rho^2 + z'^2\right)^{3/2}} \hat{a}_z$$

$d\vec{E}_\rho$ $\quad$ $d\vec{E}_z$



$$\therefore \overrightarrow{dE} = \hat{a}_\rho dE_\rho - \hat{a}_z dE_z$$

Electric Field due to a Line Charge (contd.)

Now:

$$dE_\rho = \int_{z'=-\infty}^{z'=\infty} \frac{\rho_l \rho}{4\pi\varepsilon_0 \left(\rho^2 + z'^2\right)^{3/2}} dz' = \frac{\rho_l \rho}{4\pi\varepsilon_0} \int_{z'=-\infty}^{z'=\infty} \frac{dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\Rightarrow E_\rho = \frac{\rho_l \rho}{4\pi\varepsilon_0} \left[\frac{1}{\rho^2} \frac{z'}{\left(\rho^2 + z'^2\right)^{1/2}} \right]_{z'=-\infty}^{z'=\infty}$$

$$\therefore E_\rho = \frac{\rho_l}{2\pi\varepsilon_0 \rho}$$

AND:

$$dE_z = \frac{\rho_l}{4\pi\varepsilon_0} \int_{z'=-\infty}^{z'=\infty} \frac{z' dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\therefore E_z = \frac{\rho_l}{4\pi\varepsilon_0} \times (0) = 0$$

Therefore:

$$\vec{E}\left(\bar{r}\right) = E_\rho \hat{a}_\rho - E_z \hat{a}_z = \frac{\rho_l}{2\pi\varepsilon_0 \rho} \hat{a}_\rho$$

BTW, there are multiple ways of solving this problem.
You can master this art through practice!