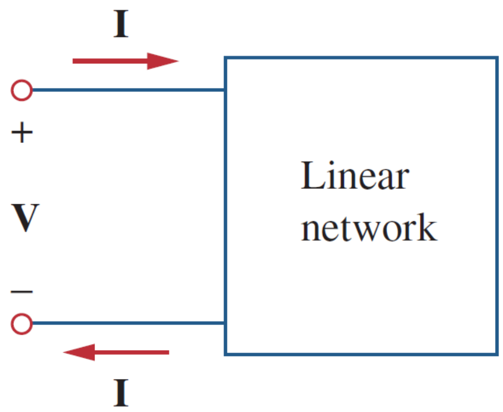
Lecture – 23 **Date: 30.10.2017**

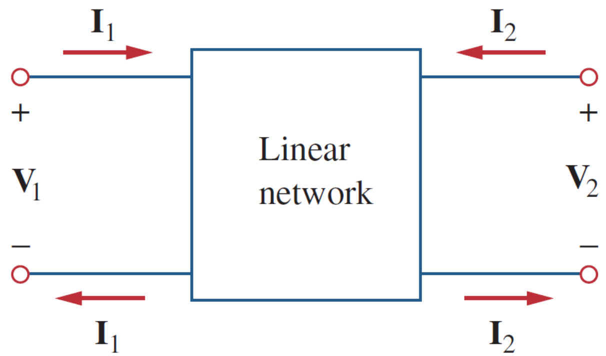
- Multi-port networks
- Impedance and Admittance Matrix
- Lossless and Reciprocal Networks

Introduction

- A pair of terminals through which a current may enter or leave a network is known as a *port*.
- Two-terminal devices or elements (such as resistors, capacitors, and inductors) result in one-port networks. Most of the circuits we have dealt with so far are two-terminal or one-port circuits.



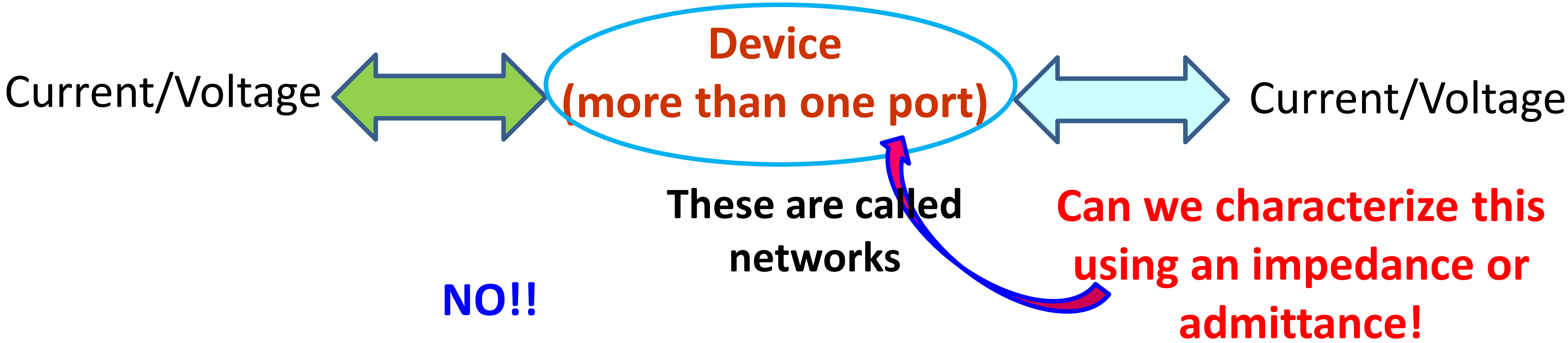
- So far, considered the voltage across or current through a single pair of terminals—such as the two terminals of a resistor, capacitor, inductor.
- We have also studied four-terminal or two-port circuits involving op amps, and transformers.



A two-port network is an electrical network with two separate ports for input and output.

2-port Networks

- **Requirement of Matrix Formulation**



NO!!

What is the way?

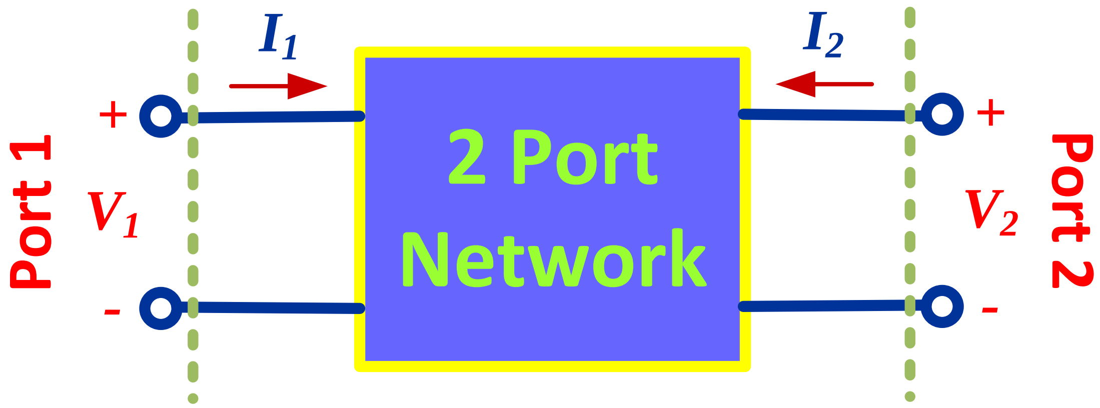
Impedance or Admittance Matrix. Right?

In principle, N by N impedance matrix completely characterizes a linear N-port device. Effectively, the impedance matrix defines a multi-port device the way a Z_L describes a single port device (e.g., a resistor)

Linear networks can be completely characterized by parameters measured at the network ports without knowing the content of the networks.

Multiport Networks

- Networks can have any number of ports – however, analysis of a 2-port, 3-port or 4-port network is sufficient to explain the theory and the associated concepts.



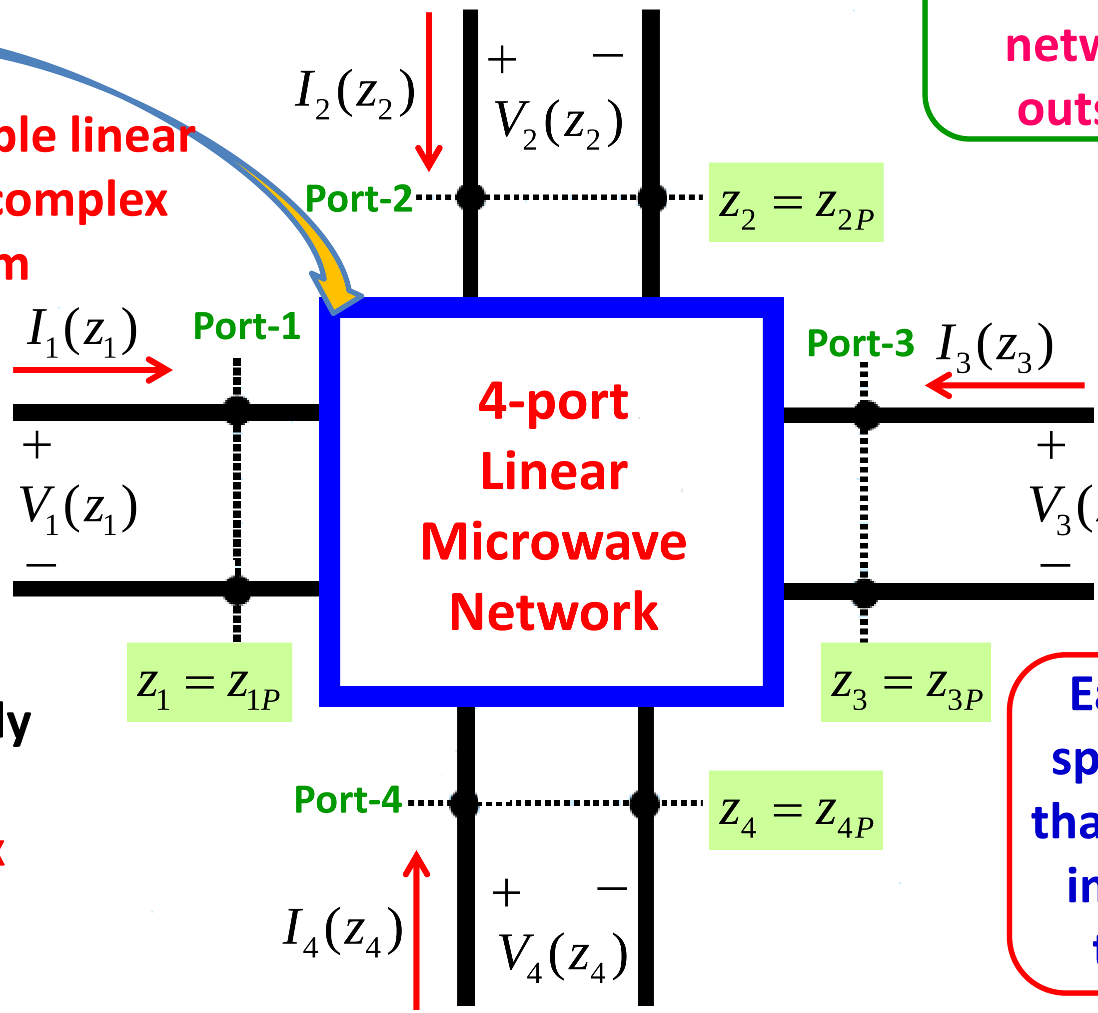
- The ports can be characterized with many parameters (Z, Y, h, S, ABCD). Each has a specific advantage.
- For 2-port Network, each parameter set is related to 4 variables:
  - 2 independent variables for excitation
  - 2 dependent variables for response

The Impedance Matrix

- Let us consider the following 4-port network:

Four identical cables used to connect this network to the outside world

This could be a simple linear device or a large/complex linear system



Either way, the network can be fully described by its **impedance matrix**

Each cable has specific location that defines input impedances to the network

The arbitrary locations are known as **ports** of the network

The Impedance Matrix (contd.)

- In principle, the current and voltages at the port-n of networks are given as:

$$V_n(z_n = z_{nP}) \qquad I_n(z_n = z_{nP})$$

- the simplified formulations are:

$$V_n = V_n(z_n = z_{nP}) \qquad I_n = I_n(z_n = z_{nP})$$

- If we want to say that there exists a non-zero current at port-1 and zero current at all other ports then we can write as:

$$I_1 \neq 0 \qquad I_2 = I_3 = I_4 = 0$$

- In order to define the elements of impedance matrix, there will be need to measure/determine the associated voltages and currents at the respective ports. Suppose, if we measure/determine current at port-1 and then voltage at port-2 then we can define:

$$Z_{21} = \frac{V_2}{I_1}$$

Trans-impedance

- Similarly, the trans-impedance parameters Z_{31} and Z_{41} are:

$$Z_{31} = \frac{V_3}{I_1} \qquad Z_{41} = \frac{V_4}{I_1}$$

- We can define other trans-impedance parameters such as Z_{34} as the ratio between the complex values I_4 (into port-4) and V_3 (at port-3), given that the currents at all other ports (1, 2, and 3) are zero.

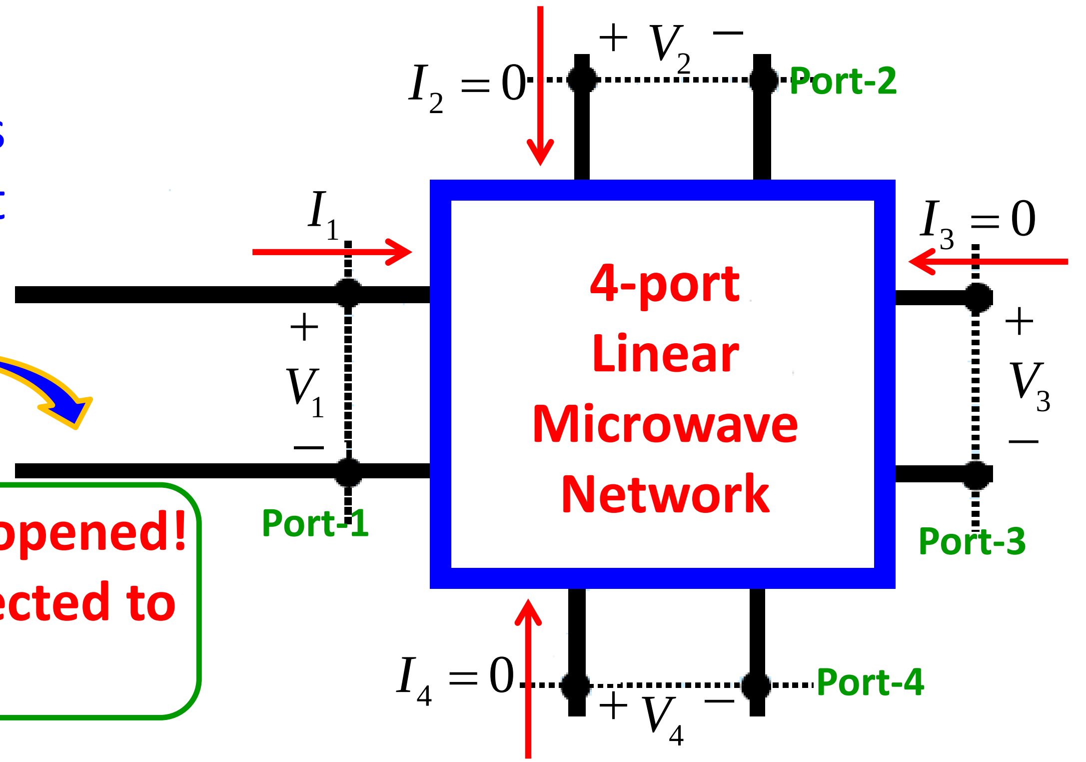
The Impedance Matrix (contd.)

- Therefore, the more generic form of trans-impedance is:

$$Z_{mn} = \frac{V_m}{I_n}$$ **(given that $I_k = 0$ for all $k \neq n$)**

How do we ensure that all but **one port** current is zero?

- Open the ports where the current needs to be zero



The ports should be opened! not the cables connected to the ports

- We can then define the respective trans-impedances as:

$$Z_{mn} = \frac{V_m}{I_n}$$ **(given that all ports $k \neq n$ are open)**

The Impedance Matrix (contd.)

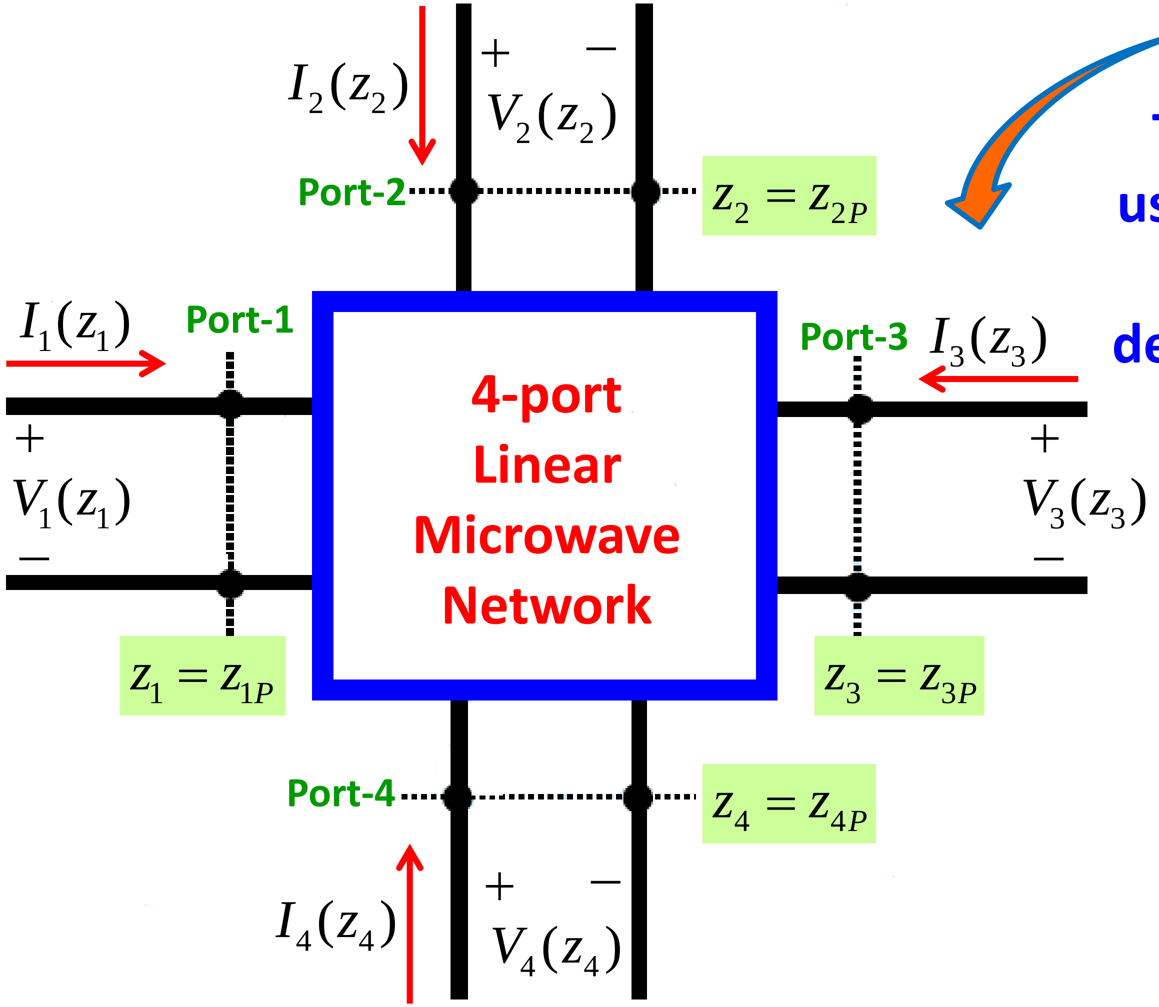
- Once we have defined the trans-impedance terms by opening various ports, it is time to formulate the impedance matrix
- Since the network is **linear**, the voltage at any port due to all the port currents is simply the coherent **sum** of the voltage at that port due to **each** of the currents
- For example, the voltage at **port-3** is: $V_3 = Z_{34}I_4 + Z_{33}I_3 + Z_{32}I_2 + Z_{31}I_1$
- Therefore we can generalize the voltage for **N-port** network as: $V_m = \sum_{n=1}^{N} Z_{mn} I_n \quad \Rightarrow \mathbf{V} = \mathbf{ZI}$
- Where **I** and **V** are vectors given as: $\mathbf{V} = [\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, ..., \mathrm{V}_\mathrm{N}]^T \qquad \mathbf{I} = [\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, ..., \mathrm{I}_\mathrm{N}]^T$
- The term **Z** is matrix given by:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{21} & & & \vdots \\ \vdots & & & \\ Z_{m1} & Z_{m2} & \cdots & Z_{mn} \end{bmatrix}$$

Impedance Matrix

The Admittance Matrix

- Let us consider the 4-port network again:



This can be characterized using admittance matrix – if currents are taken as dependent variables instead of voltages

The elements of admittance matrix are called trans-admittance parameters Y_{mn}

The Admittance Matrix (contd.)

Important

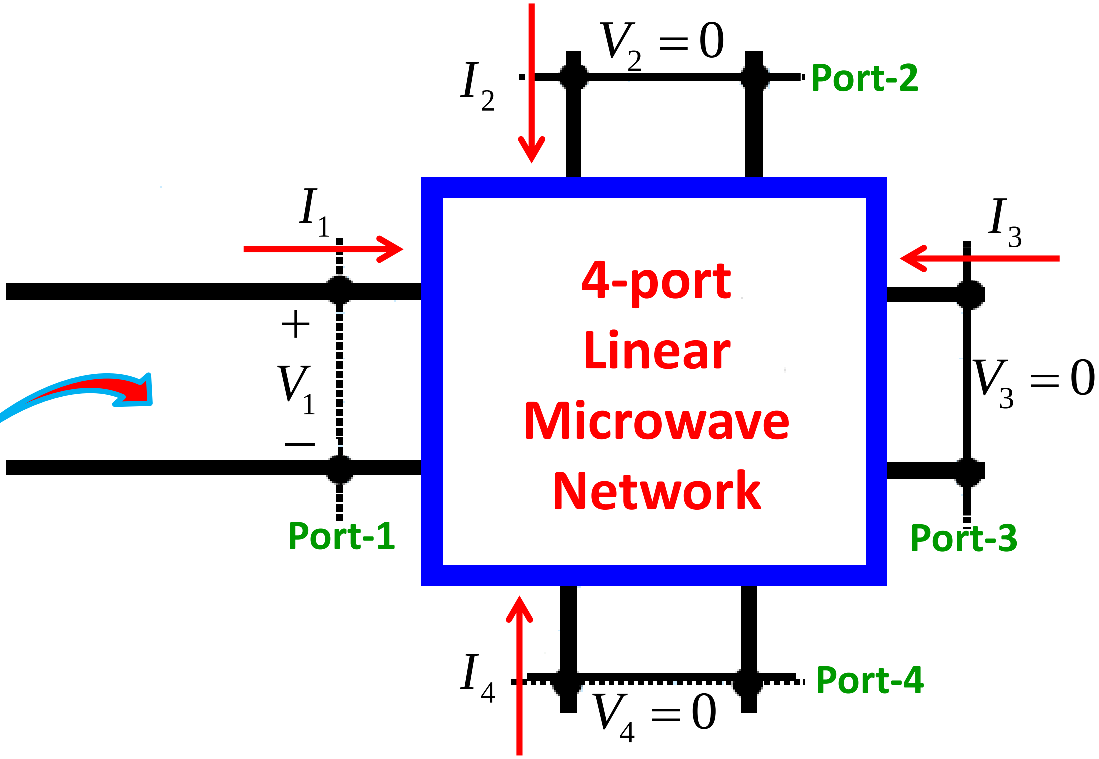
$$Y_{mn} \neq \frac{1}{Z_{mn}}$$

- The trans-admittances Y_{mn} are defined as:

$$Y_{mn} = \frac{I_m}{V_n}$$ **(given that $V_k = 0$ for all $k \neq n$)**

- It is apparent that the voltage at all but one port must be equal to zero. This can be ensured by short-circuiting the voltage ports.

The ports should be short-circuited! not the cable connected to the ports



- Now, since the network is **linear**, the current at any **one** port due to all the port voltages is simply the coherent **sum** of the currents at that port due to **each** of the port voltages.

The Admittance Matrix (contd.)

- For example, the current at **port-3** is:

$$I_3 = Y_{34}V_4 + Y_{33}V_3 + Y_{32}V_2 + Y_{31}V_1$$

- Therefore we can generalize the current for **N-port** network as:

$$I_m = \sum_{n=1}^{N} Y_{mn}V_n \;\Rightarrow\; \mathbf{I} = \mathbf{YV}$$

- Where **I** and **V** are vectors given as:

$$\mathbf{V} = \left[\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \ldots., \mathrm{V}_N\right]^T \qquad \mathbf{I} = \left[\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \ldots., \mathrm{I}_N\right]^T$$

- The term **Y** is matrix given by:

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & & & \vdots \\ \vdots & & & \\ Y_{m1} & Y_{m2} & \cdots & Y_{mn} \end{bmatrix}$$

Admittance Matrix

- The values of elements in the admittance matrix are frequency dependents and often it is advisable to describe admittance matrix as:

$$\boldsymbol{Y}(\omega) = \begin{bmatrix} Y_{11}(\omega) & Y_{12}(\omega) & \cdots & Y_{1n}(\omega) \\ Y_{21}(\omega) & & & \vdots \\ \vdots & & & \\ Y_{m1}(\omega) & Y_{m2}(\omega) & \cdots & Y_{mn}(\omega) \end{bmatrix}$$

The Admittance Matrix (contd.)

You said that: $Y_{mn} \neq \dfrac{1}{Z_{mn}}$

Is there any relationship between admittance and impedance matrix of a given device?

Answer: Let us see if we can figure it out!

$$\mathbf{I} = \mathbf{YV}$$

- Recall that we can determine the inverse of a matrix. Denoting the matrix inverse of the admittance matrix as $\mathbf{Y}^{-1}$, we find:

$$\Rightarrow \mathbf{Y}^{-1}\mathbf{I} = \mathbf{Y}^{-1}\left(\mathbf{YV}\right) \quad \mathbf{Y}^{-1}\mathbf{I} = \left(\mathbf{Y}^{-1}\mathbf{Y}\right)\mathbf{V} \quad \mathbf{Y}^{-1}\mathbf{I} = \mathbf{V}$$

- We also know: $\mathbf{V} = \mathbf{ZI}$

$$\mathbf{Z} = \mathbf{Y}^{-1} \quad \textbf{OR} \quad \mathbf{Y} = Z^{-1}$$

Reciprocal and Lossless Networks

- We can **classify** multi-port devices or networks as either **lossless** or **lossy**; **reciprocal** or **non-reciprocal**. Let's look at each classification individually.

Lossless Network

- A **lossless** network or device is simply one that **cannot** absorb power. This does **not** mean that the delivered power at **every port** is zero; rather, it means the total power flowing **into** the **device** must equal the total power **exiting** the **device**.

- A lossless device exhibits an impedance matrix with an interesting **property**. Perhaps not surprisingly, we find for a lossless device that the **elements** of its impedance matrix will be **purely reactive**:

$$\text{Re}(Z_{mn}) = 0$$

For a lossless device

- If the device is lossy, then the elements of the impedance matrix must have **at least** one element with a real (i.e., resistive) component.
- Furthermore, we can similarly say that if the elements of an **admittance** matrix are **all** purely imaginary (i.e., $\text{Re}\{Y_{mn}\} = 0$), then the device is lossless.

Reciprocal and Lossless Networks (contd.)

Reciprocal Network

- Ideally, most **passive, linear** components will turn out to be **reciprocal**—regardless of whether the designer **intended** it to be or not!
- Reciprocity is a tremendously important characteristic, as it greatly **simplifies** an impedance or admittance matrix!
- Specifically, we find that a reciprocal device will result in a **symmetric** impedance and admittance **matrix**, meaning that:

$$Z_{mn} = Z_{nm} \qquad Y_{mn} = Y_{nm}$$

For a reciprocal device

- **For example**, we find for a reciprocal device that $Z_{23} = Z_{32}$, and $Y_{12} = Y_{21}$.

Reciprocal and Lossless Networks (contd.)

$$\mathbf{Z}=\begin{bmatrix} j2 & 0.1 & j3 \\ -j & -1 & 1 \\ 4 & -2 & 0.5 \end{bmatrix}$$

neither lossless nor reciprocal

lossless, but not reciprocal

$$\mathbf{Z}=\begin{bmatrix} j2 & j0.1 & j3 \\ -j & -j1 & j1 \\ j4 & -j2 & j0.5 \end{bmatrix}$$

$$\mathbf{Z}=\begin{bmatrix} j2 & -j & 4 \\ -j & -1 & -j2 \\ 4 & -j2 & j0.5 \end{bmatrix}$$

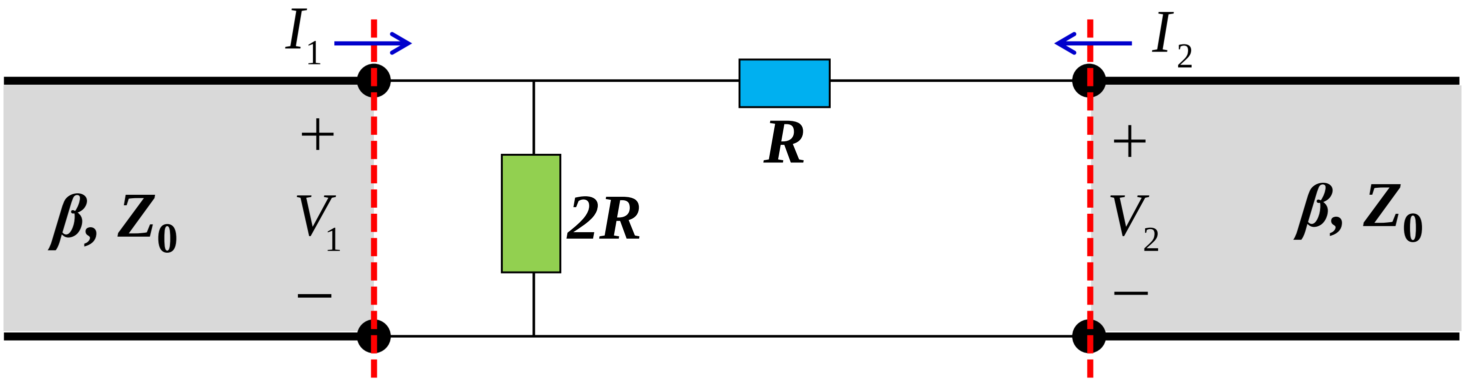
reciprocal, but not lossless

lossless and reciprocal

$$\mathbf{Z}=\begin{bmatrix} j2 & -j & j4 \\ -j & -j & -j2 \\ j4 & -j2 & j0.5 \end{bmatrix}$$

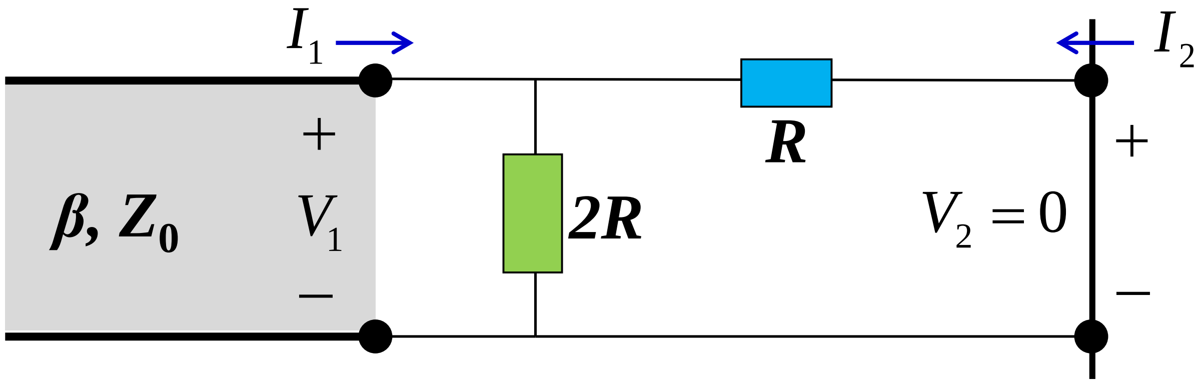
Example – 1

- determine the **Y matrix** of this two-port device.



Example – 1 (contd.)

Step-1: Place a **short** at port 2



Step-2: Determine currents I_1 and I_2

- Note that **after** the short was placed at port 2, both resistors are in **parallel**, with a potential V_1 across each

Therefore current I_1 is ⇒

$$I_1 = \frac{V_1}{2R} + \frac{V_1}{R} = \frac{3V_1}{2R}$$

- The current I_2 equals the portion of current I_1 through R but with opposite sign

$$I_2 = -\frac{V_1}{R}$$

Example – 1 (contd.)

Step-3: Determine the trans-admittances Y_{11} and Y_{21}

$$Y_{11} = \frac{I_1}{V_1} = \frac{3}{2R}$$

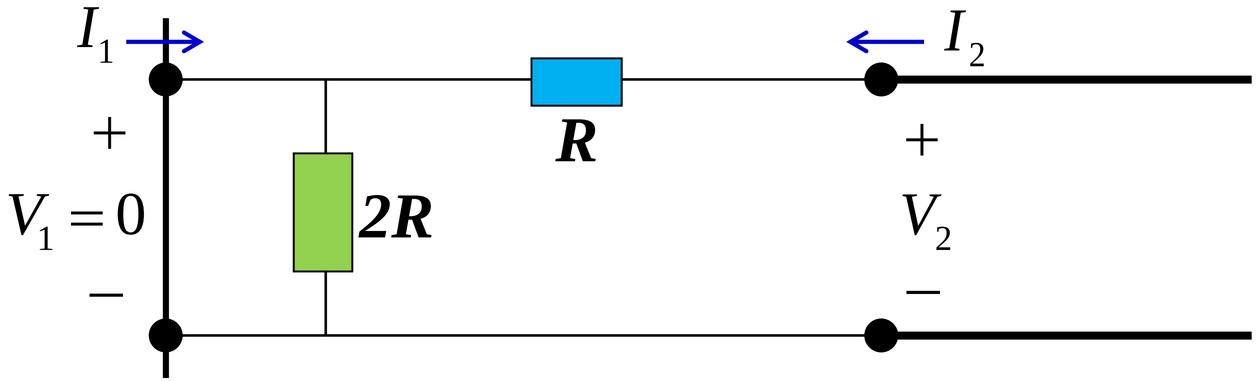
$$Y_{21} = \frac{I_2}{V_1} = -\frac{1}{R}$$

Note that Y_{21} is real and negative

This is **still** a valid physical result, **although** you will find that the **diagonal** terms of an impedance or admittance matrix (e.g., Y_{22} , Z_{11}, Y_{44}) will **always** have a real component that is **positive**

To find the **other two** trans-admittance parameters, we must **move** the short and then **repeat** each of our previous steps!

Example – 1 (contd.)



Step-1:

Place a **short** at port 1

Step-2: Determine currents I_1 and I_2

- Note that **after** a short was placed at port **1**, resistor 2R has **zero** voltage across it—and thus **zero current** through it!

Therefore:

$$I_2 = \frac{V_2}{R}$$

$$I_1 = -I_2 = -\frac{V_2}{R}$$

Step-3:

Determine the trans-admittances Y_{12} and Y_{22}

$$Y_{12} = \frac{I_1}{V_2} = -\frac{1}{R}$$

$$Y_{22} = \frac{I_2}{V_2} = \frac{1}{R}$$

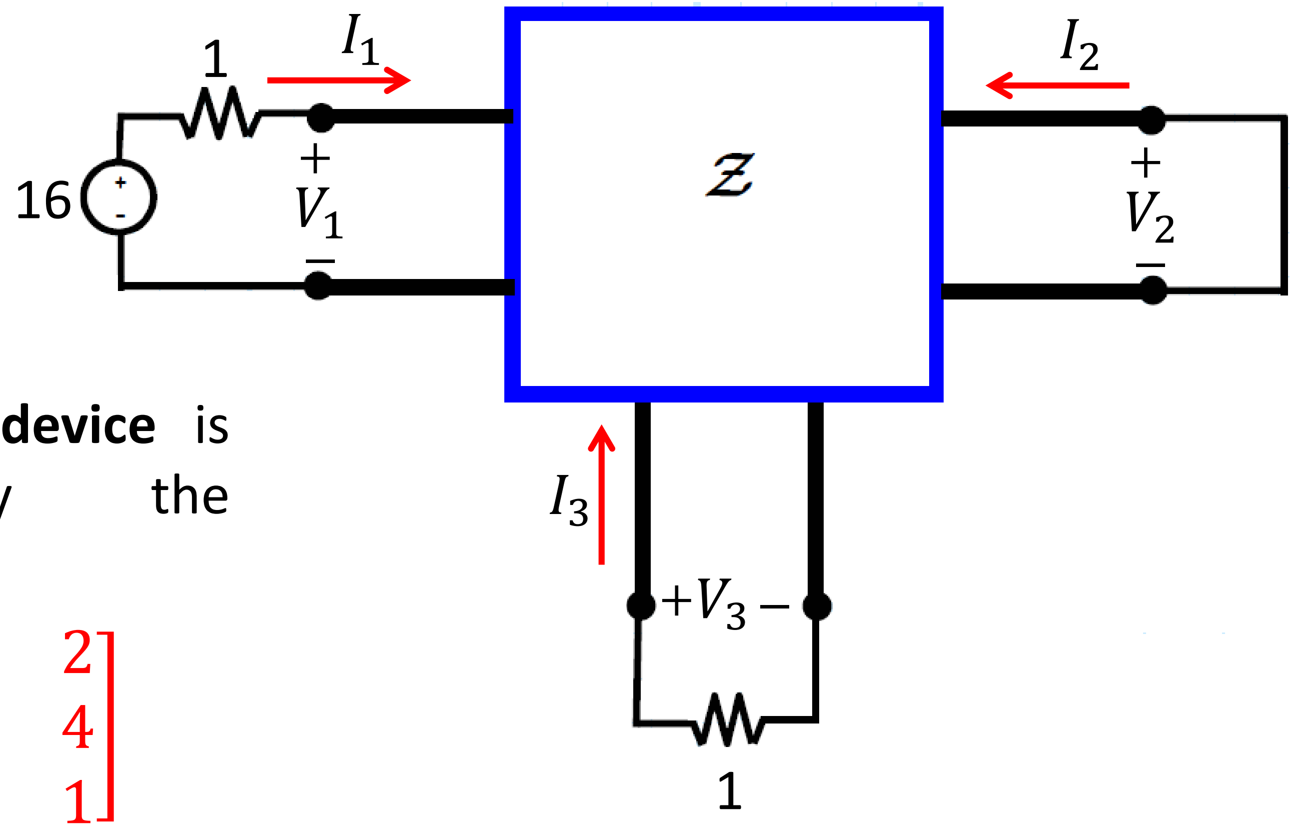
Therefore the admittance matrix is:

$$\mathbf{Y} = \begin{bmatrix} 3/2R & -1/R \\ -1/R & 1/R \end{bmatrix}$$

Is it lossless or reciprocal?

Example – 2

- Consider this circuit:



- Where the 3-port **device** is characterized by the **impedance matrix**:

$$\mathbf{Z}=\begin{bmatrix}2 & 1 & 2\\ 1 & 1 & 4\\ 2 & 4 & 1\end{bmatrix}$$

- determine all port **voltages** V_1, V_2, V_3 and all **currents** I_1, I_2, I_3.