Lecture – 20 **Date: 23.10.2017**

- Fourier Series and Applications

Laplace Transform and Circuit Analysis : Self Study

Fourier Series and Circuit Applications

- So far, we have learnt the analysis of circuits with sinusoidal sources.
- However, many circuits are driven by non-sinusoidal periodic functions. The steady-state response of a circuit to a non-sinusoidal periodic excitation requires the application of a Fourier series, ac phasor analysis, and the superposition principle.
- The procedure usually involves four steps.

Steps for Applying Fourier Series:

1. Express the excitation as a Fourier series.
2. Transform the circuit from the time domain to the frequency domain.
3. Find the response of the dc and ac components in the Fourier series.
4. Add the individual dc and ac responses using the superposition principle.

Fourier Series

- a non-sinusoidal periodic function can be expressed as an infinite sum of sinusoidal functions.

$$f(t) = f(t + nT)$$

n: an integer

T: time period of the function

- *Fourier theorem* allows simplification of any practical periodic function of frequency ω_0 as an infinite sum of sine or cosine functions that are integral multiples of ω_0.

$$f(t) = a_0 + a_1 \cos\omega_0 t + b_1 \sin\omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

$$\boxed{f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}}$$

$\omega_0 = \frac{2\pi}{T}$: Fundamental Frequency

Fourier Series

The Fourier series of a periodic function $f(t)$ is a representation that resolves $f(t)$ into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

- The conditions for $f(t)$ to yield a convergent Fourier series are:
  1. $f(t)$ is single-valued everywhere.
  2. $f(t)$ has a finite number of finite discontinuities in any one period.
  3. $f(t)$ has a finite number of maxima and minima in any one period.
  4. The integral $\int_{t_0}^{t_0+T} |f(t)| dt < \infty$ for any t_0
- The process of determining the coefficients a_0, a_n, and b_n is called *Fourier analysis.* The following trigonometric integrals are very helpful in Fourier analysis.

$$\int_0^T \sin n\omega_0 t \, dt = 0 \qquad \int_0^T \cos n\omega_0 t \, dt = 0 \qquad \int_0^T \sin n\omega_0 t \cos m\omega_0 t \, dt = 0$$

$$\int_0^T \sin n\omega_0 t \sin m\omega_0 t \, dt = 0, \qquad \int_0^T \cos n\omega_0 t \cos m\omega_0 t \, dt = 0, \qquad (m \neq n)$$

Fourier Series

$$\int_0^T \sin^2 n\omega_0 t\, dt = \frac{T}{2} \qquad \int_0^T \cos^2 n\omega_0 t\, dt = \frac{T}{2}$$

- Use the trigonometric integrals to determine the coefficients.

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt \qquad a_n = \frac{2}{T}\int_0^T f(t) \cos n\omega_0 t\, dt \qquad b_n = \frac{2}{T}\int_0^T f(t) \sin n\omega_0 t\, dt$$

- An alternative form of ***f(t)*** is the *amplitude-phase* form:

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) \quad \left.\begin{matrix} a_n = A_n \cos \phi_n \\ b_n = -A_n \sin \phi_n \end{matrix}\right.$$

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n} \quad \Rightarrow \quad A_n\underline{/\phi_n} = a_n - jb_n$$

The plot of the amplitude A_n of the harmonics versus $n\omega_0$ is called the *amplitude spectrum* of $f(t)$; the plot of the phase φ_n versus $n\omega_0$ is the *phase spectrum* of $f(t)$; Both the amplitude and phase spectra form the *frequency spectrum* of *f*(*t*).

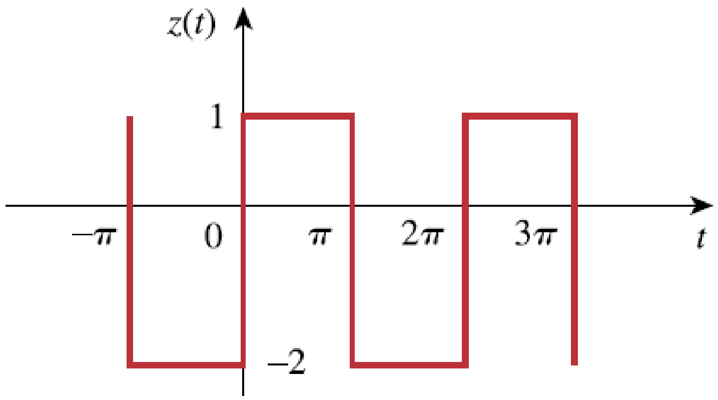
Fourier Series

The frequency spectrum of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency.

the Fourier analysis is also a mathematical tool for finding the spectrum of a periodic signal

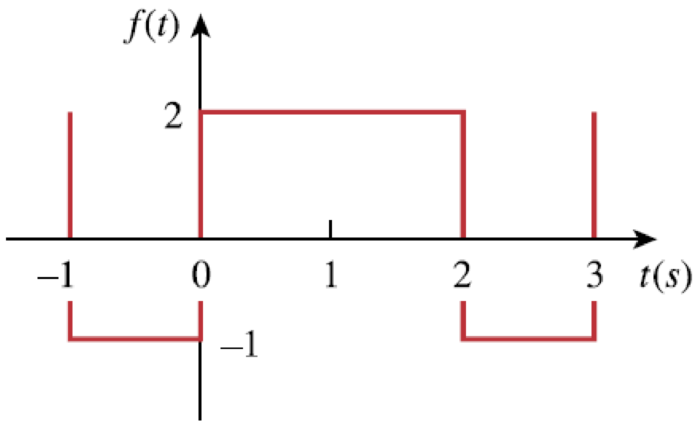
Example – 1

Obtain the Fourier series expansion for the waveform:



Example – 2

Determine the Fourier series for the periodic function:

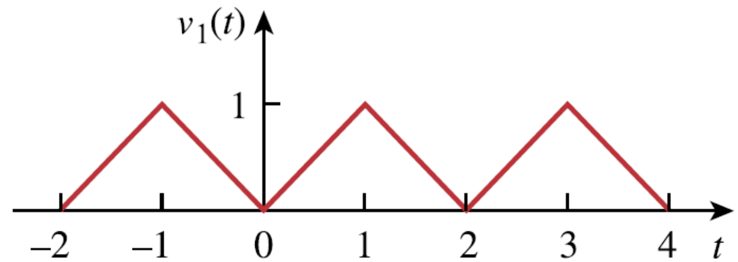


Example – 3

Express the Fourier series: $$f(t) = 10 + \sum_{n=1}^{\infty} \frac{4}{n^2+1}\cos 10nt + \frac{1}{n^3}\sin 10nt$$

(a) in a cosine and angle form.
(b) in a sine and angle form.

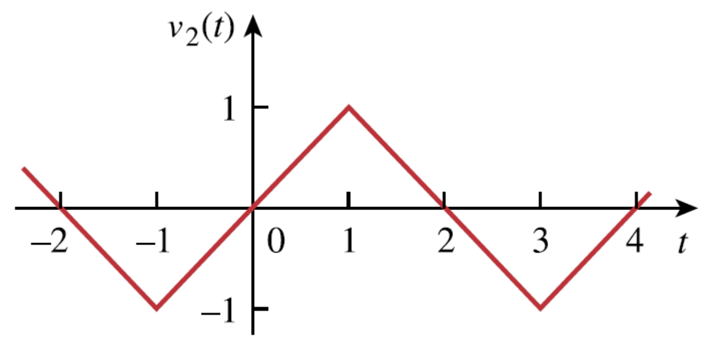
Example – 4



(a)

The waveform in Fig. (a) has the following Fourier series:

$$v_1(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \cdots\right) \text{V}$$

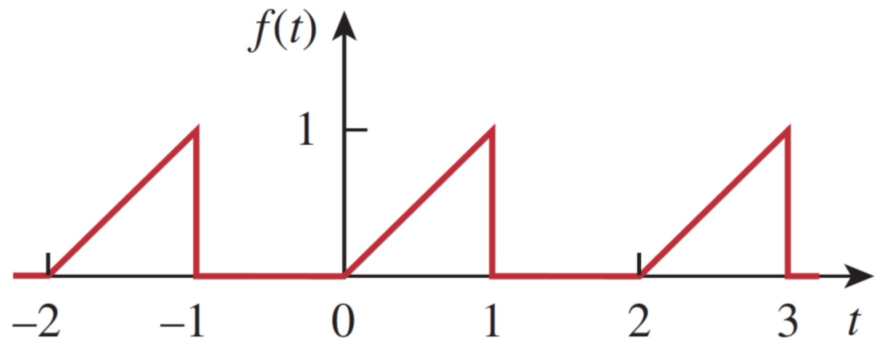


(b)

Obtain the Fourier series for Fig. (b).

Example – 5

Obtain the Fourier series for this periodic function and plot the amplitude and phase spectra.



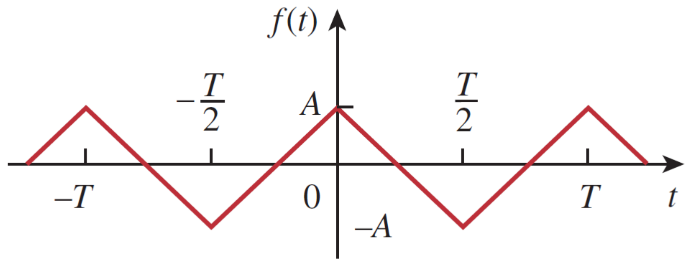
Symmetry Considerations

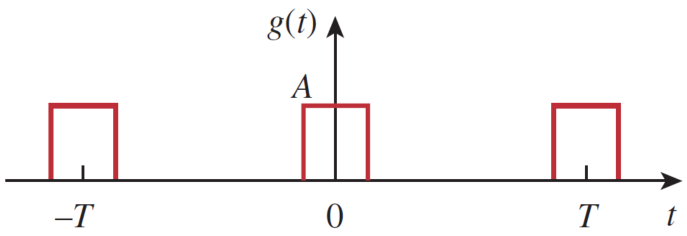
- For some functions, only some of the coefficients exist.
- Is there any method whereby one can know in advance that some Fourier coefficients would be zero?
- This will eliminate the unnecessary work involved in the tedious process of calculating them.

Such a method does exist; it is based on recognizing the existence of symmetry. Here we discuss three types of symmetry:
(1) even symmetry, (2) odd symmetry, (3) half-wave symmetry.

Even Symmetry

$f(t) = f(-t)$: Even Function   Examples are $t^2, t^4, \cos(t)$ etc.





Main property of even function, $f_e(t)$: $$\int_{-T/2}^{T/2} f_e(t)\,dt = 2\int_0^{T/2} f_e(t)\,dt$$

Fourier Coefficients for even function: $$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

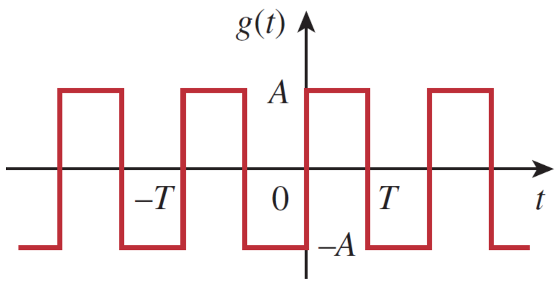
$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt \qquad b_n = 0$$

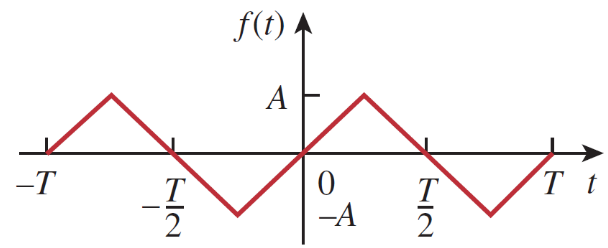
Leads to *Fourier cosine series*

Odd Symmetry

$f(-t) = -f(t)$: Odd Function

Examples are $t, t^3, \sin(t)$ etc.





Main property of odd function, $f_o(t)$:

$$\int_{-T/2}^{T/2} f_o(t)\,dt = 0$$

Fourier Coefficients for odd function:

$$a_0 = 0, \qquad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

Leads to *Fourier sine series*

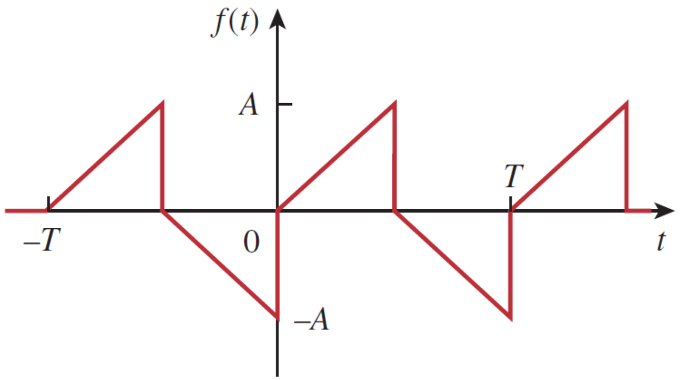
this makes sense as the sine function is itself an odd function. Also, there is no dc term for the Fourier series expansion of an odd function.

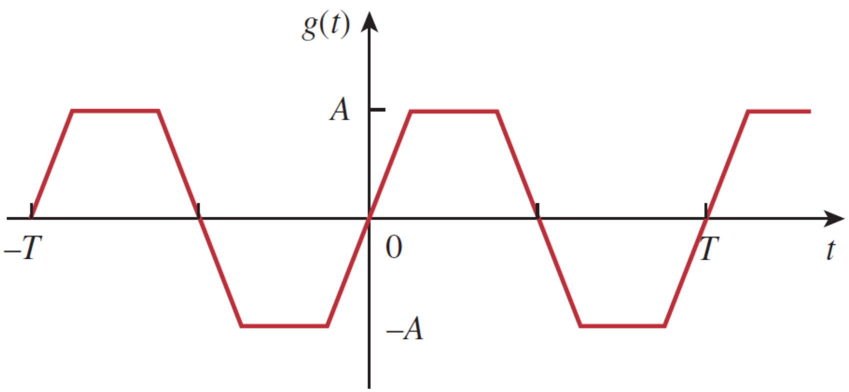
Half-Wave Symmetry

- A function is half-wave symmetric if: $$f\left(t-\frac{T}{2}\right)=-f(t)$$

Also Odd Symmetry

each half-cycle is the mirror image of the next half-cycle.





$$a_0 = 0$$

$$a_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\, dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

$$b_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin n\omega_0 t\, dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

The Fourier series of a half-wave symmetric function contains only odd harmonics.

Example – 6

Determine if these functions are even, odd, or neither.

(a) $1 + t$ (b) $t^2 - 1$ (c) $\cos n\pi t \sin n\pi t$

(d) $\sin^2 \pi t$ (e) e^{-t}

Example – 7

Calculate the Fourier coefficients for this function: