Lecture – 24 **Date: 31.10.2017**

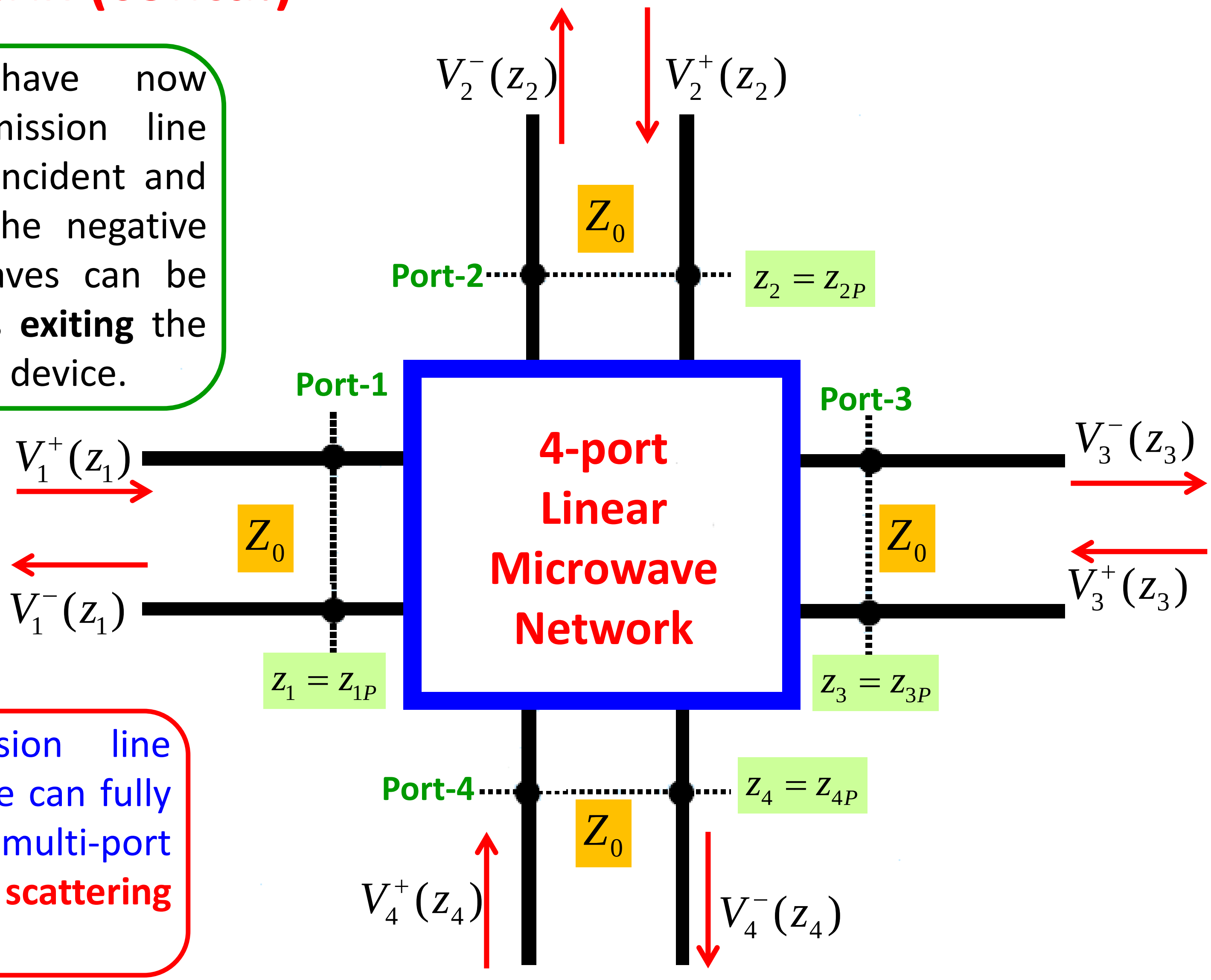
- Multi-port networks (Contd.), Scattering Matrix
- Matched, Lossless, and Reciprocal Networks

Scattering Matrix

- At **"low"** frequencies, a **linear** device or network can be fully characterized using an **impedance or admittance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.
- But, at high frequencies, it is **not feasible** to measure total currents and voltages!
- Instead, we can measure the **magnitude** and **phase** of each of the two transmission line **waves** $V^{+}(z)$ and $V^{-}(z)$ → enables determination of relationship between the incident and reflected waves at **each** device terminal to the incident and reflected waves at **all** other terminals
- These relationships are completely represented by the **scattering matrix** that **completely** describes the behavior of a linear, multi-port device at a **given frequency** ω, and a given line impedance Z_0

Scattering Matrix (contd.)

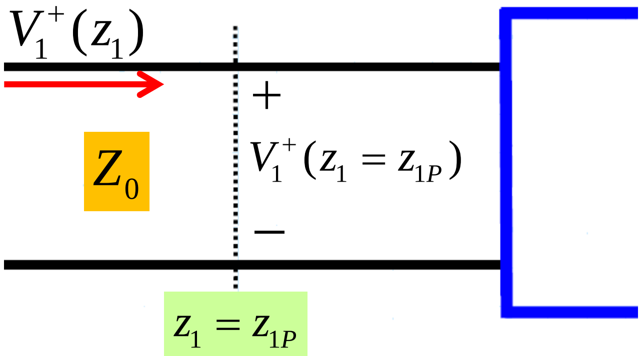
Note that we have now characterized transmission line activity in terms of incident and "reflected" waves. The negative going "reflected" waves can be viewed as the waves **exiting** the multi-port network or device.



Viewing transmission line activity this way, we can fully characterize a multi-port device by its **scattering parameters**!

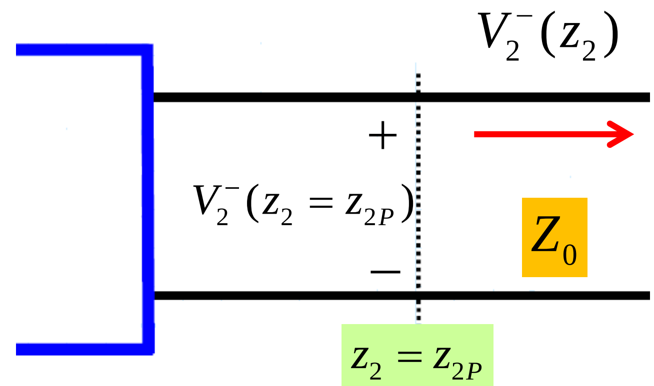
Scattering Matrix (contd.)

- Say there exists an **incident** wave on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).



Say we measure/determine the voltage of the wave flowing **into port 1,** at the port 1 **plane** (i.e., determine $V_1^+(z_1 = z_{1P})$).

Say we then measure/determine the voltage of the wave flowing **out** of **port 2,** at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).



The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is known as the **scattering parameter** S_{21}

Scattering Matrix (contd.)

Therefore:

$$S_{21} = \frac{V_2^-(z_2 = z_{2P})}{V_1^+(z_1 = z_{1P})} = \frac{V_2^- e^{+j\beta z_{2P}}}{V_1^+ e^{-j\beta z_{1P}}} = \frac{V_2^-}{V_1^+} e^{+j\beta(z_{2P}+z_{1P})}$$

Similarly:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})}$$

$$S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

- We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3) and $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4), given that the input to all other ports (1,2, and 3) are zero
- Thus, more **generally**, the ratio of the wave incident on port **n** to the wave emerging from port **m** is:

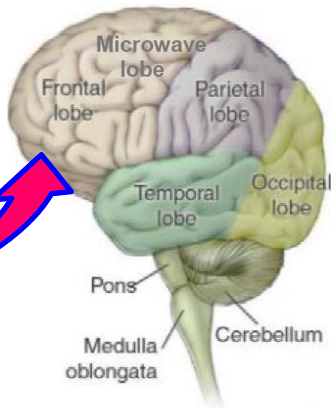
$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \qquad V_k^+(z_k) = 0 \quad \textbf{for all } k \neq n$$

Scattering Matrix (contd.)

- Note that, frequently the port positions are assigned a **zero** value (e.g., z_{1P}=0, z_{2P}=0). This of course **simplifies** the scattering parameter calculation:

$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_m^+ e^{+j\beta 0}}{V_n^- e^{-j\beta 0}} = \frac{V_m^+}{V_n^-}$$

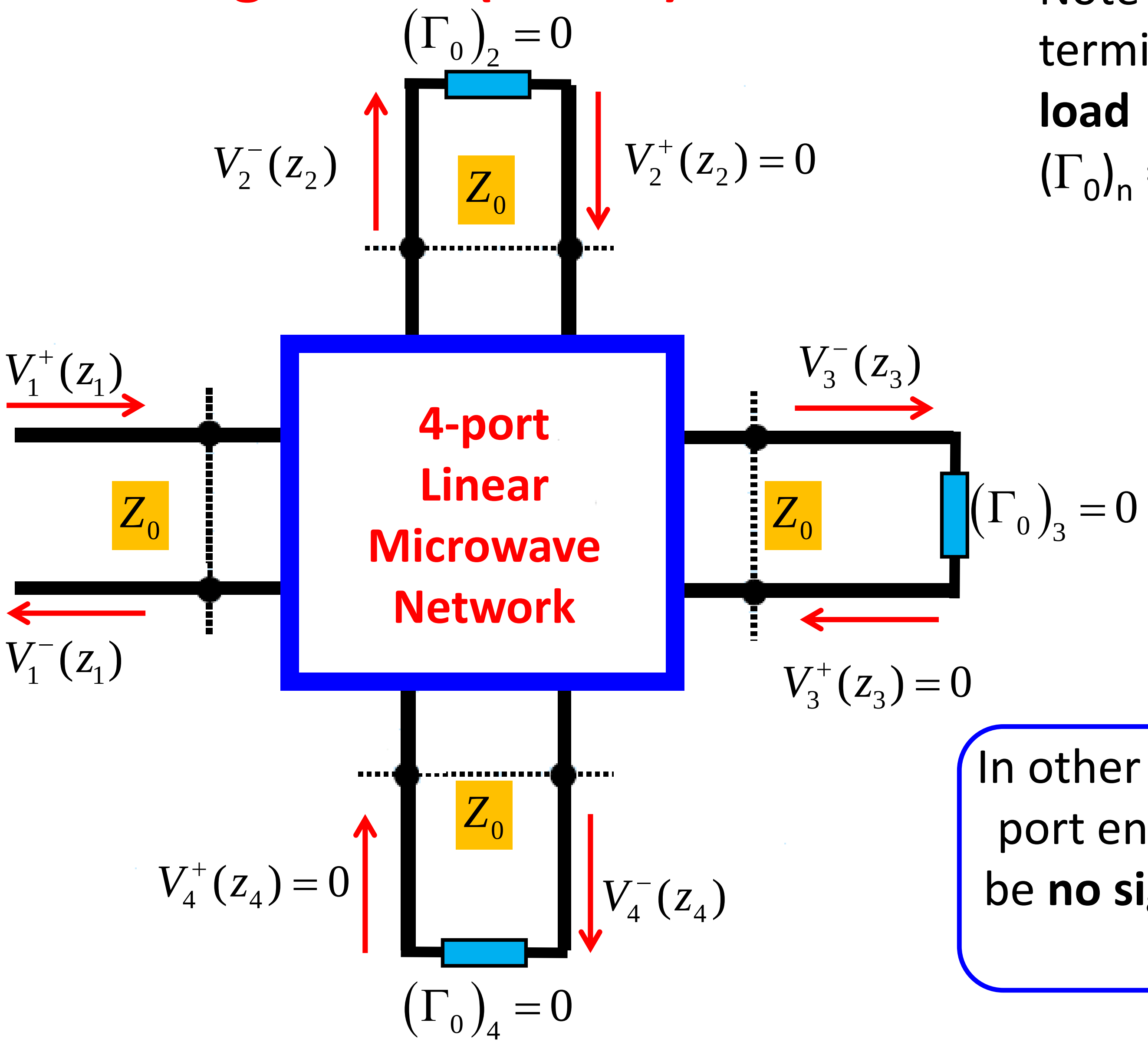
- We will **generally assume** that the port locations are defined as z_{nP}=0, and thus use the **above** notation. But **remember** where this expression came from!



Q: How do we ensure that **only one** incident wave is non-zero ?

A: Terminate all other ports with a **matched load**!

Scattering Matrix (contd.)



- Note that **if** the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $(\Gamma_0)_n = 0$ and therefore:

$$V_n^+(z_n) = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Scattering Matrix (contd.)

Just between you and me, I think you've messed this up! **In all** previous slides you said that if $\Gamma_0 = 0$, the wave in the **minus** direction would be zero:

$$V^{-}(z) = 0 \qquad \text{if} \qquad \Gamma_0 = 0$$

but just **now** you said that the wave in the **positive** direction would be zero:

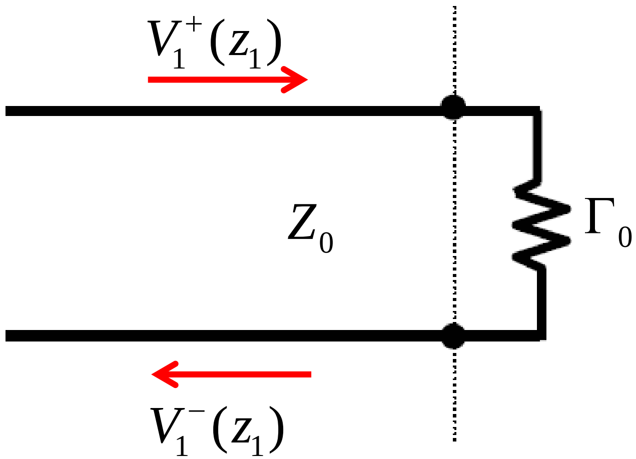
$$V^{+}(z) = 0 \qquad \text{if} \qquad \Gamma_0 = 0$$

Obviously, there is **no way** that **both** statements can be correct!

Scattering Matrix (contd.)

Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we **originally** analyzed this case:

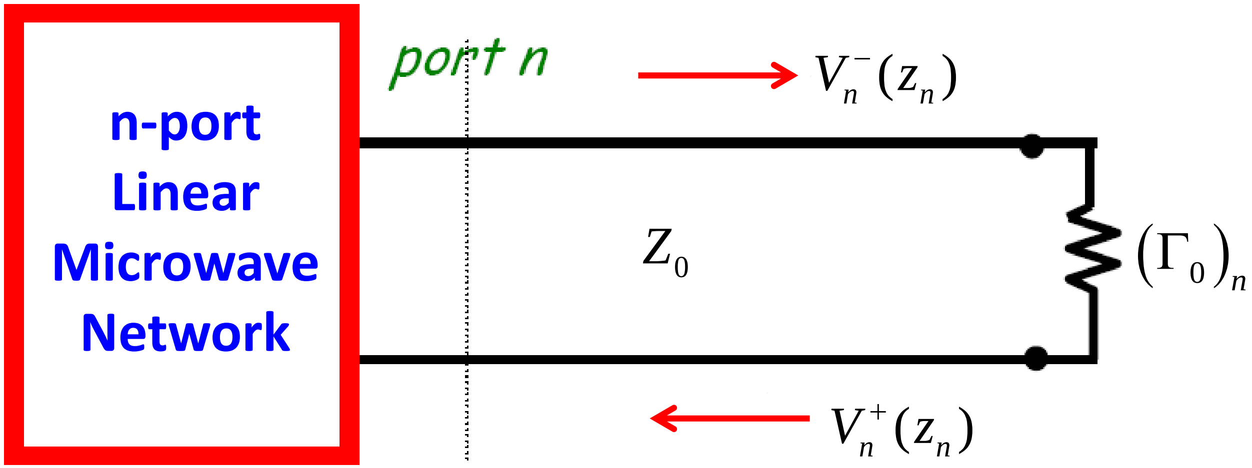


$V^-(z) = 0$ if $\Gamma_0 = 0$

In this original case, the wave **incident** on the load is $V^+(z)$ (**plus** direction), while the **reflected** wave is $V^-(z)$ (**minus** direction).

Scattering Matrix (contd.)

Contrast this with the case we are **now** considering:



- For this current case, the situation is **reversed**. The wave incident on the load is **now** denoted as $V_n^-(z_n)$ (coming **out** of port n), while the wave reflected off the load is **now** denoted as $V_n^+(z_n)$ (going **into** port n).

Scattering Matrix (contd.)

- **back** to our discussion of **S-parameters**. We found that **if** z_{nP} = 0 for all ports n, the scattering parameters could be directly written in terms of wave **amplitudes** V_n^+ and V_m^-

$$S_{mn} = \frac{V_m^-}{V_n^+} \qquad V_k^+(z_k) = 0$$

for all k ≠ n

- Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_m^-}{V_n^+}$$

(for all ports, except port *n*, are terminated in matched loads)

- One more **important** note—notice that for the ports terminated in matched loads (i.e., those ports with **no** incident wave), the voltage of the exiting **wave** is also the **total** voltage!

$$V_m(z_m) = V_m^+ e^{-j\beta z_m} + V_m^- e^{+j\beta z_m} = 0 + V_m^- e^{+j\beta z_m} = V_m^- e^{+j\beta z_m}$$

⇐ For all terminated ports!

Scattering Matrix (contd.)

- We can use the scattering matrix to determine the solution for a more **general** circuit—one where the ports are **not** terminated in matched loads!
- Since the device is **linear**, we can apply **superposition**. The output at any port due to **all** the incident waves is simply the coherent **sum** of the output at that port due to **each** wave!
- For example, the **output** wave at port 3 can be determined by (assuming $z_{nP} = 0$):

$$V_3^- = S_{34}V_4^+ + S_{33}V_3^+ + S_{32}V_2^+ + S_{31}V_1^+$$

- More **generally**, the output at port *m* of an N-port device is:

$$V_m^- = \sum_{n=1}^{N} S_{mn}V_n^+ \qquad z_{nP} = 0$$

- This expression of Scattering parameter can be written in **matrix** form as:

$$\mathbf{V^- = SV^+}$$

Scattering Matrix (contd.)

$$V^- = SV^+$$

Scattering Matrix

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1n} \\ S_{21} & & & \vdots \\ \vdots & & & \\ S_{m1} & S_{m2} & \dots & S_{mn} \end{bmatrix}$$

- The scattering matrix is N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_0 describes a single-port device (e.g., a load)!
- The values of the scattering matrix for a particular device or network, like Γ_0, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\boldsymbol{S(\omega)} = \begin{bmatrix} S_{11}(\omega) & S_{12}(\omega) & \dots & S_{1n}(\omega) \\ S_{21}(\omega) & & & \vdots \\ \vdots & & & \\ S_{m1}(\omega) & S_{m2}(\omega) & \dots & S_{mn}(\omega) \end{bmatrix}$$

- Also realize that—also just like Γ_0—the scattering matrix is dependent on **both** the **device/network** and the Z_0 value of the **cable connected** to it.
- Thus, a device connected to cables with Z_0 =50Ω will have a **completely different scattering matrix** than that same device connected to transmission lines with Z_0 =100Ω

Matched, Lossless, Reciprocal Devices

- A device can be **lossless** or **reciprocal**. In addition, we can also classify it as being **matched.**
- Let's examine **each** of these three characteristics, and how they relate to the **scattering matrix.**

Matched Device

A matched device is another way of saying that the **input impedance** at each port is **equal to Z_0** when **all other** ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will come out from a port if a signal is incident on that port (but **only** that port!).

- **In other words:** $V_m^- = S_{mm} V_m^+ = 0$ For all m → **When all the ports 'm' are matched**

- It is apparent that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

$$\mathbf{S} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

Matched, Lossless, Reciprocal Devices (contd.)

Lossless Device

- For a lossless device, all of the power that is delivered to each device port must eventually find its way **out**!
- In other words, power is not **absorbed** by the network—no power to be **converted to heat**!
- The **power incident** on some port m is related to the amplitude of the **incident wave** (V_m^+) as:

$$P_m^+ = \frac{\left|V_m^+\right|^2}{2Z_0}$$

- The power of the **wave exiting** the port is:

$$P_m^- = \frac{\left|V_m^-\right|^2}{2Z_0}$$

- power absorbed by that port is the **difference** of the incident power and reflected power:

$$\Delta P_m = P_m^+ - P_m^- = \frac{\left|V_m^+\right|^2}{2Z_0} - \frac{\left|V_m^-\right|^2}{2Z_0}$$

Matched, Lossless, Reciprocal Devices (contd.)

- For an N-port device, the total incident power is:

$$P^{+} = \sum_{m=1}^{N} P_m^{+} = \frac{1}{2Z_0} \sum_{m=1}^{N} \left|V_m^{+}\right|^2$$

$$\left|V_m^{+}\right|^2 = \left(\mathbf{V}^{+}\right)^H \mathbf{V}^{+}$$

$(\mathbf{V}^{+})^H$ is the conjugate transpose of the row vector $\mathbf{V}^{+}$

$$P^{+} = \sum_{m=1}^{N} P_m^{+} = \frac{\left(\mathbf{V}^{+}\right)^H \mathbf{V}^{+}}{2Z_0}$$

Similarly, the total reflected power

$$P^{-} = \sum_{m=1}^{N} P_m^{-} = \frac{\left(\mathbf{V}^{-}\right)^H \mathbf{V}^{-}}{2Z_0}$$

- Recall that the incident and reflected wave amplitudes are **related** by the **scattering matrix** of the device as:

$$\mathbf{V}^{-} = \mathbf{S}\mathbf{V}^{+}$$

- Therefore:

$$P^{-} = \frac{\left(\mathbf{V}^{-}\right)^H \mathbf{V}^{-}}{2Z_0} = \frac{\left(\mathbf{V}^{+}\right)^H \mathbf{S}^H \mathbf{S}\mathbf{V}^{+}}{2Z_0}$$

- Therefore the **total power delivered** to the N-port device is:

$$\Delta P = P^{+} - P^{-} = \frac{\left(\mathbf{V}^{+}\right)^H \mathbf{V}^{+}}{2Z_0} - \frac{\left(\mathbf{V}^{+}\right)^H \mathbf{S}^H \mathbf{S}\mathbf{V}^{+}}{2Z_0}$$

$$\Rightarrow \Delta P = \frac{\left(\mathbf{V}^{+}\right)^H}{2Z_0}\left(\boldsymbol{I} - \mathbf{S}^H \mathbf{S}\right)\mathbf{V}^{+}$$

Matched, Lossless, Reciprocal Devices (contd.)

- For a lossless device: **ΔP=0** $\Rightarrow \dfrac{(\mathbf{V}^+)^H}{2Z_0}(\boldsymbol{I}-\mathbf{S}^H\mathbf{S})\mathbf{V}^+ = 0$ ⇐ **For all V⁺**
- Therefore: $\boldsymbol{I}-\mathbf{S}^H\mathbf{S}=0$ ⇒ $\Rightarrow \mathbf{S}^H\mathbf{S}=\boldsymbol{I}$

a special kind of matrix known as a **unitary matrix**

> If a network is **lossless**, then its scattering matrix **S** is **unitary**

- **How to recognize a unitary matrix?**

The **columns** of a unitary matrix form an **orthonormal set**!

Example:

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

each **column** of the scattering matrix will have a **magnitude equal to one**

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \qquad \textbf{For all n}$$

inner product (i.e., dot product) of **dissimilar columns** must be **zero**

$$\sum_{m=1}^{N} S_{mi}S_{mj}^* = S_{1i}S_{1j}^* + S_{2i}S_{2j}^* + \dots + S_{Ni}S_{Nj}^* = 0 \qquad \textbf{For all } i \neq j$$

dissimilar columns are orthogonal

Matched, Lossless, Reciprocal Devices (contd.)

- For example, for a lossless **three-port** device: say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{|V_1^+|^2}{2Z_0}$$

- and the power **exiting** the device at each port is:

$$P_m^- = \frac{|V_m^-|^2}{2Z_0} = \frac{|S_{m1}V_1^+|^2}{2Z_0} = |S_{m1}|^2 P_1^+$$

- The **total** power exiting the device is therefore:

$$P^- = P_1^- + P_2^- + P_3^- = |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+$$

$$\Rightarrow P^- = \left(|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2\right) P_1^+$$

- Since this device is **lossless**, the incident power (**only** on port 1) is **equal** to exiting power (i.e, $P^- = P_1^+$). This is true **only** if:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1$$

- Of course, this will be true if the incident wave is placed on **any** of the **other** ports of this lossless device:

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$

$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1$$

Matched, Lossless, Reciprocal Devices (contd.)

- We can state in general then that: $\sum_{m=1}^{N} |S_{mn}|^2 = 1$ For all n
- In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.
- An **example** of a (unitary) scattering matrix for a 4-port **lossless** device is:

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1/2 & j\sqrt{3}/2 & 0 \\ 1/2 & 0 & 0 & j\sqrt{3}/2 \\ j\sqrt{3}/2 & 0 & 0 & 1/2 \\ 0 & j\sqrt{3}/2 & 1/2 & 0 \end{bmatrix}$$

Reciprocal Device

- Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.
- For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

Matched, Lossless, Reciprocal Devices (contd.)

- For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$S^T = S$$

where T indicates transpose.

- An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\boldsymbol{S} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10-j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$

Example – 3

- A **lossless, reciprocal** 3-port device has S-parameters of $S_{11} = {}^{1}/_{2}$, $S_{31} = {}^{1}/_{\sqrt{2}}$, and $S_{33} = 0$. It is likewise known that all scattering parameters are **real**.

→ Find the remaining **6** scattering parameters.

Q: This problem is clearly **impossible**—you have not provided us with sufficient **information**!

A: Yes I have! Note I said the device was **lossless** and **reciprocal**!

Example – 3 (contd.)

- Start with what we **currently** know:

$$\mathbf{S}=\begin{bmatrix} 1/2 & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ 1/\sqrt{2} & S_{32} & 0 \end{bmatrix}$$

- As the device is **reciprocal**, we then also know:

$$S_{12} = S_{21} \qquad S_{13} = S_{31} = 1/\sqrt{2} \qquad S_{32} = S_{23}$$

- And therefore:

$$\mathbf{S}=\begin{bmatrix} 1/2 & S_{21} & 1/\sqrt{2} \\ S_{21} & S_{22} & S_{32} \\ 1/\sqrt{2} & S_{32} & 0 \end{bmatrix}$$

- Now, since the device is **lossless**, we know that:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1 \;\Rightarrow\; (1/2)^2 + |S_{21}|^2 + (1/\sqrt{2})^2 = 1$$

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1 \;\Rightarrow\; |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$

$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1 \;\Rightarrow\; (1/2)^2 + |S_{32}|^2 + (1/\sqrt{2})^2 = 1$$

Columns have unit magnitude

Example – 3 (contd.)

$$0 = S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* = \frac{1}{2}S_{12}^* + S_{21}S_{22}^* + \frac{1}{\sqrt{2}}S_{32}^*$$

$$0 = S_{11}S_{13}^* + S_{21}S_{23}^* + S_{31}S_{33}^* = \frac{1}{2}\frac{1}{\sqrt{2}} + S_{21}S_{32}^* + \frac{1}{\sqrt{2}}(0)$$

$$0 = S_{12}S_{13}^* + S_{22}S_{23}^* + S_{32}S_{33}^* = S_{21}\left(\frac{1}{\sqrt{2}}\right) + S_{22}S_{32}^* + S_{32}(0)$$

Dissimilar columns are **orthogonal**

We can simplify these expressions and can further simplify them by using the fact that the elements are all **real**, and therefore $S_{21} = S_{21}{}^*$ (etc.).

Q: I count the simplified expressions and find 6 equations yet only a paltry 3 unknowns. Your typical buffoonery appears to have led to an over-constrained condition for which there is **no** solution!

Example – 3 (contd.)

A: Actually, we have **six** real equations and **six** real unknowns, since scattering element has a magnitude and phase. In this case we know the values are **real**, and thus the phase is either $0°$ or $180°$(i.e., $e^{j0} = 1$ or $e^{j\pi} = -1$); however, we do not know which one!

- the scattering matrix for the given **lossless**, **reciprocal** device is:

$$\mathbf{S} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$