<u>Lecture – 15</u> **<u>Date: 26.09.2017</u>**

- Frequency Response (Contd.)

Introduction

- In sinusoidal circuit analysis, we learnt how to find voltages and currents in a circuit with a constant frequency source.
- For **fixed amplitude** and **variable frequency** of the sinusoidal source → one can obtain the circuit's *frequency response*.
- Frequency Response is a complete description of the sinusoidal steady-state behavior of a circuit **as a function of frequency**.
- Sinusoidal Steady-State frequency responses are of significance in many applications, especially in communications and control systems.
- Electric Filters utilizes Freq Response → eliminate signals with unwanted frequencies and pass signals of the desired frequencies.
- Filters are used in radio, TV, and telephone systems to separate one broadcast frequency from another.

Transfer Function

- transfer function (also called the *network function*) → useful analytical tool for finding the frequency response → represented by $H(\omega)$.
- Circuit's frequency response is the plot of $H(\omega)$ when ω varies ($0 \to \infty$).

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}}$$

frequency-dependent ratio of a phasor output **Y**(ω) to a phasor input **X**(ω).

- Since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

Being a complex quantity, $\boldsymbol{H}(\omega)$ has a magnitude $H(\omega)$ and a phase φ.

Transfer Function (contd.)

- transfer function of a circuit can be obtained by first converting it to frequency-domain equivalent by replacing resistors, inductors, and capacitors with their impedances R, $j\omega L$ and $^{1}/_{j\omega C}$.
- use any circuit technique(s) to obtain the appropriate expressions.
- Can be simplified to: $\mathbf{H}(\omega) = \dfrac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}$

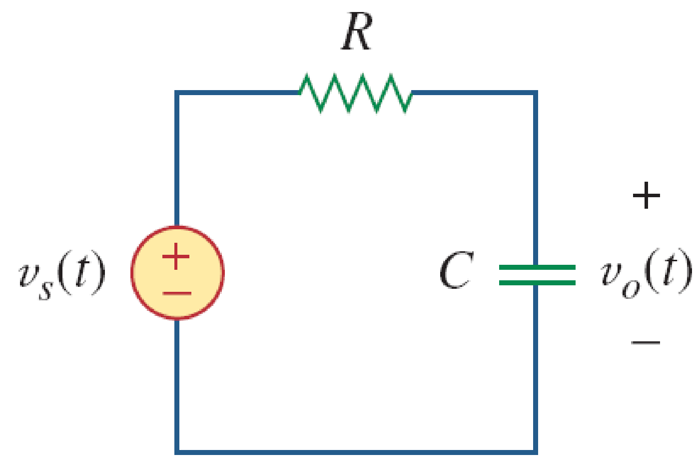
The roots of $\boldsymbol{N}(\omega)$ are called the *zeros* and are usually represented as $j\omega = z_1, z_2, \ldots$ ...Similarly, the roots of $\boldsymbol{D}(\omega)$ are the *poles* and are represented as $j\omega = p_1, p_2, \ldots$ ....

A zero is a value that results in a zero value of the function. A pole is a value for which the function is infinite.

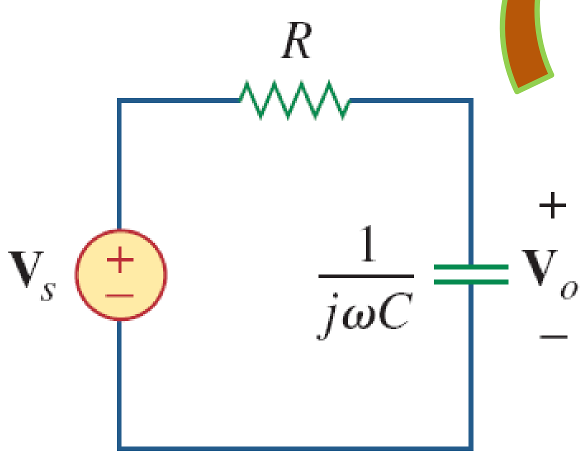
To avoid complex algebra, it is expedient to replace temporarily $j\omega$ with s when working with **H**(ω) and replace s with $j\omega$ at the end.

Example – 1

Find the transfer function $\frac{V_0}{V_s}$ and the corresponding frequency response of this *RC* circuit. Assume, $v_s = V_m cos\omega t$.



The frequency-domain equivalent of the circuit:



the transfer function

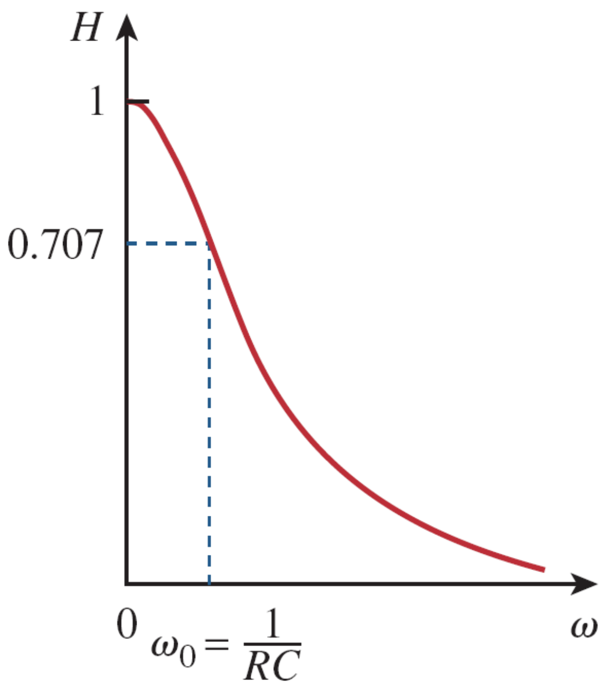
$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

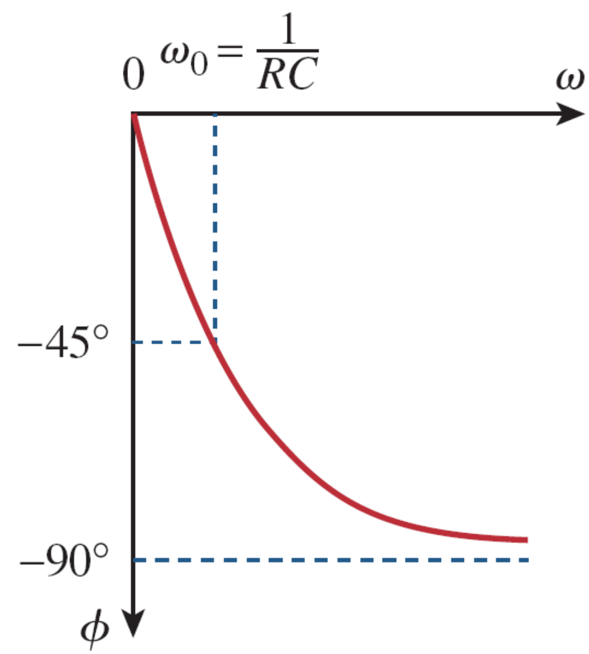
$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

where $\omega_0 = \frac{1}{RC}$. For plotting *H* and φ for $0 < \omega < \infty$, we need values at some critical points.

Example – 1 (contd.

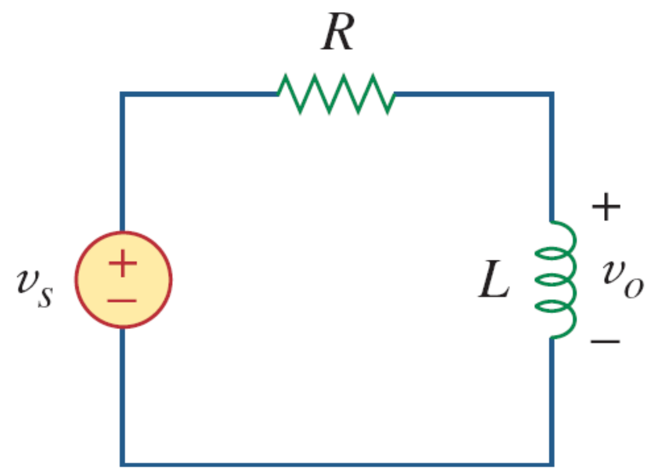
ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	$-84°$
1	0.71	$-45°$	20	0.05	$-87°$
2	0.45	$-63°$	100	0.01	$-89°$
3	0.32	$-72°$	∞	0	$-90°$



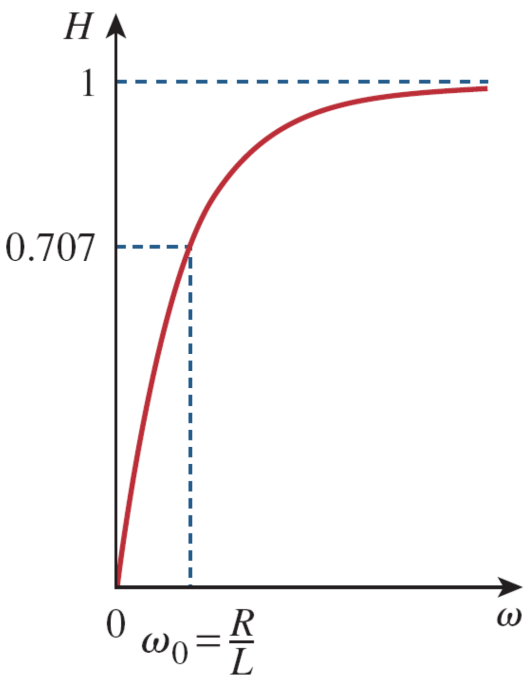


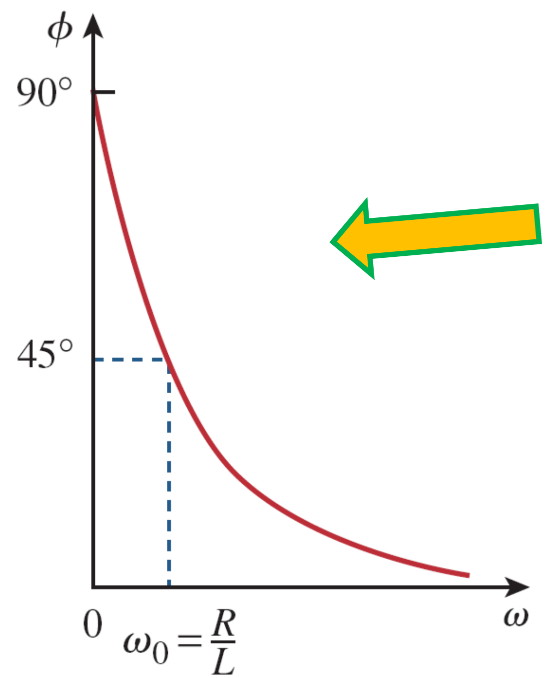
Example – 2

Find the transfer function $\frac{V_0}{V_s}$ and the corresponding frequency response of this *RL* circuit. Assume, $v_s = V_m cos\omega t$.



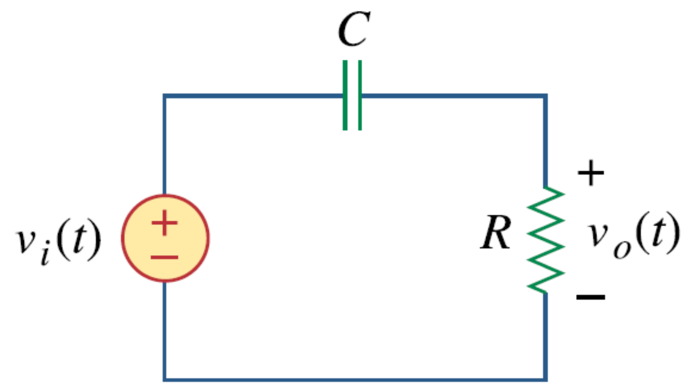
$$H(\omega) = \frac{V_0}{V_S} = \frac{j\omega L}{R + j\omega L}$$





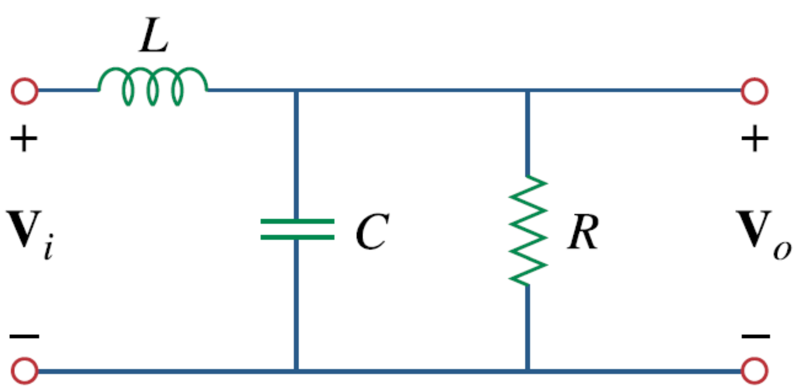
Example – 3

Find the transfer function $\frac{V_0}{V_s}$ and the corresponding frequency response of this RC circuit. Assume, $\omega_0 = \frac{1}{RC}$.

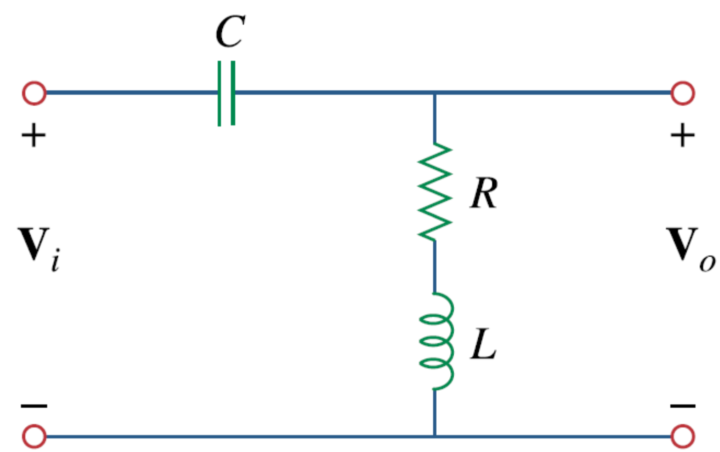


Example – 4

Find the transfer function $\frac{V_0}{V_i}$ of the following circuits.



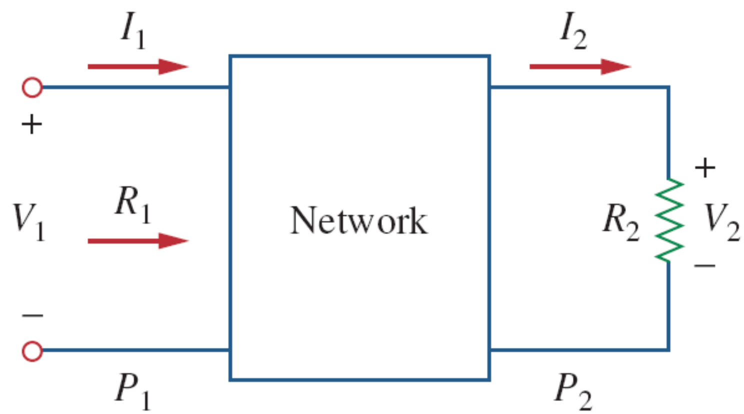
(a)



(b)

Decibel Scale

- It is not always easy to get a quick plot of the magnitude and phase of the transfer function.
- A more systematic way of obtaining the frequency response is to use Bode plots (based on logarithms).
- The bel is used to measure the ratio of two levels of power or power gain *G:*



$$G = \text{number of bels} = log_{10}\frac{P_2}{P_1}$$

- The *decibel* (dB) provides us with a unit of less magnitude. It is 1/10th of a bel:

$$G_{dB} = 10log_{10}\frac{P_2}{P_1}$$

For $P_2 = P_1$, the gain is 0dB. If $P_2 = 2P_1$ then gain = 3dB while for $P_2 = 0.5P_1$ the gain is -3dB

- In voltage or current ratio: $G_{dB} = 20log_{10}\frac{V_2}{V_1}$ $\quad G_{dB} = 20log_{10}\frac{I_2}{I_1}$

Decibel Scale (contd.)

- We use $\mathbf{10log_{10}}$ for power and $\mathbf{20log_{10}}$ for voltage or current, because of the square relationship between them.
- The ***dB*** value is a logarithmic measurement of the *ratio* of one variable to another *of the same type* → can only be applied when the transfer function *H* is expressed as ratio of same quantities.
- so far we only used voltage and current magnitudes in above equations. Negative signs and angles will be handled independently.

> With this in mind, we can apply the concepts of logarithms and decibels to construct Bode plots.

Example – 5

Calculate |**H**(ω)| if H_{dB} equals: (a) $0.05\ dB$ (b) $-6.2\ dB$ (c) $104.7\ dB$

Example – 6

Determine the magnitude (in dB) and the phase (in degrees) of **H**(ω) at ω = 1 if **H**(ω) equals:

(a) 0.05 (b) 125 (c) $\frac{10j\omega}{2+j\omega}$ (d) $\frac{3}{1+j\omega}+\frac{6}{2+j\omega}$

Bode Plot

- Obtaining the Freq Response from $H(\omega)$ is extremely tedious as the frequency range required in frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis.
- Further, there is a more systematic way of locating the important features of the magnitude and phase plots of the transfer function using semilogarithmic plots known as ***Bode Plot***.
- In ***Bode Plot***, the *magnitude in decibels* is plotted against the *logarithm of the frequency*; on a separate plot, the *phase in degrees* is plotted against the *logarithm of the frequency*.

$$\mathbf{H} = H\underline{/\phi} = He^{j\phi}$$

$$\ln \mathbf{H} = \ln H + \ln e^{j\phi} = \ln H + j\phi$$

the real part is a function of the magnitude while the imaginary part is the phase.

- In a Bode plot, magnitude is plotted in decibels (dB) vs frequency while the phase is plotted in degrees vs frequency on a semilog scale.

Bode Plot (contd.)

- A generic transfer function:

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}\,(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

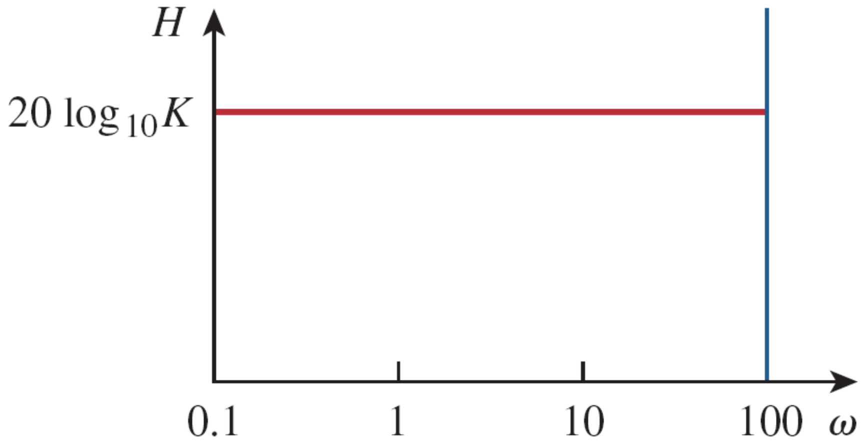
This generic formulation may include up to seven types of different factors that can appear in various combinations.

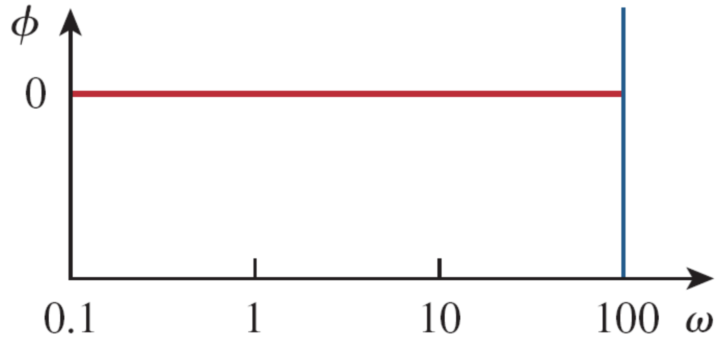
- A gain K
- A pole $(j\omega)^{-1}$ or zero $(j\omega)$ at the origin
- A simple pole $1/\left(1+j\omega/p_1\right)$ or zero $\left(1+j\omega/z_1\right)$
- A quadratic pole $1/\left[1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]$ or zero $\left[1+\frac{j2\zeta_1\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]$

- The Bode plot is constructed by plotting each factor separately and then by subsequent addition.

Bode Plot (contd.)

<u>Constant term:</u> For K, the magnitude is $20log_{10}K$ and the phase angle is 0°



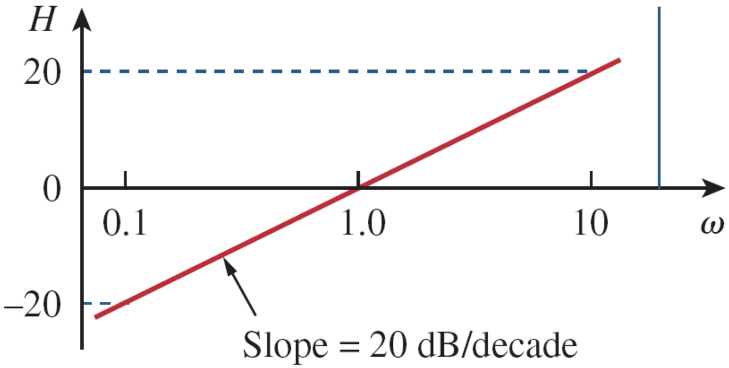


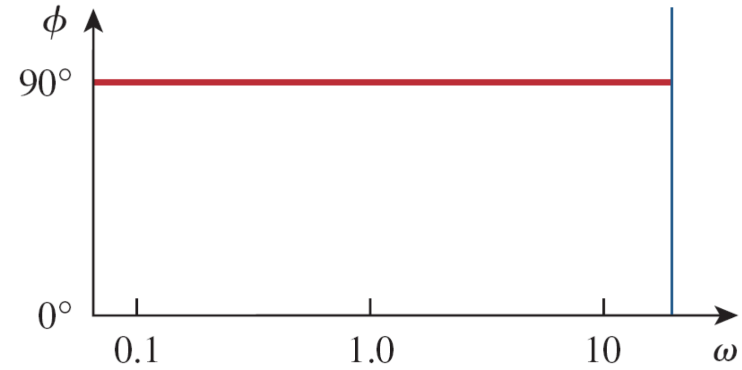
For negative K, the magnitude remains $20log_{10}|K|$ but the phase angle is $\pm 180°$

<u>Pole/zero at the origin:</u> For the zero $(j\omega)$ at the origin, the magnitude is $20log_{10}(\omega)$ and the phase is 90°.

- **slope of the magnitude plot is 20dB/decade, while the phase is constant with frequency.**

Bode Plot (contd.)





The Bode plots for the pole $(j\omega)^{-1}$ are similar except that the slope of the magnitude plot is -20dB/decade while the phase is $-90°$. In general, for $(j\omega)^N$ where *N* is an integer, the magnitude plot will have a slope of 20*N* dB/decade, while the phase is 90*N* degrees.

Simple pole/zero: $\left(1 + {}^{j\omega}/_{z_1}\right)$

$$H_{\text{dB}} = 20\log_{10}\left|1 + \frac{j\omega}{z_1}\right|$$

$= 0 \qquad$ as $\omega \to 0$

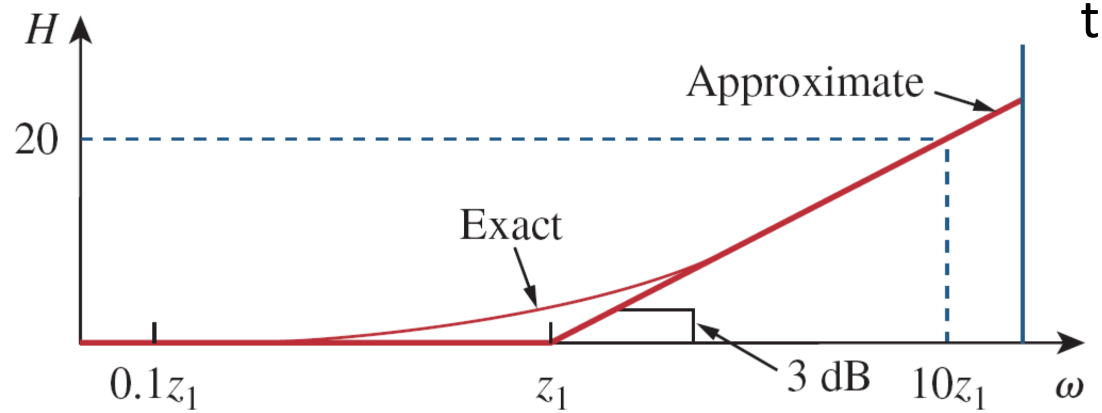
$= 20log_{10}\frac{\omega}{z_1} \qquad$ as $\omega \to \infty$

- It shows that we can approximate the magnitude as zero (a straight line with zero slope) for small values of ω and by a straight line with slope 20 dB/decade for large values of ω.

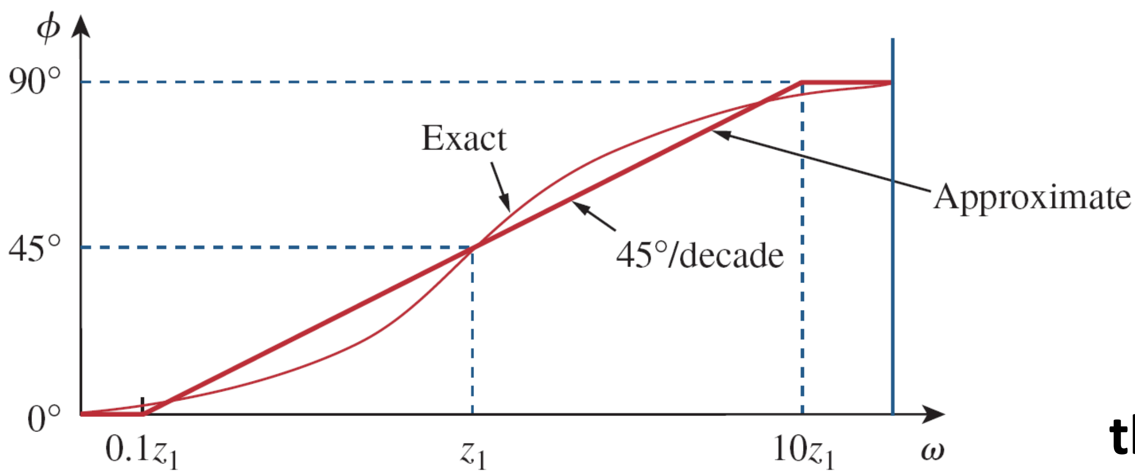
Bode Plot (contd.)

- The frequency $\omega = z_1$ where the two asymptotic lines meet is called the *corner frequency* or *break frequency*.

$$\phi = \tan^{-1}\left(\frac{\omega}{z_1}\right) = \begin{cases} 0, & \omega = 0 \\ 45°, & \omega = z_1 \\ 90°, & \omega \to \infty \end{cases}$$



the approximate plot is close to the actual plot except at the break frequency ($\omega = z_1$), where the deviation is $20log_{10}|1 + j1| = 20log_{10}|\sqrt{2}| \approx 3dB$.



As a straight-line approximation we let:

$\varphi \cong 0°$ for $\omega \leq \frac{z_1}{10}$

$\varphi \cong 45°$ for $\omega = z_1$

$\varphi \cong 90°$ for $\omega \geq 10z_1$

the straight-line plot has a slope of 45° per decade.

Bode Plot (contd.)

- The Bode plots for the pole $^{1}/_{\left(1+{}^{j\omega}/_{p_1}\right)}$ are similar except that the corner frequency is at $\omega = p_1$, the magnitude has a slope of –20 dB/decade and the phase has a slope of $-45°$ per decade.

Quadratic pole/zero:

$$^{1}/_{\left[1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]}$$

$$H_{\mathrm{dB}} = -20\log_{10}\left|1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad 0$$

$$\text{as} \quad \omega \rightarrow 0$$

$$H_{\mathrm{dB}} = -20\log_{10}\left|1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right| \quad \Rightarrow \quad -40\log_{10}\frac{\omega}{\omega_n}$$

$$\text{as} \quad \omega \rightarrow \infty$$

Clearly, the amplitude plot consists of two straight asymptotic lines: one with zero slope for $\omega < \omega_n$ and the other with slope $-40\ dB$/decade for $\omega > \omega_n$ with ω_n as the corner frequency.