ECE 215

Lecture – 7 **Date: 22.08.2017**

- Phasors
- AC Circuits: Impedance and Admittance, Kirchoff's Laws, Phase Shifter

Phasors (contd.)

As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

Phasor domain is also called frequency domain

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ)$$

$$= \mathrm{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \mathrm{Re}(j\omega \mathbf{V} e^{j\omega t})$$

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega\mathbf{V}$$

(Time domain) (Phasor domain)

$$\int v\, dt \quad \Leftrightarrow \quad \frac{\mathbf{V}}{j\omega}$$

(Time domain) (Phasor domain)

The differences between $v(t)$ and **V** should be understood:

1. $v(t)$ is the *instantaneous or time domain* representation, while **V** is the *frequency or phasor domain* representation.
2. $v(t)$ is time dependent, while **V** is not.
3. $v(t)$ is always real with no complex term, while **V** is generally complex.

Example – 8: If $f(\phi) = \cos\phi + j\sin\phi$, show that $f(\phi) = e^{j\phi}$.

Example – 9: Find the phasors corresponding to the following signals:

(a) $v(t) = 21\cos(4t - 15^o)$ V
(b) $i(t) = -8\sin(10t + 70^o)$ mA
(c) $v(t) = 120\sin(10t - 50^o)$ V
(d) $i(t) = -60\cos(30t + 10^o)$ mA

Example – 10: Obtain the sinusoids corresponding to each of the following phasors:

(a) $\mathbf{V}_1 = 60\angle 15^o$ V, $\omega = 1$
(b) $\mathbf{V}_2 = 6 + j8$ V, $\omega = 40$
(c) $\mathbf{I}_1 = 2.8e^{-j\pi/3}$ A, $\omega = 377$
(d) $\mathbf{I}_2 = -0.5 - j1.2$ A, $\omega = 10^3$

Example – 11: Simplify the following:

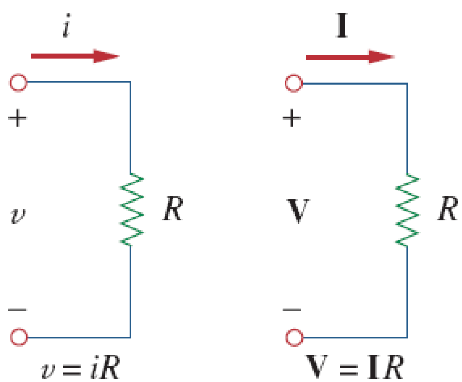
(a) $f(t) = 5\cos(2t + 15^o) - 4\sin(2t - 30^o)$
(b) $g(t) = 8\sin t + 4\cos(t + 50^o)$
(c) $h(t) = \int_0^t (10\cos 40t + 50\sin 40t)dt$

Example – 12: Using phasors, determine $i(t)$ in the following equations:

(a) $2\dfrac{di}{dt} + 3i(t) = 4\cos(2t - 45^o)$

(b) $10\int i\ dt + \dfrac{di}{dt} + 6i(t) = 5\cos(5t + 22^o)$

Phasor Relationships for Circuit Elements

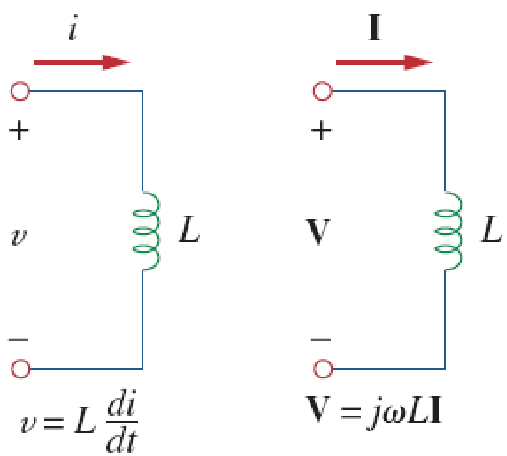


If the current through a resistor R is $i = I_m(cos\omega t + \varphi)$, then the voltage across it is given by Ohm's law as:

$$v = iR = RI_m\cos(\omega t + \phi) \qquad \mathbf{V} = RI_m\angle\phi$$

$$\mathbf{V} = R\mathbf{I}$$

∴ voltage-current relation for the resistor in the phasor domain continues to be Ohm's law



For the inductor L, assume current $i = I_m(cos\omega t + \varphi)$, then the voltage across it is:

$$v = L\frac{di}{dt} = -\omega LI_m\sin(\omega t + \phi)$$

$$v = \omega LI_m\cos(\omega t + \phi + 90^\circ)$$

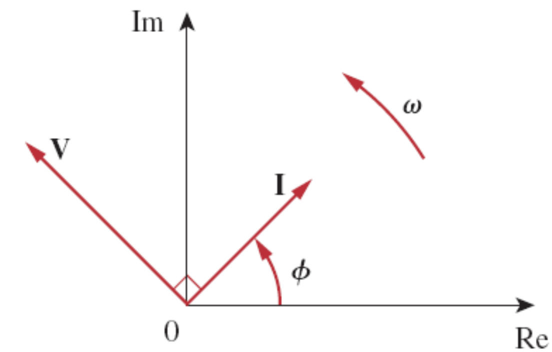
Phasor Relationships for Circuit Elements

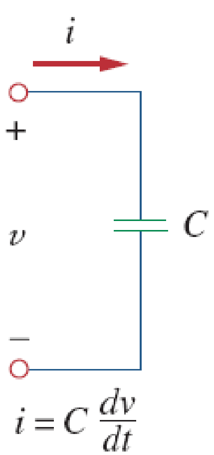
$$v = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

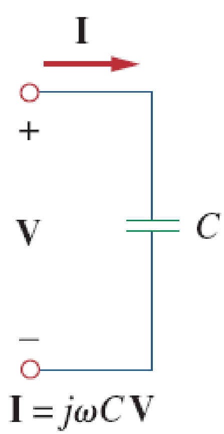
$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \underline{/\phi + 90^\circ}$$

$$\mathbf{V} = j\omega L \mathbf{I}$$

the voltage has a magnitude of $\omega L I_m$ and a phase of φ. The voltage and current are 90^o out of phase. Specifically, the current lags the voltage by 90^o.

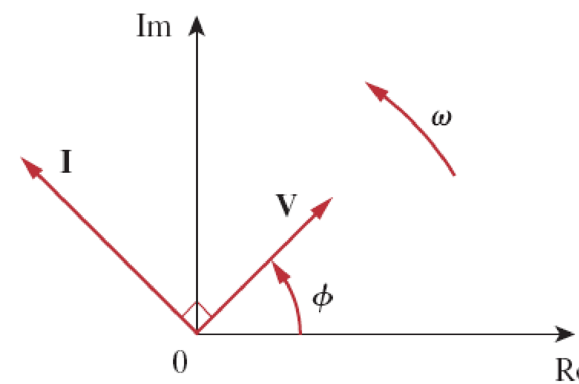






$$\mathbf{I} = j\omega C \mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

the current leads the voltage by 90^o.



Example – 13

What is the instantaneous voltage across a 2μF capacitor when the current through it is $i = 4\sin(10^6 t + 25^o)\, A$?

Example – 14

A voltage $v(t) = 100\cos(60t + 20^o)$V is applied to a parallel combination of a 40kΩ resistor and a 50μF capacitor. Find the steady-state currents through the resistor and the capacitor.

Example – 15

A series *RLC* circuit has *R* = 80 Ω, *L* = 240mH, and *C* = 5mF. If the input voltage is $v(t) = 100\cos(2t)$, find the current flowing through the circuit.

Impedance and Admittance

- we know: $\mathbf{V} = R\mathbf{I}$, $\mathbf{V} = j\omega L\mathbf{I}$, $\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$ $\dfrac{\mathbf{V}}{\mathbf{I}} = R$, $\dfrac{\mathbf{V}}{\mathbf{I}} = j\omega L$, $\dfrac{\mathbf{V}}{\mathbf{I}} = \dfrac{1}{j\omega C}$

- we express Ohm's law in phasor form: $\mathbf{Z} = \dfrac{\mathbf{V}}{\mathbf{I}}$ or $\mathbf{V} = \mathbf{Z}\mathbf{I}$

where **Z** is a frequency-dependent quantity known as *impedance*, measured in ohms. It is the ratio of the phasor voltage **V** to the phasor current **I**, measured in Ω.

$$\mathbf{Z}_L = j\omega L \text{ and } \mathbf{Z}_C = -j/\omega C$$

- For $\omega = 0$(i.e., dc sources): $Z_L = 0$ and $Z_C \rightarrow \infty$.
- the inductor acts like a short circuit, while the capacitor acts like an open circuit.

- For $\omega \rightarrow \infty$(i.e., high frequencies): $Z_L \rightarrow \infty$ and $Z_C = 0$.
- the inductor acts like an open circuit, while the capacitor acts like a short circuit.

- complex quantity $\mathbf{Z} = R + jX$ → $\mathbf{Z} = |\mathbf{Z}| \underline{/\theta}$

$$R = |\mathbf{Z}|\cos\theta, \quad X = |\mathbf{Z}|\sin\theta \qquad |\mathbf{Z}| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R}$$

Impedance and Admittance

- It is sometimes (parallel circuits) convenient to work with the reciprocal of impedance, known as *admittance* ***(Y)***.

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \Rightarrow \mathbf{Y} = G + jB$$

$$G = \frac{R}{R^2 + X^2}$$

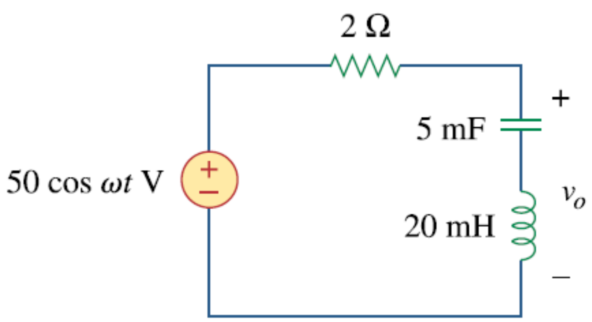
$$B = -\frac{X}{R^2 + X^2}$$

Example – 1

A linear network has a current input $4\cos(\omega t + 20°)$A and a voltage output $10\cos(\omega t + 110°)$V. Determine the associated impedance.

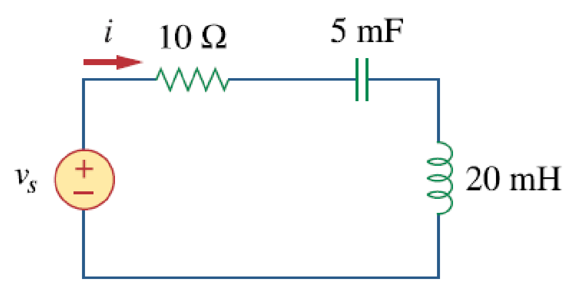
Example – 2

What value of ω will cause the forced response v_0 in this circuit to be zero?



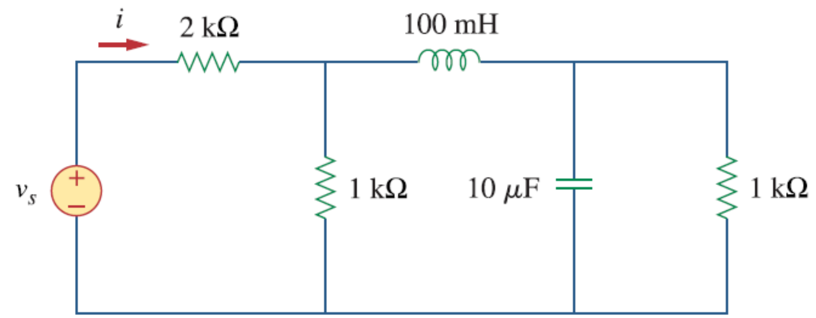
Example – 3

Find current i in this circuit, when $v_s(t) = 50cos200t$ V.



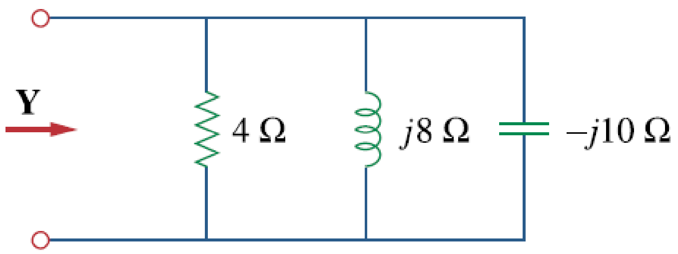
Example – 4

Find current i in this circuit, whe[n] $v_s(t) = 60\cos(200t - 10°)$ V.



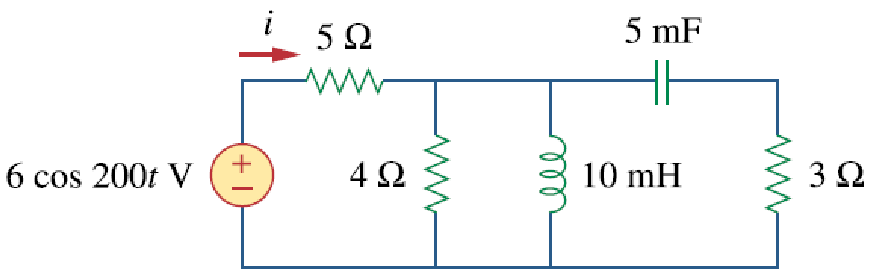
Example – 5

Determine the admittance **Y** for this circuit.



Example – 6

Find current i in this circuit.



Kirchoff's Laws

- cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws.
- Therefore, need to express them in the frequency domain.

Kirchoff's Laws

For KVL, let $v_1, v_2, \ldots \ldots .., v_n$ are the voltages around a closed loop.

$$v_1 + v_2 + \cdots + v_n = 0 \Rightarrow V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0$$

$$\mathrm{Re}(V_{m1}e^{j\theta_1}e^{j\omega t}) + \mathrm{Re}(V_{m2}e^{j\theta_2}e^{j\omega t}) + \cdots + \mathrm{Re}(V_{mn}e^{j\theta_n}e^{j\omega t}) = 0$$

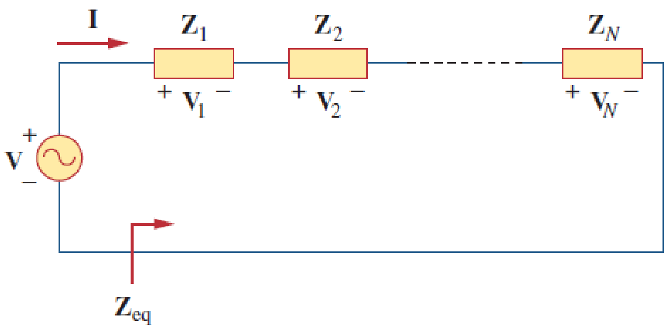
$$\mathrm{Re}[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n})e^{j\omega t}] = 0$$

$$\Rightarrow \mathrm{Re}[(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n)e^{j\omega t}] = 0 \Rightarrow \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Kirchhoff's voltage law holds for phasors.

Similarly, one can prove that Kirchhoff's current law holds in the frequency domain

Impedance Combinations

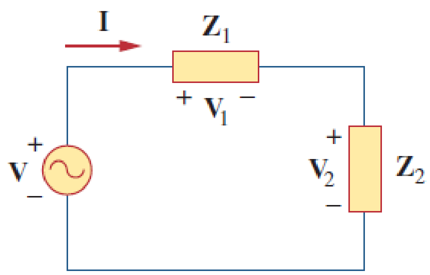


$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

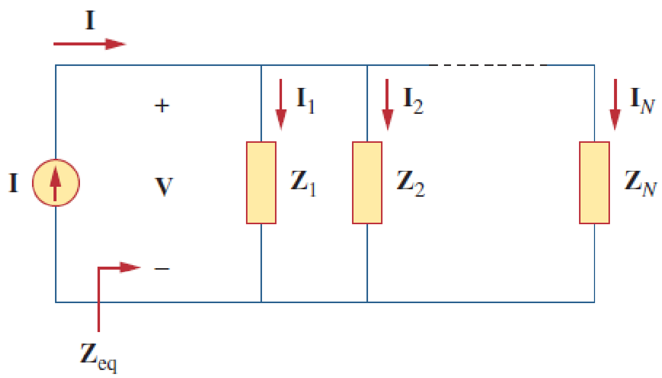
This is similar to the series connection of resistances.

Impedance Combinations



$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad \mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

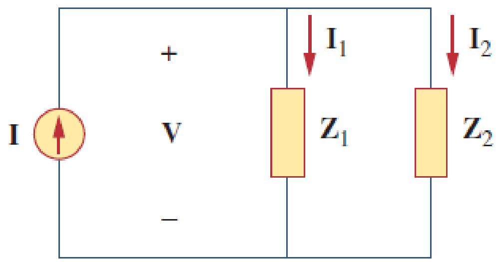
voltage-division relationship



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

$$\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N$$



$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

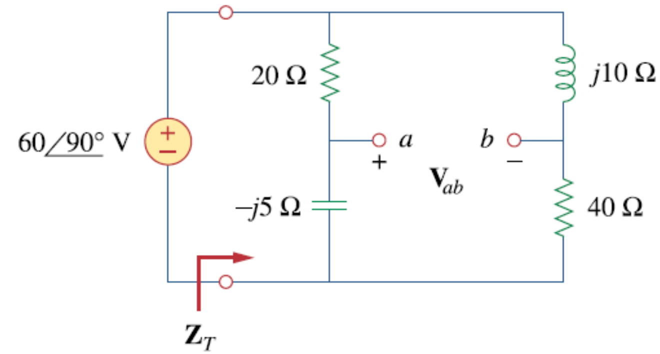
$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

the *current-division* principle.

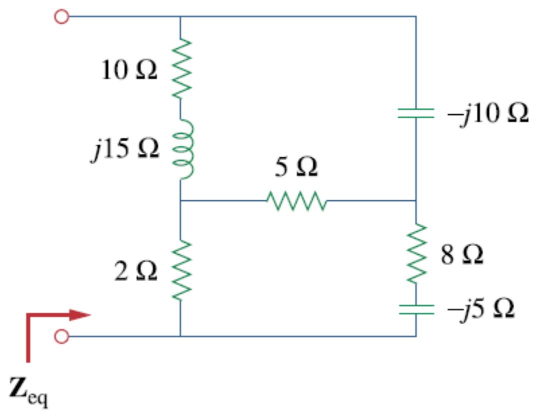
Example – 7

For this circuit, calculate $\mathbf{Z}_T$ and $\mathbf{V}_{ab}$.



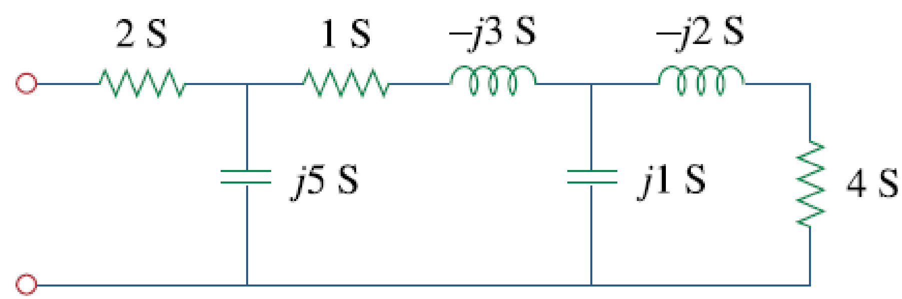
Example – 9

Find the equivalent impedance of this circuit.



Example – 8

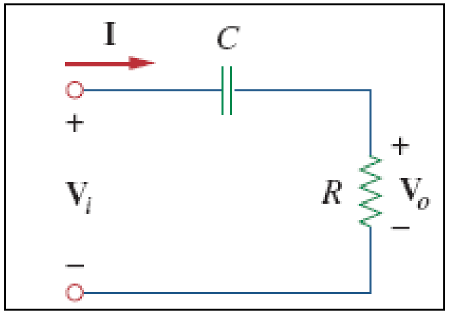
Find the equivalent admittance $\mathbf{Y}_{eq}$ of this circuit.

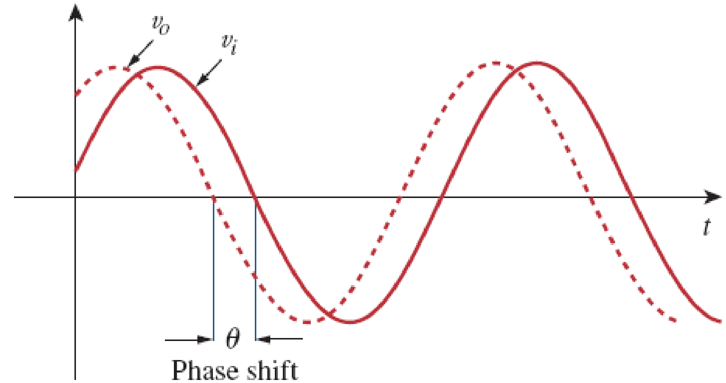


Phase Shifter

- A phase-shifting circuit is used for correcting undesirable phase shift present in a circuit.
- It is also used for the creation of desired phase shifts.
- *RC and RL circuits are extremely useful for this purpose.*

Phase Shifter





the circuit current **I** leads the applied voltage by some phase angle θ, where $0 < \theta < 90^{o}$ depending on the values of *R* and *C*.

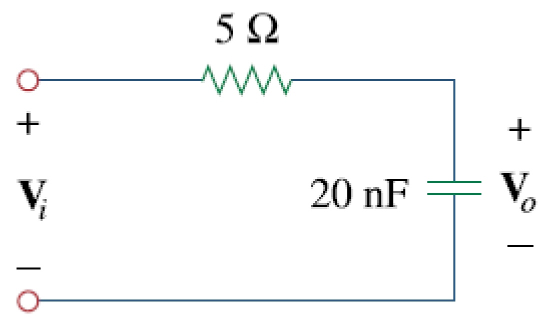
$$\mathbf{Z} = R + jX_C, \quad \Rightarrow \quad \theta = \tan^{-1}\frac{X_C}{R}$$

the amount of phase shift depends on the values of *R*, *C*, and the operating frequency.

- These simple single stage RC circuits are generally not used in practice.
- These RC circuits also work as voltage dividers. Therefore, as the phase shift approaches 90^{o} the output voltage approaches zero. For this reason, these simple *RC* circuits are used only when small amounts of phase shift are required.
- For large phase shifts, the *RC* networks are cascaded. This provides a total phase shift equal to the sum of the individual phase shifts.

Example – 10

For this RC circuit:
(a) Calculate the phase shift at 2 MHz.
(b) Find the frequency where the phase shift is 45^o.



Example – 11

A coil with impedance 8 + *j*6 Ω is connected in series with a capacitive reactance *X*. The series combination is connected in parallel with a resistor *R*. Given that the equivalent impedance of the resulting circuit is $5\angle 0^o$ Ω, find the value of *R* and *X*.

Example – 12

Consider this phase-shifting circuit.

(a) $\mathbf{V}_o$ when *R* is maximum
(b) $\mathbf{V}_o$ when *R* is minimum
(c) the value of *R* that will produce a phase shift of 45^o.