Lecture – 11 **Date: 05.09.2017**

- Magnetically Coupled Circuits: Mutual Inductance, Energy in Coupled Circuits

Magnetically Coupled Circuits

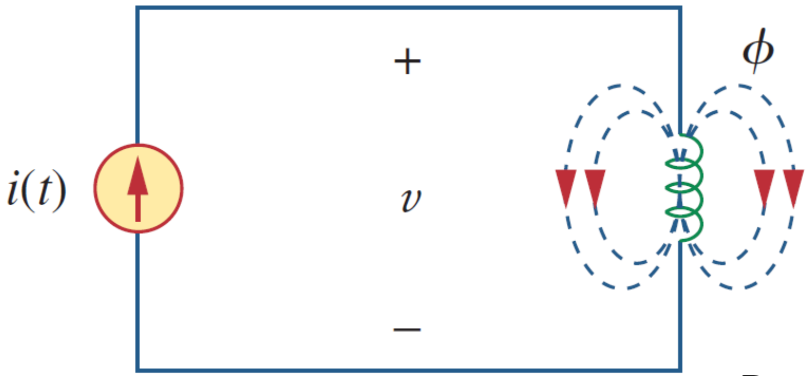
- So far, we have considered ***conductively coupled*** circuits → as one loop affects the neighboring loop through current conduction.
- When two loops with / without contacts between them affect each other through the magnetic field generated by one of them, they are said to be *magnetically coupled*.
- Transformer is based on the concept of magnetic coupling → It uses magnetically coupled coils to transfer energy from one circuit to another
- These are used in power systems for stepping up or stepping down ac voltages or currents.
- They are used in electronic circuits such as radio and television receivers for varied purposes such as impedance matching, isolating one part of a circuit from another, and for stepping up or down ac voltages and currents.

Mutual Inductance

When two inductors (or coils) are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter. This phenomenon is known as *mutual inductance*.

Magnetically Coupled Circuits (contd.)

Lets consider the following:



a single inductor (a coil with *N* turns)

- The flow of current $\boldsymbol{i}$ through the coil leads to the presence of a magnetic flux $\boldsymbol{\varphi}$ around it.
- According to Faraday's law, the voltage $\boldsymbol{v}$ induced in the coil is:

$$v = N\frac{d\varphi}{dt}$$

But the flux is produced by current and hence any change in flux is due to change in the current

$$v = N\frac{d\varphi}{dt} \Rightarrow v = N\frac{d\varphi}{di}\frac{di}{dt} \Rightarrow v = L\frac{di}{dt}$$

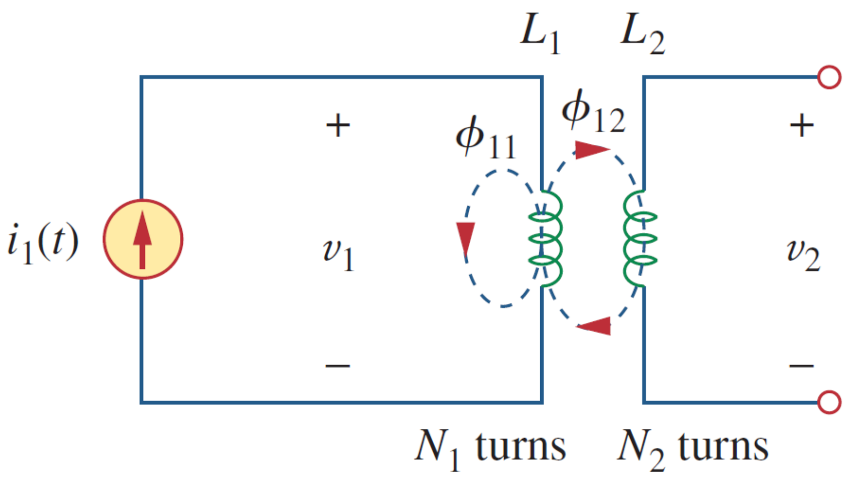
the voltage-current relationship for the inductor

$$L = N\frac{d\varphi}{di}$$

***L** is called the self-inductance*, because it relates the voltage induced in a coil by a time-varying current in the same coil

Magnetically Coupled Circuits (contd.)

Now consider the following:



- For the sake of simplicity, assume that the second inductor carries no current.
- The magnetic flux φ_1 due to i_1 in coil 1 has two components: φ_{11} links only coil 1 whereas φ_{12} links both coils.

$$\varphi_1 = \varphi_{11} + \varphi_{12}$$

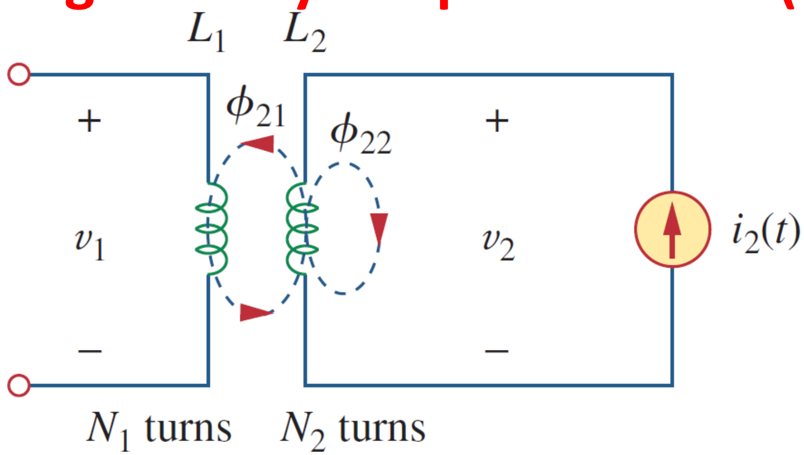
- the two coils are physically separated, they are said to be *magnetically coupled.*

$$v_1 = N_1 \frac{d\varphi_1}{dt} \Rightarrow v_1 = N_1 \frac{d\varphi_1}{di_1}\frac{di_1}{dt} \Leftarrow v_1 = L_1 \frac{di_1}{dt} \Leftarrow L_1 = N_1 \frac{d\varphi_1}{di_1}$$

$$v_2 = N_2 \frac{d\varphi_{12}}{dt} \Rightarrow v_2 = N_2 \frac{d\varphi_{12}}{dt}\frac{di_1}{dt} \Leftarrow v_2 = M_{21} \frac{di_1}{dt} \Leftarrow M_{21} = N_2 \frac{d\varphi_{12}}{di_1}$$

L_1 Self Inductance of coil 1, M_{21} is mutual inductance of coil 2

Magnetically Coupled Circuits (contd.)



- now let current i_2 flow in coil 2, while coil 1 carries no current
- The magnetic flux φ_2 due to i_2 in coil 2 has two components: φ_{22} links only coil 2 whereas φ_{21} links both coils.

$$\varphi_2 = \varphi_{22} + \varphi_{21}$$

$$v_2 = N_2 \frac{d\varphi_2}{dt} = N_2 \frac{d\varphi_2}{di_2}\frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

$$v_1 = N_1 \frac{d\varphi_{21}}{dt} = N_1 \frac{d\varphi_{21}}{di_2}\frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

M_{12}: Mutual Inductance of coil 1, L_2: Self Inductance of coil 2

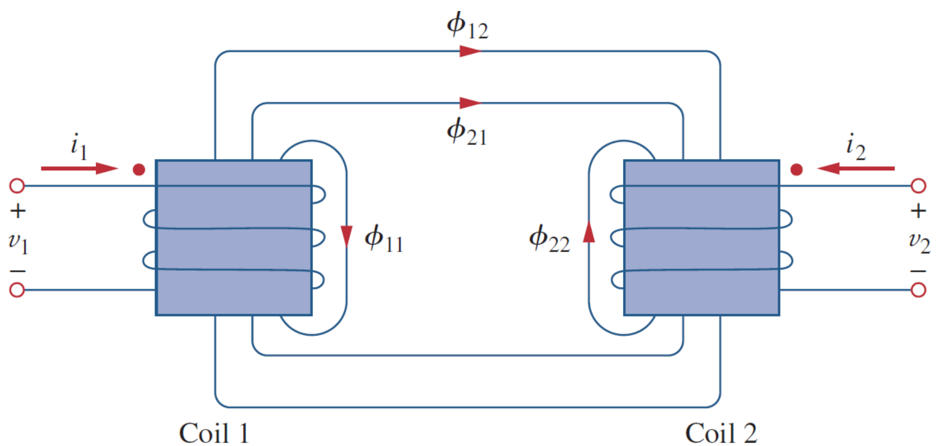
In general, $M_{21} = M_{12} = M$ ⇦ Mutual Inductance between the coils

Summary: the mutual inductance results if a voltage is induced by a time-varying current in another circuit. It is the property of an inductor to produce a voltage in reaction to a time-varying current in another inductor near it.

Mutual inductance is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

Mutual Inductance

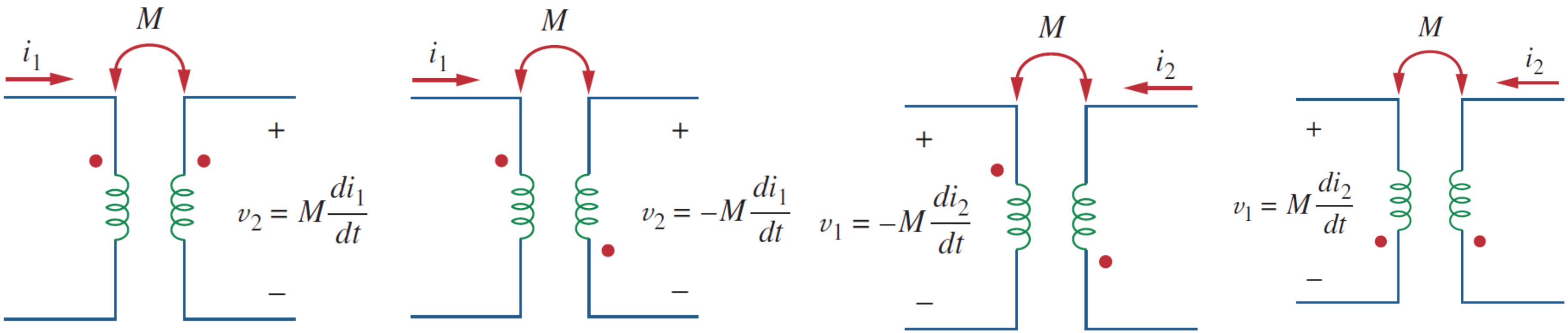
- Mutual inductance *M* is always a positive quantity, the mutual voltage $M\frac{di}{dt}$ may be negative or positive.
- The choice of the correct polarity for $M\frac{di}{dt}$ is made by examining the orientation or particular way in which both coils are physically wound and applying Lenz's law in conjunction with the right-hand rule.
- Since it is inconvenient to show the construction details of coils on a circuit schematic, *dot convention* in circuit analysis is adopted. Here, a dot is placed at one end of each of the two magnetically coupled coils to indicate the direction of the magnetic flux if current enters that dotted terminal of the coil.



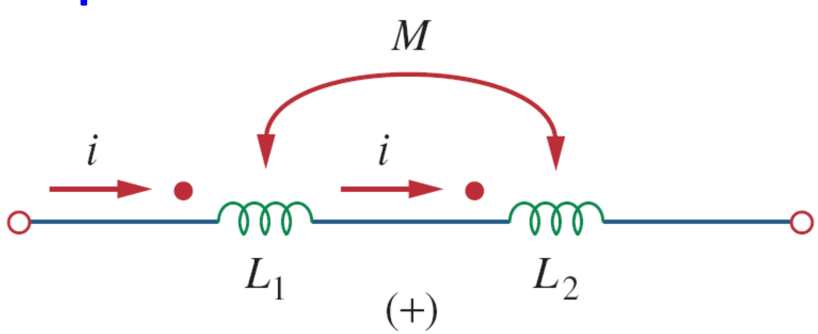
- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.

Mutual Inductance (contd.)

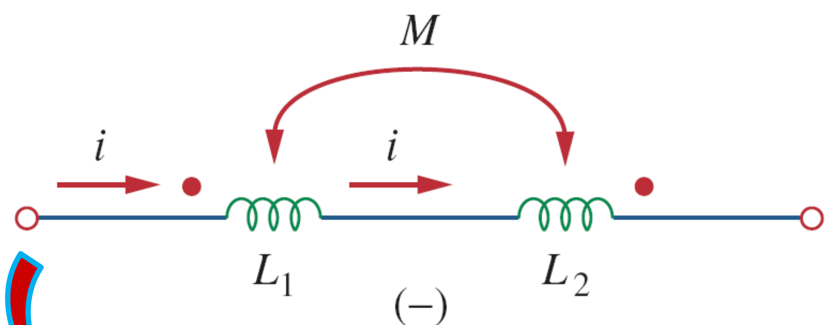
- If current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the 2nd coil is negative at the dotted terminal of the 2nd coil.



coupled coils in series:



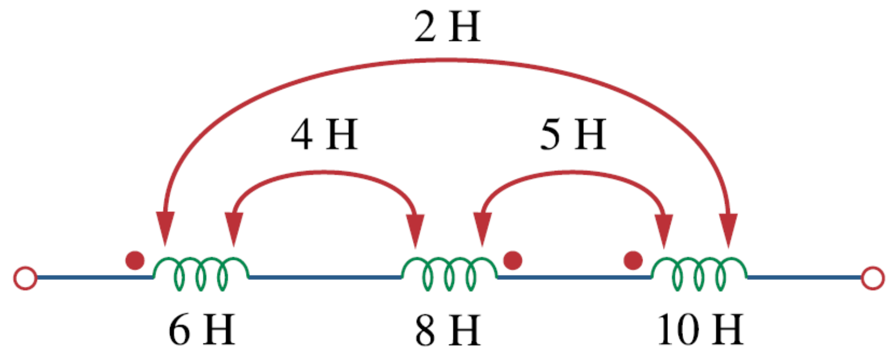
$$L = L_1 + L_2 + 2M$$



$$L = L_1 + L_2 - 2M$$

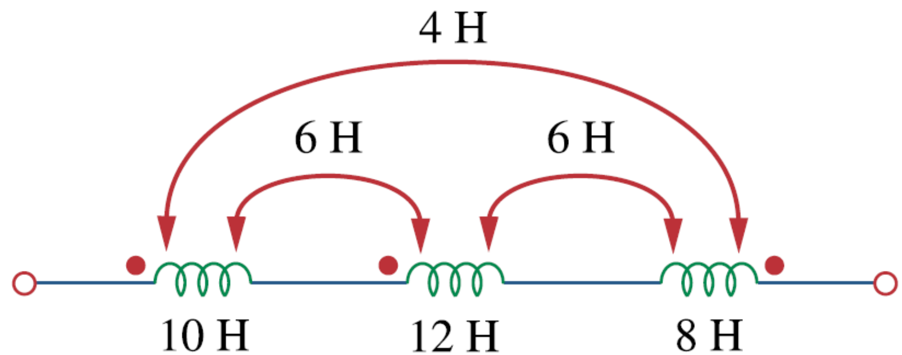
Example – 1

Calculate the total inductance.



Example – 2

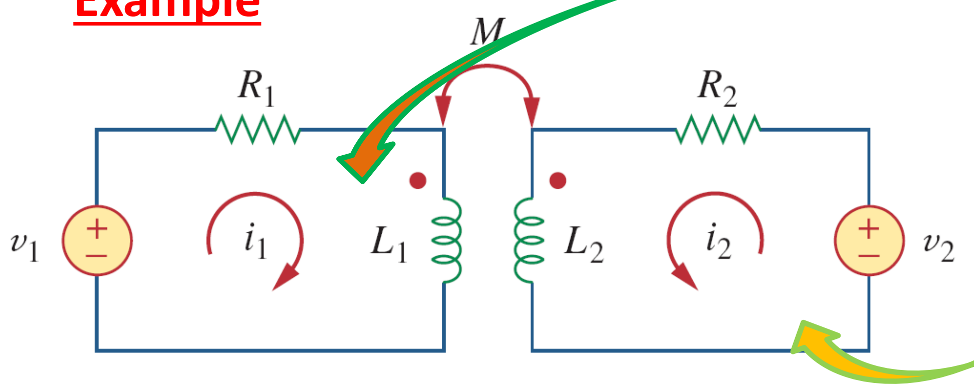
Calculate the total inductance.



Example – 3

Two coils connected in series-aiding fashion have a total inductance of 250 mH. When connected in a series-opposing configuration, the coils have a total inductance of 150mH. If the inductance of one coil (L_1) is three times the other, find L_1, L_2, and M.

Example



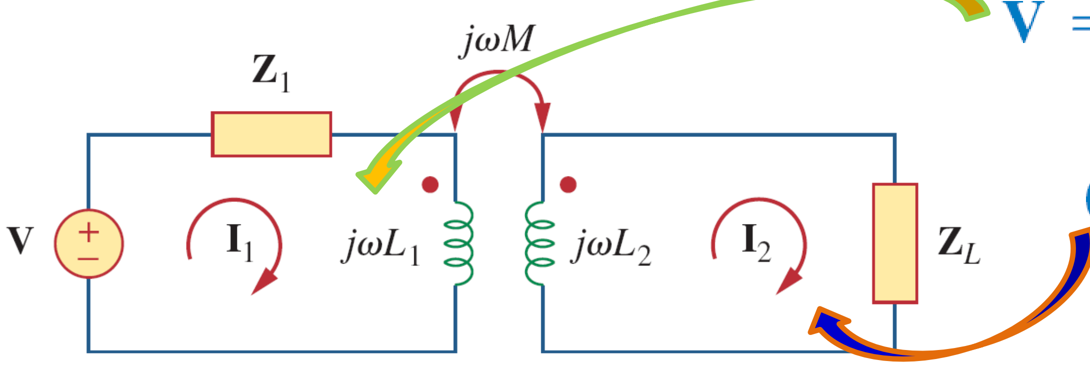
$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

- The KVL expressions in the frequency domain:

$$\mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M \mathbf{I}_2 \qquad \mathbf{V}_2 = j\omega M \mathbf{I}_1 + (R_2 + j\omega L_2)\mathbf{I}_2$$

- **Another example:**



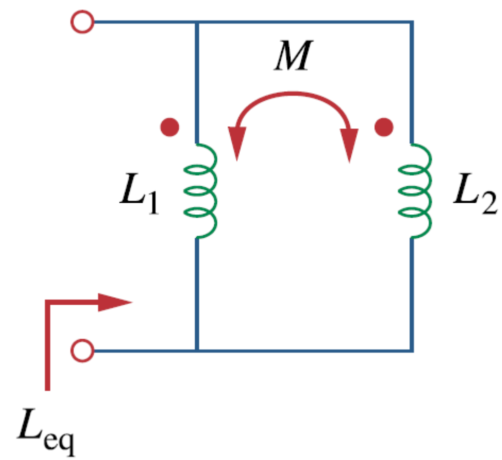
$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M \mathbf{I}_2$$

$$0 = -j\omega M \mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$$

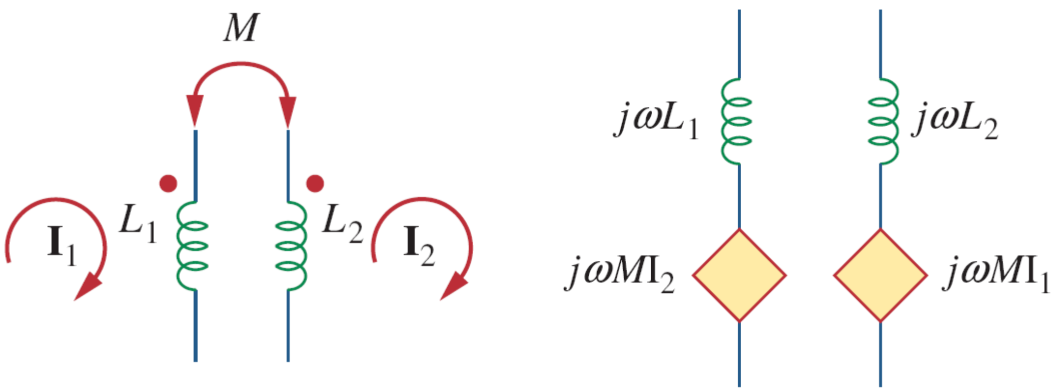
Example – 4

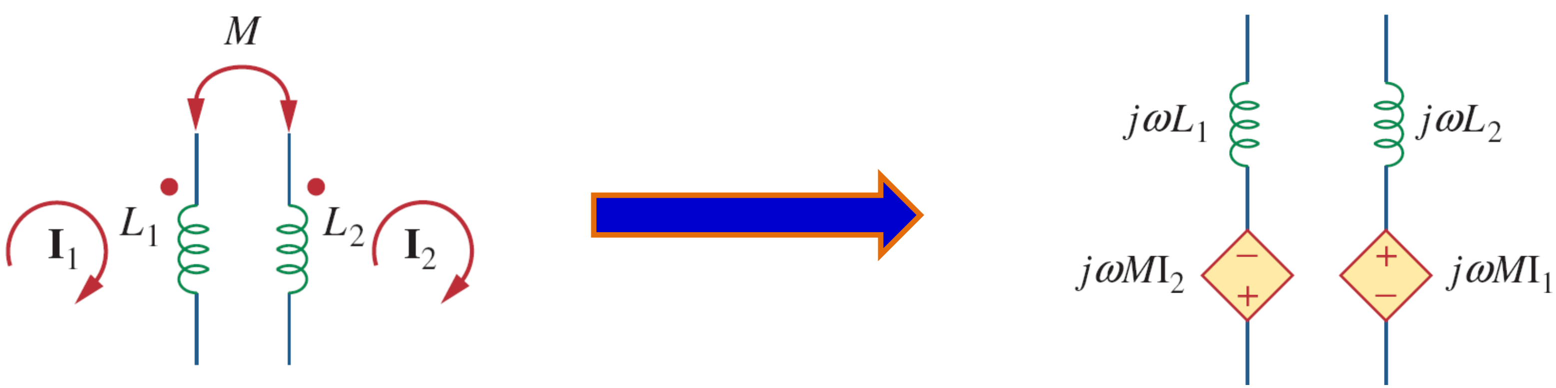
For the coupled coils, show that:

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$



- Often, solving mutually coupled circuits requires tracking of two or more steps made at once regarding the sign and values of the mutually induced voltages.



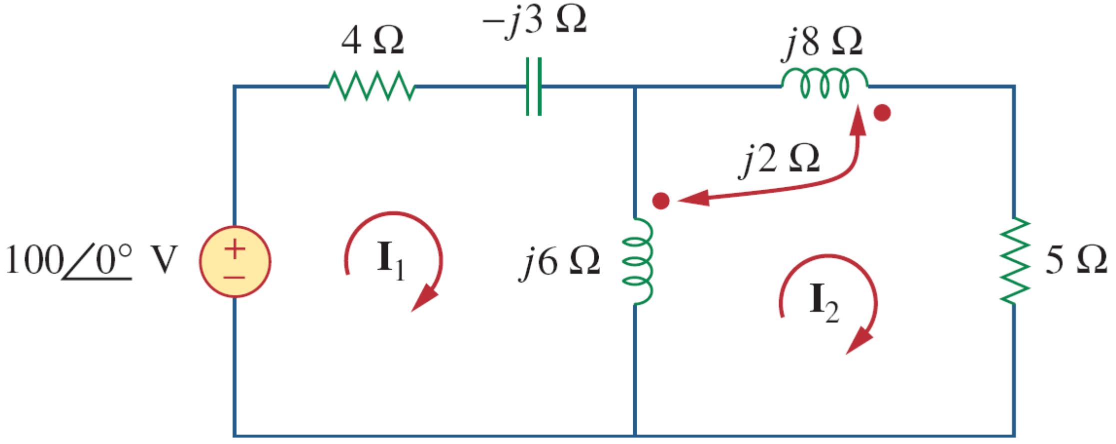


Mutual Inductance (contd.)

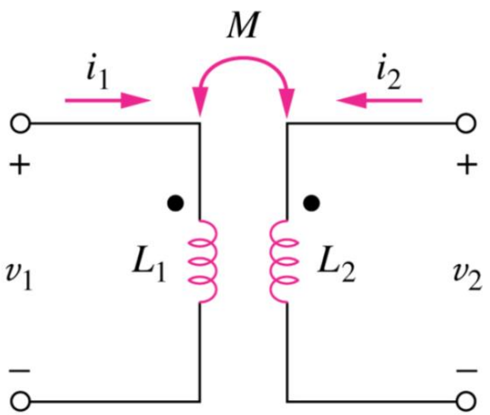
In this course, we are not concerned with the determination of the mutual inductances of the coils and their dot placements. We assume that the mutual inductance and the placement of the dots are the "givens'' of the circuit problem, like the circuit components *R*, *L*, and *C*.

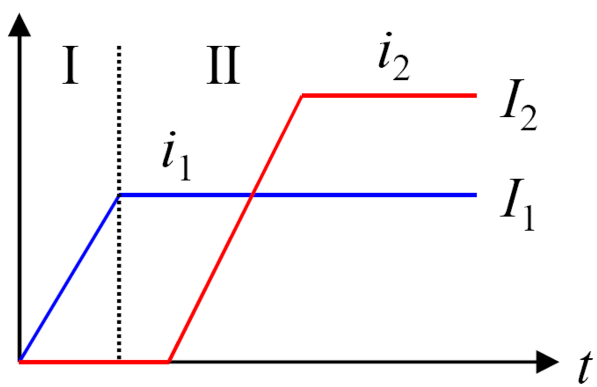
Example – 5

Calculate the mesh currents in this circuit.



Energy in Coupled Circuits





- To find the stored energy let $i_1 = I_1$ and $i_2 = I_2$:

Step-1: $i_2 = 0$ and i_1 increases from 0 to $I_1 \rightarrow$ the power in the circuit is

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt}$$

the energy stored in the circuit

$$w_1 = \int p_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2$$

Step-2: $i_1 = I_1$ and i_2 increases from 0 to I_2 $\rightarrow$ the power in the circuit is

$$p_2(t) = i_1 v_1 + i_2 v_2 = I_1 M_{12} \frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt}$$

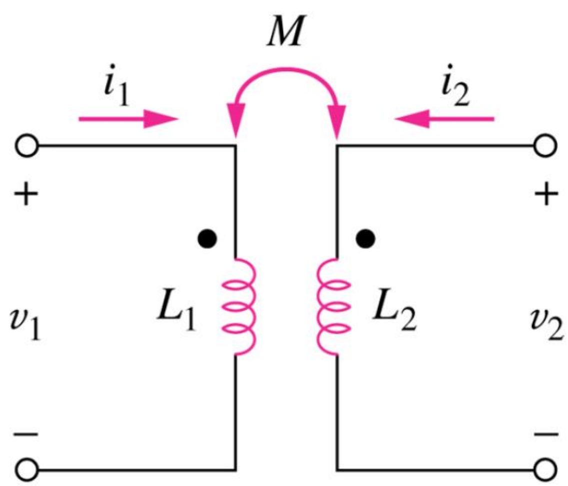
the energy stored in the circuit $= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$

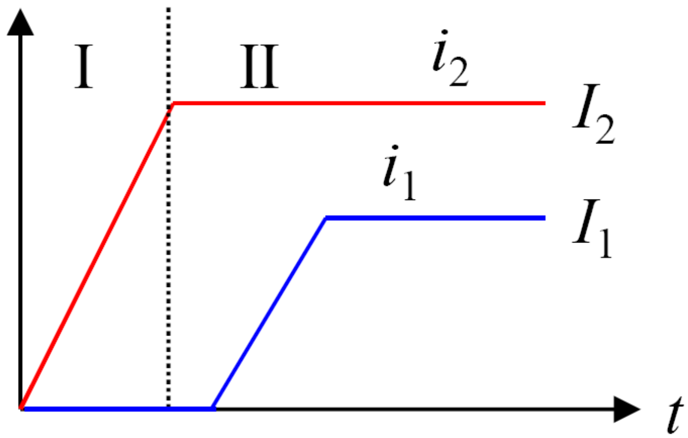
Energy in Coupled Circuits (contd.)

- The total energy stored in the coils when both and have reached constant i_1 and i_2 values is:

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$

Alternatively:





$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2$$

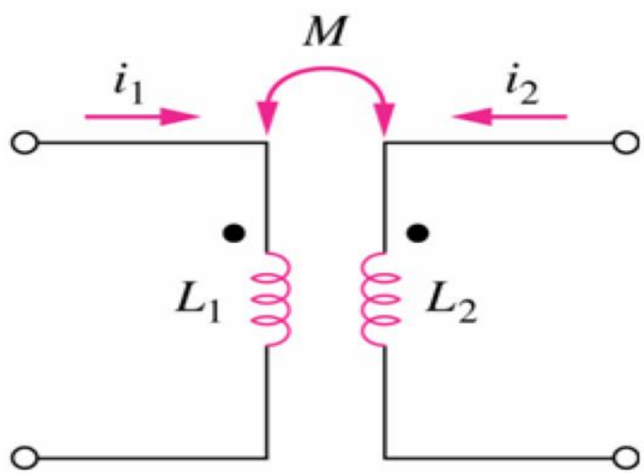
- But the total energy stored should be the same regardless of how we reach the final conditions.

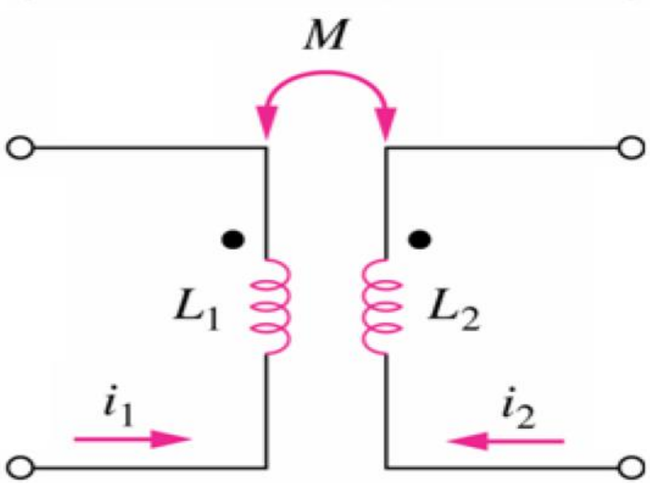
$$M_{12} = M_{21} = M$$

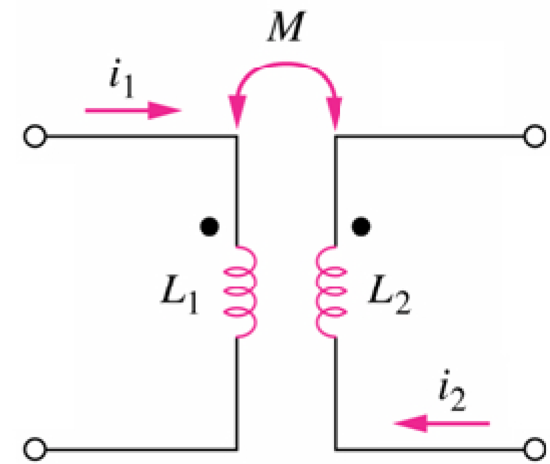
$$\Rightarrow \quad w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

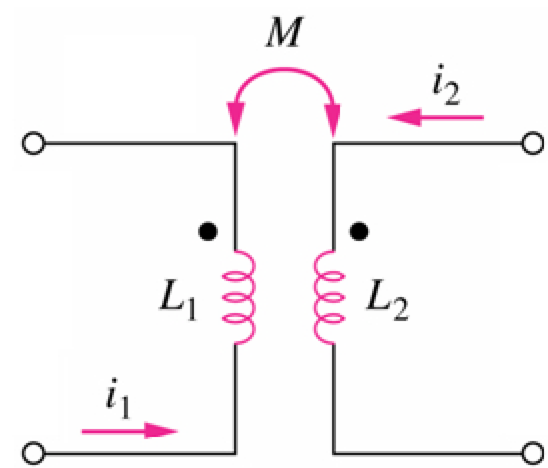
Energy in Coupled Circuits (contd.)

- This equation is applicable when the currents both enter the dotted terminals or undotted terminals.









In general:

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2$$

Energy in Coupled Circuits (contd.)

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 \geq 0$$

$$\frac{1}{2}(i_1\sqrt{L_1} - i_2\sqrt{L_2})^2 + i_1 i_2(\sqrt{L_1 L_2} - M) \geq 0$$

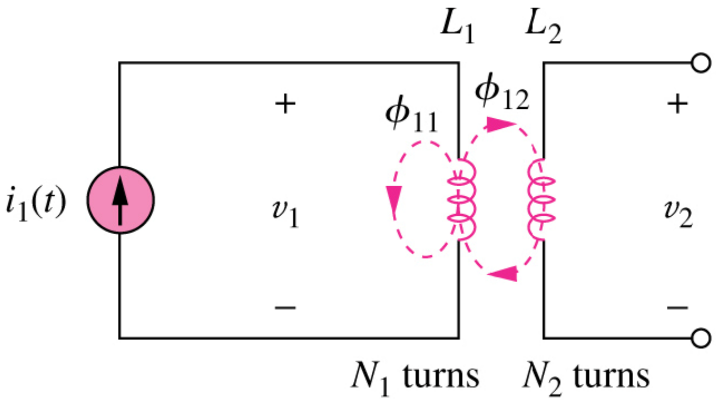
$$\sqrt{L_1 L_2} - M \geq 0 \quad \Rightarrow \quad M \leq \sqrt{L_1 L_2}$$

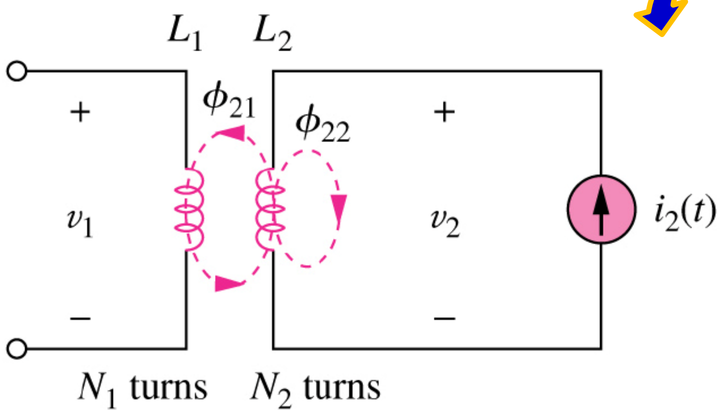
- The extent to which the mutual inductance *M* approaches the upper limit is specified by the *coefficient of coupling k*:

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

The coupling coefficient is the fraction of the total flux emanating from one coil that links the other coil.

Energy in Coupled Circuits (contd.)





$$k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{12}}{\phi_{11} + \phi_{12}}$$

$$k = \frac{\phi_{21}}{\phi_2} = \frac{\phi_{21}}{\phi_{21} + \phi_{22}}$$

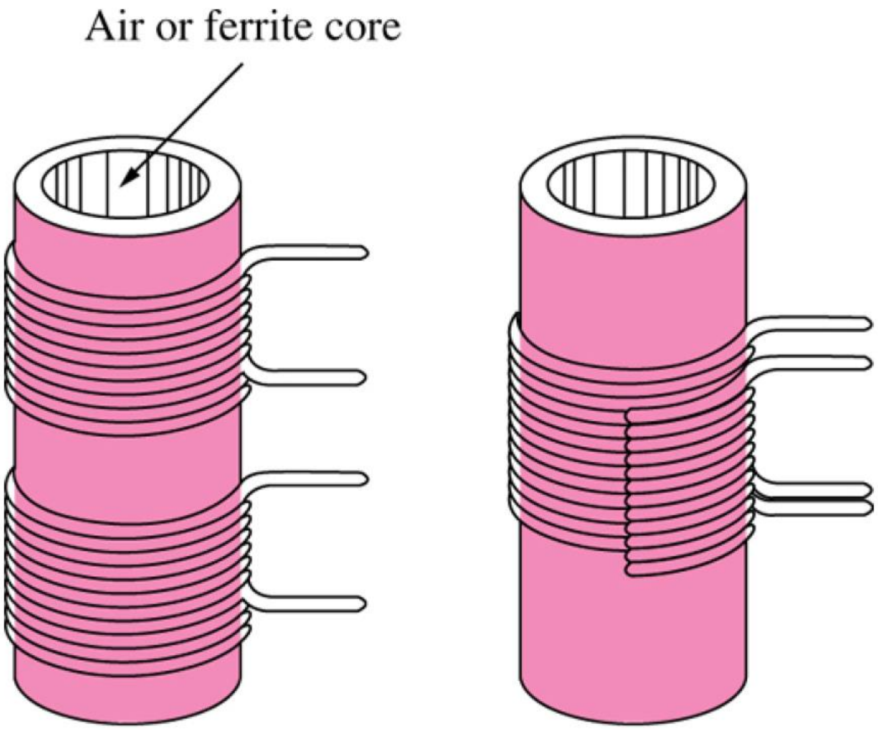
If $k = 1$, then we have 100 percent coupling, i.e., the entire flux produced by one coil links another coil:

$$\phi_{11} = \phi_{22} = 0$$

The coupling coefficient k is a measure of the magnetic coupling between two coils: $0 \leq k \leq 1$

Energy in Coupled Circuits (contd.)

- We expect k to depend on the closeness of the two coils, their core, their orientation, and their windings.



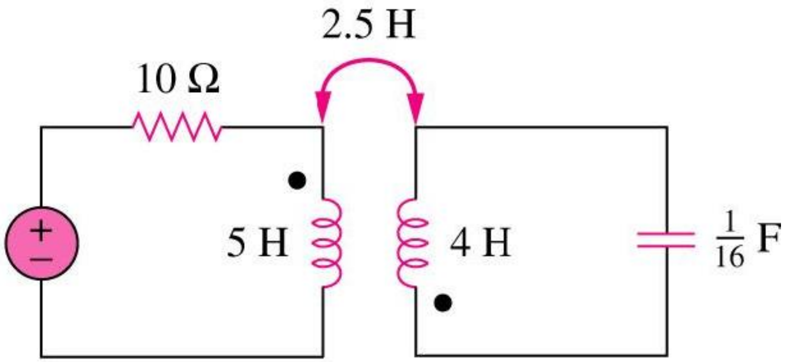
$k < 0.5$ $\qquad$ $k > 0.5$

The air-core transformers used in radio frequency circuits are loosely coupled, whereas iron-core transformers used in power systems are tightly coupled.

Example – 6

Determine the coupling coefficient and calculate the energy stored in the coupled inductors at time $t = 1s$.

$$v = 60\cos(4t + 30°)\ \text{V}$$



Example – 7

Given, L_1=40mH, L_2=5mH, and coupling coefficient k=0.6. Find $i_1(t)$ and $v_2(t)$, given that $v_1(t) = 10cos\omega t$ and $i_2(t) = 2sin\omega t$, $\omega = 2000\frac{rad}{s}$.