Lecture – 10 Date: 04.02.2016

- Conservative and Solenoidal Vector Fields
- Charge, Charge Density, Total Charge

The Conservative Vector Field

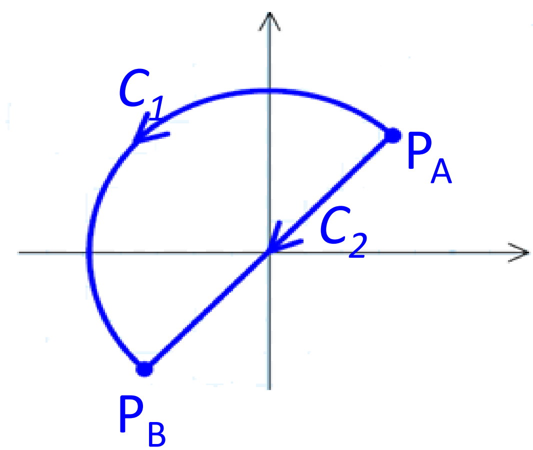
- Of all possible vector fields $\vec{A}(\bar{r})$, there is a subset of vector fields called **conservative** fields. A conservative vector field is a vector field that can be expressed as the **gradient** of some scalar field $g(\bar{r})$:

$$\vec{C}(\bar{r}) = \Delta g(\bar{r})$$

In other words, the gradient of **any** scalar field **always** results in a conservative field!

- A conservative field has the interesting property that its line integral is dependent on the **beginning** and **ending** points of the contour **only**! In other words, for the two contours:

$$\int_{C_1} \vec{C}(\bar{r}).\overline{dl} = \int_{C_2} \vec{C}(\bar{r}).\overline{dl}$$



- We therefore say that the line integral of a conservative field **is path independent**.

The Conservative Vector Field (contd.)

- This path independence is evident when considering the **integral identity**:

$$\int_C \nabla g(\bar{r}).\overline{dl} = g\left(\bar{r} = \bar{r}_B\right) - g\left(\bar{r} = \bar{r}_A\right)$$

position vector $\bar{r}_B$ denotes the **ending** point (P_B) of contour C, and $\bar{r}_A$ denotes the **beginning** point (P_A). $g(\bar{r} = \bar{r}_B)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_B$, and $g(\bar{r} = \bar{r}_A)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_A$.

- For **one** dimension, the above identity simply reduces to the familiar expression:

$$\int_{x_a}^{x_b} \frac{\partial g(x)}{\partial x} dx = g\left(x = x_b\right) - g\left(x = x_a\right)$$

- Since **every** conservative field can be written in terms of the **gradient** of a scalar field, we can use this identity to conclude:

$$\int_C \vec{C}(\bar{r}).\overline{dl} = \int_C \nabla g(\bar{r}).\overline{dl}$$

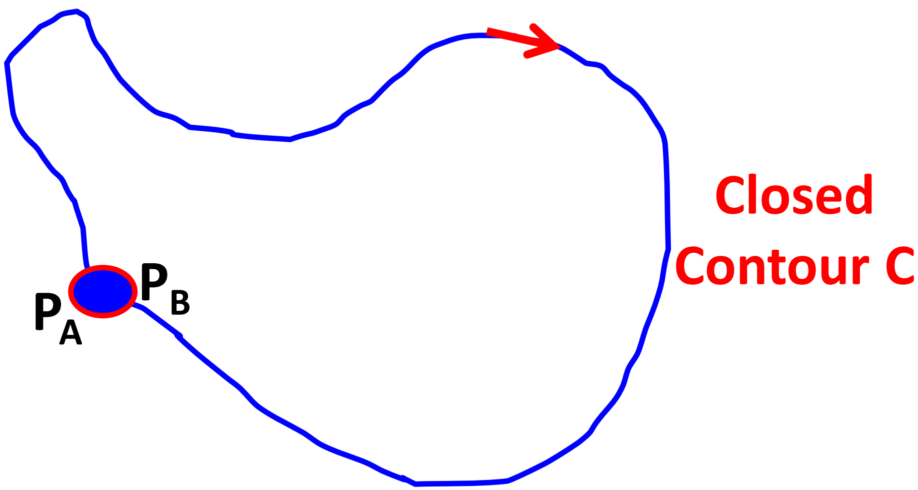
$$\therefore \int_C \vec{C}(\bar{r}).\overline{dl} = g\left(\bar{r} = \bar{r}_B\right) - g\left(\bar{r} = \bar{r}_A\right)$$

Consider then what happens then if we integrate over a **closed** contour.

The Conservative Vector Field (contd.)

Q: What the heck is a closed contour ??

A: A closed contour's beginning and ending is the **same** point! e.g.,



A contour that is **not** closed is referred to as an **open** contour.

- Integration over a closed contour is **denoted** as:

$$\oint_C \vec{A}(\bar{r}).\overline{dl}$$

- The integration of a **conservative** field over a **closed** contour is therefore:

$$\oint_C \vec{C}(\bar{r}).\overline{dl} == \oint_C \nabla g(\bar{r}).\overline{dl}$$

$$= g\left(\bar{r} = \bar{r}_B\right) - g\left(\bar{r} = \bar{r}_A\right)$$

$$= 0$$

This result is due to the fact that $\bar{r}_A = \bar{r}_B$ ⟹ $g\left(\bar{r} = \bar{r}_B\right) = g\left(\bar{r} = \bar{r}_A\right)$

The Conservative Vector Field (contd.)

- Let's **summarize** what we know about a **conservative** vector field:

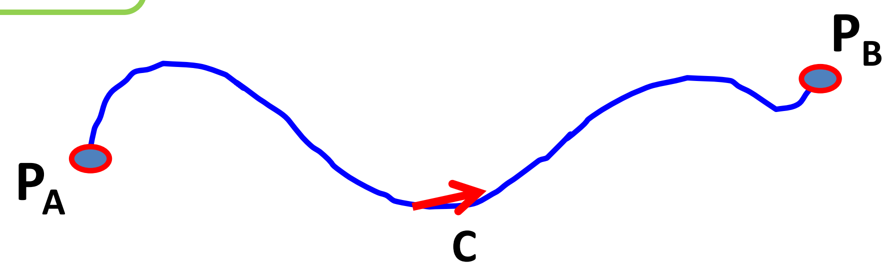
1. A conservative vector field can always be expressed as the **gradient** of a **scalar** field.
2. The gradient of **any** scalar field is therefore a conservative vector field.
3. Integration over an **open** contour is dependent **only** on the value of scalar field $g(\bar{r})$ at the beginning and ending points of the contour (i.e., integration is **path independent**).
4. Integration of a conservative vector field over any **closed** contour is always equal to **zero**.

Example – 1

- Consider the conservative vector field: $\vec{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$
- Evaluate the contour integral: $\int_C \vec{A}(\bar{r}).\overline{dl}$

where $\vec{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$

and contour C is:



- The **beginning** of contour C is the point denoted as: $\bar{r}_A = 3\hat{a}_x - \hat{a}_y + 4\hat{a}_z$
- while the **end** point is denoted with position vector: $\bar{r}_B = -3\hat{a}_x - 2\hat{a}_z$

Note that ordinarily, this would be an **impossible** problem for **us** to do!

Example – 1 (contd.)

- we note that vector field $\vec{A}(\bar{r})$ is **conservative,** therefore:

$$\int_C \vec{A}(\bar{r}).\overline{dl} = \int_C \nabla g(\bar{r}).\overline{dl} \Rightarrow = g\left(\bar{r} = \bar{r}_B\right) - g\left(\bar{r} = \bar{r}_A\right)$$

- For this problem, it is evident that: $g(\bar{r}) = \left(x^2 + y^2\right)z$

- Therefore, $g(\bar{r} = \bar{r}_A)$ is the **scalar** field evaluated at $x = 3, y = -1, z = 4$; while $g(\bar{r} = \bar{r}_B)$ is the **scalar** field evaluated at at $x = -3, y = 0, z = -2$.

$$g\left(\bar{r} = \bar{r}_A\right) = \left((3)^2 + (-1)^2\right)4 = 40 \qquad g\left(\bar{r} = \bar{r}_B\right) = \left((-3)^2 + (0)^2\right)\left(-2\right) = -18$$

Therefore:

$$\int_C \vec{A}(\bar{r}).\overline{dl} = \int_C \nabla g(\bar{r}).\overline{dl} \Rightarrow = -18 - 40 = -58$$

The Curl of Conservative Fields

- Recall that every **conservative** field can be written as the gradient of some scalar field: $\vec{C}(\bar{r}) = \nabla g(\bar{r})$

Therefore:

$$C_x(\bar{r}) = \frac{\partial g(\bar{r})}{\partial x} \qquad C_y(\bar{r}) = \frac{\partial g(\bar{r})}{\partial y} \qquad C_z(\bar{r}) = \frac{\partial g(\bar{r})}{\partial z}$$

- Consider now the **curl of a conservative field**: $\nabla \times \vec{C}(\bar{r}) = \nabla \times \nabla g(\bar{r})$

- Recall that if $\vec{C}(\bar{r})$ is expressed using the **Cartesian** coordinate system, the curl of $\vec{C}(\bar{r})$ is:

$$\nabla \times \vec{C}(\bar{r}) = \left[\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right]\hat{a}_z$$

- Likewise, the **gradient** of $g(\bar{r})$ is:

$$\nabla \times \vec{C}(\bar{r}) = \left[\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right]\hat{a}_z$$

- Combining the two results:

$$\nabla \times \nabla g(\bar{r}) = \nabla \times \vec{C}(\bar{r}) = \left[\frac{\partial^2 g(\bar{r})}{\partial y \partial z} - \frac{\partial^2 g(\bar{r})}{\partial z \partial y}\right]\hat{a}_x + \left[\frac{\partial^2 g(\bar{r})}{\partial z \partial x} - \frac{\partial^2 g(\bar{r})}{\partial x \partial z}\right]\hat{a}_y + \left[\frac{\partial^2 g(\bar{r})}{\partial x \partial y} - \frac{\partial^2 g(\bar{r})}{\partial y \partial x}\right]\hat{a}_z$$

The Curl of Conservative Fields (contd.)

- We know: $$\frac{\partial^2 g(\bar{r})}{\partial y \partial z} = \frac{\partial^2 g(\bar{r})}{\partial z \partial y}$$
- each component of $\nabla \times \nabla g(\bar{r})$ is then equal to **zero**, and we can say: $$\nabla \times \nabla g(\bar{r}) = \nabla \times \vec{C}(\bar{r}) = 0$$

→ The **curl** of every **conservative** field is **equal to zero** !

Q: Are there some **non-**conservative fields whose curl is also equal to zero?
A: NO! The curl of a conservative field, and **only** a conservative field, is equal to **zero**.

- Thus, we have way to **test** whether some vector field $\vec{A}(\bar{r})$ is conservative: **evaluate its curl**!
  1. If the result **equals zero**—the vector field **is** conservative.
  2. If the result is **non-zero**—the vector field **is not** conservative.

The Curl of Conservative Fields (contd.)

- Let’s again **recap** what we’ve learnt about **conservative** fields:

1. The line integral of a conservative field is **path independent**.
2. Every conservative field can be expressed as the **gradient** of some scalar field.
3. The gradient of **any** and **all** scalar fields is a conservative field.
4. The line integral of a conservative field around any **closed** contour is equal to zero.
5. The **curl** of every conservative field is equal to **zero**.
6. The **curl** of a vector field is zero **only** if it is conservative.

The Solenoidal Vector Field

1. We know that a **conservative** vector field $\vec{C}(\bar{r})$ can be identified from its curl, which is always equal to zero:

$$\nabla \times \vec{C}(\bar{r}) = 0$$

- Similarly, there is **another** type of vector field $\vec{S}(\bar{r})$, called a **solenoidal** field, whose **divergence** always equals zero:

$$\nabla . \vec{S}(\bar{r}) = 0$$

Moreover, it should be noted that **only** solenoidal vector have zero divergence! Thus, zero divergence is a **test** for determining if a given vector field is solenoidal.

We sometimes refer to a solenoidal field as a **divergenceless** field.

The Solenoidal Vector Field (contd.)

2. Recall that **another** characteristic of a **conservative** vector field is that it can be expressed as the **gradient** of some **scalar** field (i.e., $\vec{C}(\bar{r})=\nabla g(\bar{r})$).

- Solenoidal vector fields have a **similar** characteristic! Every solenoidal vector field can be expressed as the **curl** of some other vector field (say $\vec{A}(\bar{r})$).

$$\vec{S}(\bar{r}) = \nabla \times \vec{A}(\bar{r})$$

- Additionally, it is important to note that **only** solenoidal vector fields can be expressed as the curl of some other vector field.

The curl of **any** vector field **always** results in a solenoidal field!

- Note if we **combine** these two previous equations, we get a **vector identity**:

$$\nabla . \nabla \times \vec{A}(\bar{r}) = 0$$

a result that is always true for **any** and **every** vector field $\vec{A}(\bar{r})$.

The Solenoidal Vector Field (contd.)

3. Now, let's recall the **divergence theorem**:

$$\iiint_v \nabla.\vec{A}(\bar{r})dv = \oiint_S \vec{A}(\bar{r}).\overline{ds}$$

- **If** the vector field $\vec{A}(\bar{r})$ is **solenoidal**, we can write this theorem as:

$$\iiint_v \nabla.\vec{S}(\bar{r})dv = \oiint_S \vec{S}(\bar{r}).\overline{ds}$$

But the divergence of a solenoidal field is **zero:**

$$\nabla.\vec{S}(\bar{r}) = 0$$

As a result, the **left** side of the divergence theorem is zero, and we can conclude that:

$$\oiint_S \vec{S}(\bar{r}).\overline{ds} = 0$$

In other words the **surface** integral of **any** and **every** solenoidal vector field across a **closed** surface is equal to zero.

- Note this result is **analogous** to evaluating a line integral of a conservative field over a closed contour:

$$\oint_C \vec{C}(\bar{r}).\overline{dl} = 0$$

The Solenoidal Vector Field (contd.)

- Lets **summarize** what we know about **solenoidal** vector fields:

1. **Every** solenoidal field can be expressed as the **curl** of some **other** vector field.
2. The curl of **any** and **all** vector fields always results in a solenoidal vector field.
3. The **surface integral** of a solenoidal field across any **closed** surface is equal to **zero.**
4. The **divergence** of every solenoidal vector field is equal to **zero**.
5. The divergence of a vector field is zero **only** if it is **solenoidal**.

Maxwell's Equations

$$\nabla.\vec{D} = \rho_v$$ $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ $$\nabla.\vec{B} = 0$$ $$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- Under the static conditions the Maxwell's equations become:

$$\nabla.\vec{D} = \rho_v$$ $$\nabla \times \vec{E} = 0$$ $$\nabla.\vec{B} = 0$$ $$\nabla \times \vec{H} = \vec{J}$$

Electric and Magnetic fields become decoupled under static conditions

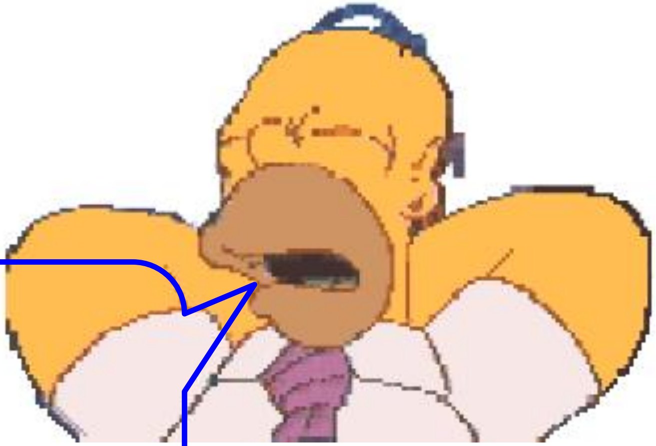
Enables us to study electricity and magnetism as distinct separate phenomena

We refer the study of electric and magnetic phenomena under static conditions as ***electrostatics*** and ***magnetostatics***

Maxwell's Equations (contd.)

The experience gained through studying electrostatics and magnetostatics phenomena will prove invaluable in tackling the more involved concepts which deal with time-varying fields

- **Oh yes!** We do not study electrostatics just as a prelude to the study of time-varying fields.
- **Electrostatics** is an **important concept** in its own right.
- Many electronics devices and systems are based on the principles of electrostatics.
- **Examples include:** x-ray machines, oscilloscopes, ink-jet electrostatic printers, liquid crystal displays, copy machines, micro-electro-mechanical switches (MEMS), accelerometers, and solid-state-based control devices etc.
- **Electrostatic principles** also guide the design of medical diagnostic sensors, such as the electrocardiogram, which records the heart's pumping pattern, and electroencephalogram, which records brain activity.



Q: I see ! Electrostatics is important as a distinct phenomena but not Magnetostatics. Right?

A: that is not correct! Magnetostatics is equally important and this concept is utilized in design of systems such as Loudspeakers, Door Bells, Magnetic Relays, Maglev Trains etc.

Electric Charge

- Most of classical physics can be described in terms of three fundamental units, which define our physical "reality".

Mass (e.g., Kg)

Distance (e.g., meters)

Time (e.g., seconds)

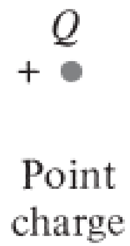
- From these fundamental units, we can define other important physical parameters such as Energy, Work, Pressure etc.
- However, these three fundamental units alone are insufficient for describing all of classical physics—we require one more to completely describe physical reality!
- This fourth fundamental unit is **Coulomb**, the unit of **electric charge**.

All **electromagnetic** phenomena can be attributed to electric charge!

Electric Charge (contd.)

- It should be noted that electric charge is **somewhat** analogous to mass. However, one important difference between mass and charge is that charge can be either **positive** or **negative**!
- Essentially, charge (like mass) is a property of **atomic particles**. Specifically, it is important to note that:
    - The charge "on" a **proton** is $+1.602 \times 10^{-19}$ C
    - The charge "on" a **neutron** is 0.0 C
    - The charge "on" an **electron** is -1.602×10^{-19} C

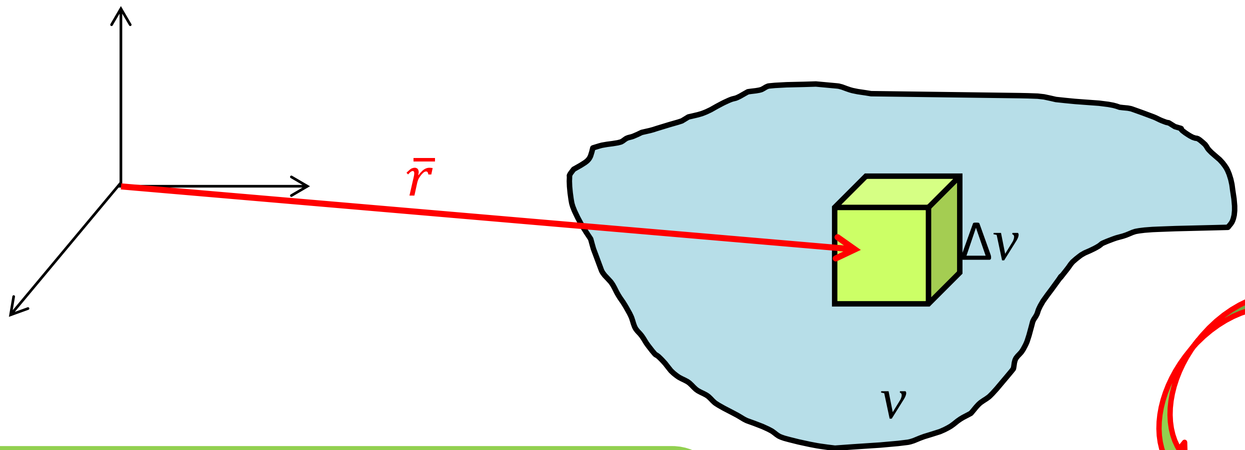
Example:



Charged particles (of all types) can be **distributed** (unevenly) across a volume, surface, or contour.

Charge Density

- In many cases, charged particles (e.g., electrons, protons, positive ions) are **unevenly distributed** throughout some volume v.
- We define **volume charge density** at a specific point $\bar{r}$ by evaluating the total net charge ΔQ in a small volume Δv surrounding the point.



$$\rho_v(\bar{r}) = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

IMPORTANT NOTE: Volume charge density indicates the **net** charge density at each point $\bar{r}$ within volume v.

Volume charge density is a **scalar field**, and is expressed with units such as **coulombs/m³**.

Charge Density (contd.)

Q: What is meant by **net** charge density ?

A: Remember, there are positively charged particles and there are negatively charged particles, and **both** can exist at the same location $\bar{r}$.

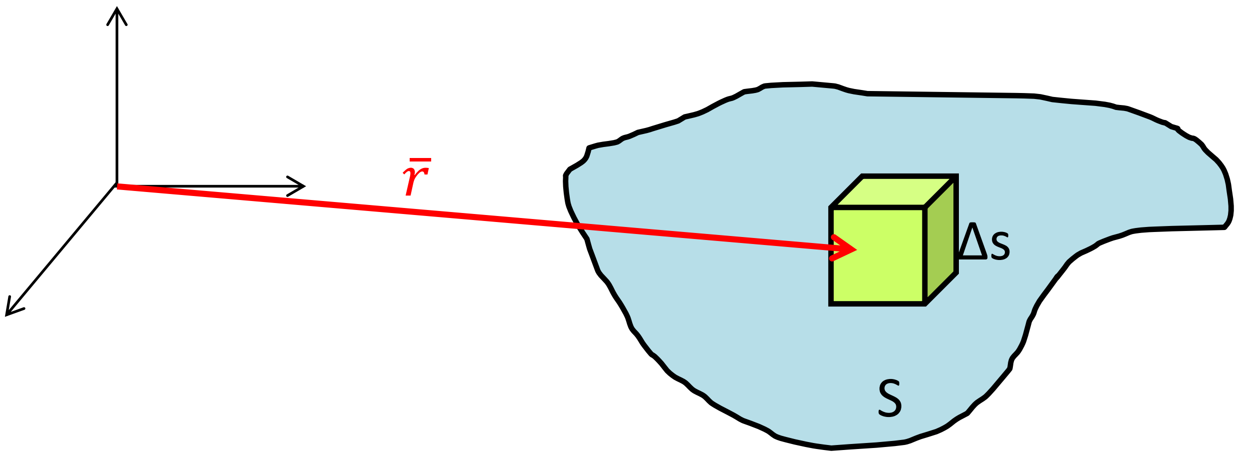
Thus, a **positive** charge density does **not** mean that **no** negatively charged particles (e.g., electrons) are present, it simply means that there is **more** positive charge than there is negative!

- It is therefore more instructive to define: $\Delta Q = \Delta Q^{+} + \Delta Q^{-}$
- **Volume** charge density can therefore be expressed as: $\rho_v(\bar{r}) \doteq \lim_{\Delta v \to 0} \frac{\Delta Q^{+} + \Delta Q^{-}}{\Delta v} = \rho_v^{+}(\bar{r}) + \rho_v^{-}(\bar{r})$
- **For example**, the charge density at some location $\bar{r}$ due to negatively charged particles might be 10.0 C/m^3, while that of positively charged particles might be 5.0 C/m^3. Therefore, the net, or **total** charge density is:

$$\rho_v^{+}(\bar{r}) + \rho_v^{-}(\bar{r}) = 5 + (-10) = -5.0C/m^3$$

Surface Charge Density

- Another possibility is that charge is unevenly distributed across some surface S. In this case, we can define a **surface charge density** as by evaluating the total charge ΔQ on a small patch of surface Δs, located at point $\bar{r}$ on surface S:



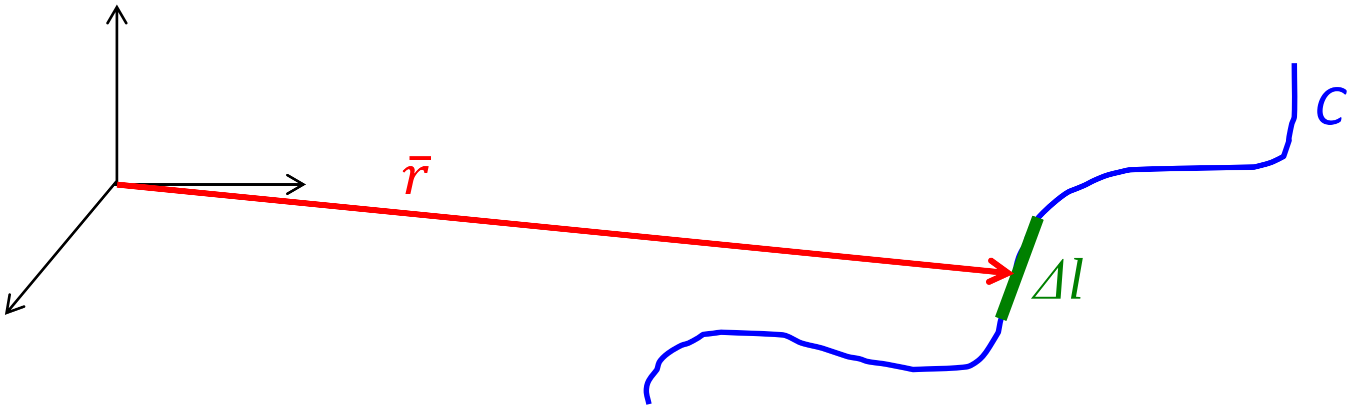
- **Surface** charge density $\rho_s(\bar{r})$ is defined as:

$$\rho_s(\bar{r}) \doteq \lim_{\Delta s \to 0} \frac{\Delta Q}{\Delta s}$$

Note the **units** for surface charge density will be **charge/area** (e.g. **C/m²**).

Line Charge Density

- Finally, let us consider the case where charge is unevenly distributed across some **contour** C. We can therefore define a **line charge density** as the charge ΔQ along a small distance Δl, located at point $\bar{r}$ of contour C.



- **Line** charge density $\rho_l(\bar{r})$ is defined as: $$\rho_l(\bar{r}) \doteq \lim_{\Delta l \to 0} \frac{\Delta Q}{\Delta l}$$

As you might expect, the units of a line charge density is charge per length (e.g., **C/m**).

Total Charge

Q: If we know charge density $\rho_v(\bar{r})$, describing the charge distribution throughout a **volume** v, can we determine the **total charge** Q contained within this volume?

A: Yes definitely! Simply **integrate** the charge density over the entire volume, and you get the **total charge** Q contained within the volume.

In other words: $Q = \iiint_v \rho_v(\bar{r})dv$

- we can determine the total charge distributed across a **surface** S by integrating the surface charge density:

$$Q = \iint_S \rho_s(\bar{r})ds$$

Q: Hey! is this **NOT** the surface integral we studied earlier.

A: True! This is a **scalar** integral; sort of a 2D version of the volume integral.

- The differential surface element ds in this integral is simply the **magnitude** of the differential surface vectors we studied earlier:

$$ds = |\overline{ds}|$$

- **For example**, if we integrate over the surface of a sphere, we use the differential surface element:

$$ds = |\overline{ds}_r| = r^2 \sin\theta d\theta d\phi$$

Total Charge (contd.)

- Finally, we can determine the total charge on **contour** C by integrating the **line charge density** $\rho_l(\bar{r})$ across the entire contour:

$$Q = \int_C \rho_l(\bar{r})\,dl$$

- The differential element *dl* is likewise related to the differential displacement vector we studied earlier:

$$dl = \left|\overline{dl}\right|$$

- **For example,** if the contour is a circle around the z-axis, then *dl* is:

$$ds = \left|\overline{d\phi}\right| = \rho d\phi$$

Example – 2

Find the total charge on a circular disc defined by $\rho \leq a$ and $z = 0$ if: $\rho_S = \rho_{S0} e^{-\rho}$ (C/m²).

$$Q = \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-\rho} \rho\, d\rho\, d\phi \quad \Rightarrow \quad Q = 2\pi\rho_{s0} \int_0^a \rho e^{-\rho} d\rho$$

$$\therefore Q = 2\pi\rho_{s0}\left[-\rho e^{-\rho} - e^{-\rho}\right]_0^a$$

Example – 3

A circular beam of charge of **radius $\boldsymbol{a}$** consists of electrons moving with a **constant speed $\boldsymbol{u}$** along the $+\boldsymbol{z}$ **direction**. The beam's axis is coincident with the z-axis and the electron **charge density is given by: $\boldsymbol{\rho_v = -c\rho^2}$ (C/m³),** where c is a constant and ρ is the radial distance from the axis of the beam. **Determine the charge density per unit length.**

Ans: $$\rho_l = -\frac{\pi c a^4}{2} C/m$$

Example – 4

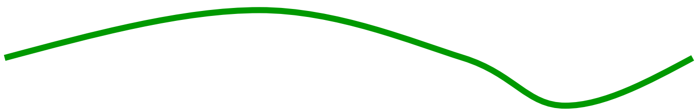
A square plate in the x–y plane is situated in the space defined by $-3m \le x \le 3m$ and $-3m \le x \le 3m$. Find the total charge on the plate if the surface charge density is given by $\rho_s = 4y^2 \left({}^{\mu C}/_{m^2} \right)$.

Electric Charge – Review

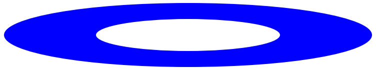
- Charge is measured in Coulombs (C). A Coulomb is a lot of charge.
- Charge comes in both positive and negative amounts.
- Charge is conserved – it can neither be created nor destroyed.

Charge can be spread out ….

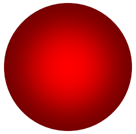
- Charge may be at a point, on a line, on a surface, or throughout a volume
- Linear charge density ρ_l units C/m
  - Multiply by length



- Surface charge density ρ_s units C/m^2
  - Multiply by area



- Charge density ρ_v units C/m^3
  - Multiply by volume



Goal of next few lectures

- Develop dexterity in applying the expressions for the electric field intensity $\vec{E}$ induced by specified distribution of charge.
- For now, our discussion will be limited to electrostatic fields generated by stationary charges.
- We will begin by considering the expression for the electric field developed by **Coulomb**.