Lecture – 6

Date: 22.08.2016

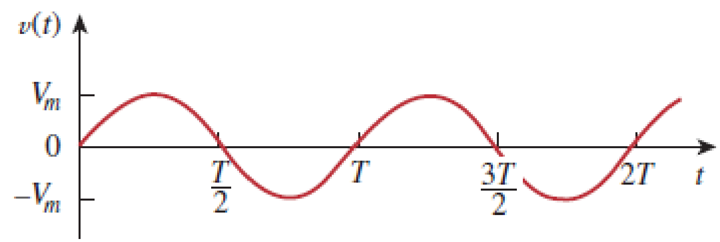
- AC Circuits: Sinusoids and Phasors

Sinusoids

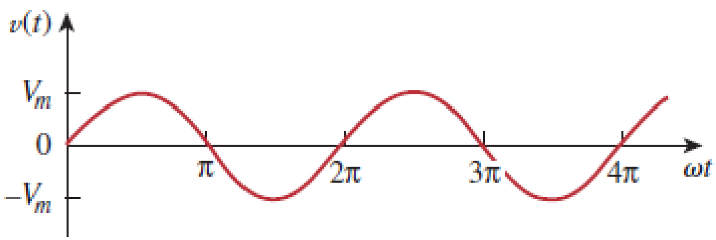
- A sinusoid is a signal that has the form of the sine or cosine function.
  - A sinusoidal current is usually referred to as *alternating current* (*ac*). Such a current reverses at regular time intervals and has alternately positive and negative values.
  - Circuits driven by sinusoidal current or voltage sources are called *ac circuits.*

- Lets consider the sinusoidal voltage: $v(t) = V_m \sin \omega t$

V_m = the *amplitude* of the sinusoid
ω = the *angular frequency* in radians/s
ωt = the *argument* of the sinusoid



As a function of time



As a function of argument

the sinusoid repeats itself every *T* seconds → *T* is the *period* of the sinusoid.

$$T = \frac{2\pi}{\omega}$$

Sinusoids (contd.)

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t+\frac{2\pi}{\omega}\right) = V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t)$$

v has the same value at $t+T$ as it does at t and is said to be *periodic*

a periodic function satisfies $f(t) = f(t+nT)$, for all t and for all integers *n*.

- The reciprocal of T is the number of cycles per second, known as the *cyclic frequency f* of the sinusoid.

$$f = \frac{1}{T}$$

$$\omega = 2\pi f$$

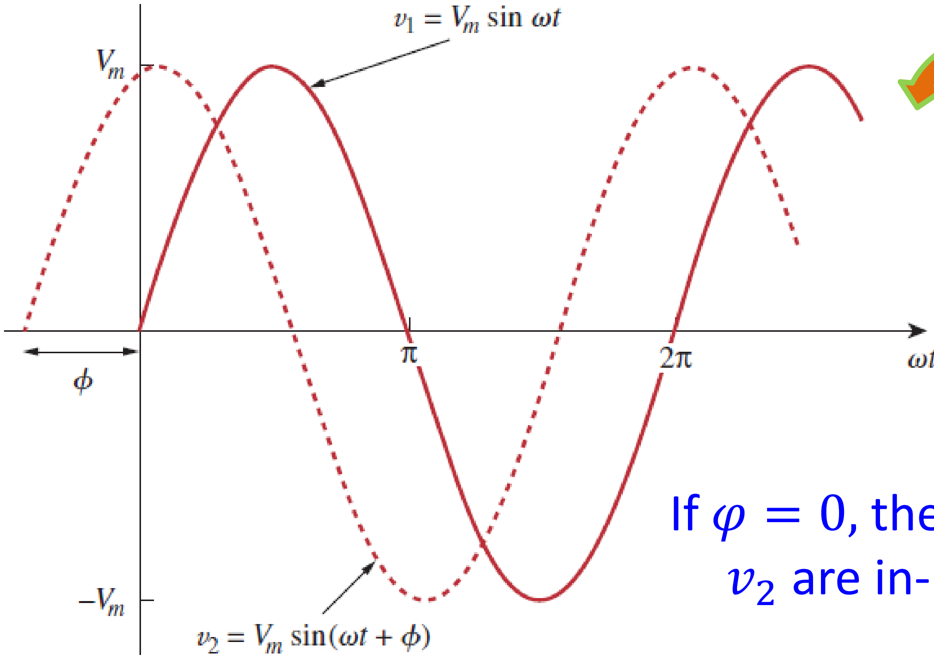
ω is in radians per second (rad/s), *f* is in hertz (Hz).

- a more general expression for the sinusoid: $v(t) = V_m \sin(\omega t + \phi)$

Where $(\omega t + \varphi)$ is the argument and φ is the *phase* and both can be in radians or degrees

Sinusoids (contd.)

- two sinusoids: $v_1(t) = V_m \sin \omega t$ $\qquad$ $v_2(t) = V_m \sin(\omega t + \phi)$



v_2 *leads* v_1 by ϕ or that v_1 *lags* v_2 by ϕ

If $\varphi \neq 0$, then v_1 and v_2 are out of phase.

If $\varphi = 0$, then v_1 and v_2 are in-phase.

they reach their minima and maxima at exactly the same time

We can compare both in this manner because they operate at the same frequency; they do not need to have the same amplitude.

Sinusoids (contd.)

- A sinusoid can be expressed in either sine or cosine form.
- When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes.
- This is achieved by using the following trigonometric identities:

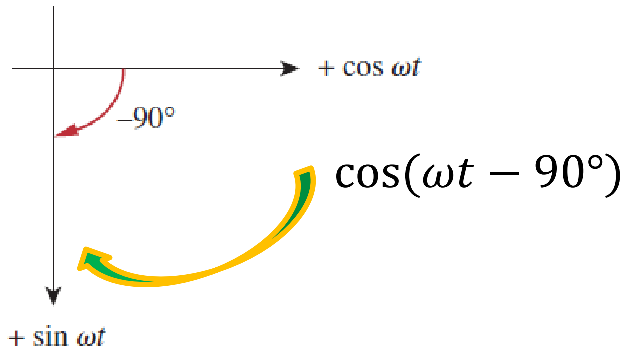
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

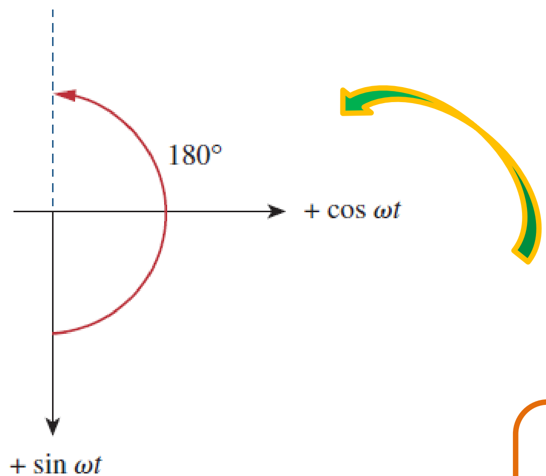
- With these identities:

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos \omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin \omega t$$

Use these to transform a sinusoid from sine form to cosine form or vice versa.

Sinusoids (contd.)





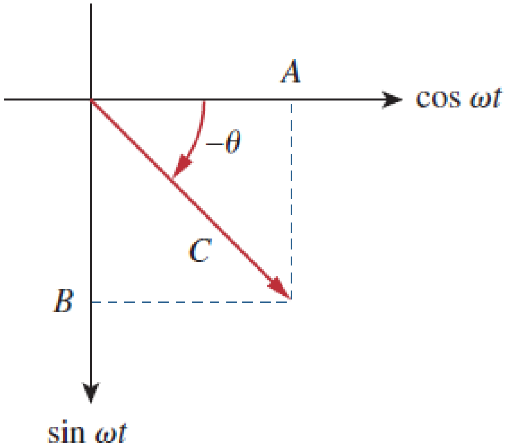
$\sin(\omega t + 180°)$

<u>Alternative Graphical Approach:</u>

- the horizontal axis represents the magnitude of cosine
- the vertical axis (pointing down) denotes the magnitude of sine.
- Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates.

graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form.

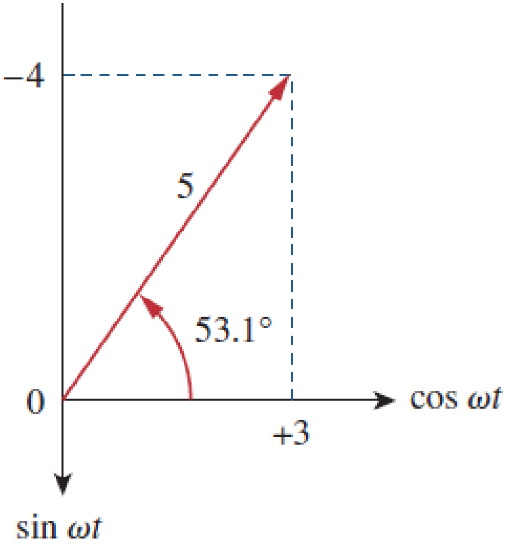
Sinusoids (contd.)



$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \theta)$$

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1}\frac{B}{A}$$

$$3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1°)$$



Do not confuse the *sine* and *cosine* axes with the axes for complex numbers. It is a natural tendency to have the vertical axis point up, however the positive direction of the sine function is pointing down.

Example – 1

A current source in a linear circuit is $i_s = 8\cos(500\pi t - 25^{o})$ A

a) What is the amplitude of the current?
b) What is the angular frequency?
c) Find the frequency of the current.
d) What is i_s at t=2ms.

Example – 2

Given $v_1 = 20\sin(\omega t + 60°)$ and $v_2 = 60\sin(\omega t - 10°)$ determine the phase angle between the two sinusoids and which one lags the other.

Example – 3

For the following pairs of sinusoids, determine which one leads and by how much.

(a) $v(t) = 10\cos(4t - 60^{o})$ and $i(t) = 4\sin(4t + 50^{o})$
(b) $v_1(t) = 4\cos(377t + 10^{o})$ and $v_2(t) = -20\cos 377t$
(c) $x(t) = 13\cos 2t + 5\sin 2t$ and $y(t) = 15\cos(2t - 11.8^{o})$

Phasors

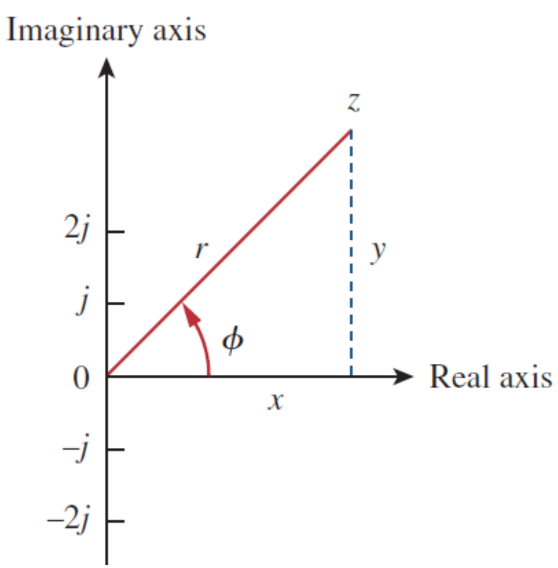
- phasor is a complex number that represents amplitude and phase of a sinusoid.
- phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources.

Complex Number:

$z = x + jy$ Rectangular form

$z = r\angle\phi$ Polar form

$z = re^{j\phi}$ Exponential form



Given x and y, we can get r and φ as:

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

if we know r and φ we can obtain x and y as

$$x = r\cos\phi, \qquad y = r\sin\phi$$

Phasors (contd.)

- Addition and subtraction of complex numbers are easier in rectangular form; multiplication and division are simpler in polar form.

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1 \qquad z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication: $z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\angle\phi_1 - \phi_2$

Reciprocal: $\dfrac{1}{z} = \dfrac{1}{r}\angle -\phi$

Square Root: $\sqrt{z} = \sqrt{r}\angle\phi/2$

Complex Conjugate: $z^* = x - jy = r\angle -\phi = re^{-j\phi}$

idea of phasor representation is based on Euler's identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$
$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

Phasors (contd.)

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t + \phi)})$$

$$v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi$$

to obtain the sinusoid corresponding to a given phasor **V**, multiply the phasor by the time factor and take the real part.

As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

Phasor domain is also called frequency domain

Phasors (contd.)

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ)$$

$$= \mathrm{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \mathrm{Re}(j\omega \mathbf{V} e^{j\omega t})$$

$$\frac{dv}{dt} \iff j\omega \mathbf{V}$$

(Time domain) (Phasor domain)

$$\int v\, dt \iff \frac{\mathbf{V}}{j\omega}$$

(Time domain) (Phasor domain)

The differences between $v(t)$ and **V** should be understood:

1. $v(t)$ is the *instantaneous or time domain* representation, while **V** is the *frequency or phasor domain* representation.

2. $v(t)$ is time dependent, while **V** is not.

3. $v(t)$ is always real with no complex term, while **V** is generally complex.

Example – 4

If $f(\phi) = \cos\phi + j\sin\phi$, show that $f(\phi) = e^{j\phi}$.

Example – 5

Find the phasors corresponding to the following signals:

(a) $v(t) = 21\cos(4t - 15^o)$ V
(b) $i(\mathrm{t}) = -8\sin(10t + 70^o)$ mA
(c) $v(t) = 120\sin(10t - 50^o)$ V
(d) $i(\mathrm{t}) = -60\cos(30t + 10^o)$ mA

Example – 6

Obtain the sinusoids corresponding to each of the following phasors:

(a) $\mathbf{V}_1 = 60\angle 15^o$ V, $\omega = 1$
(b) $\mathbf{V}_2 = 6 + j8$ V, $\omega = 40$
(c) $\mathbf{I}_1 = 2.8e^{-j\pi/3}$ A, $\omega = 377$
(d) $\mathbf{I}_2 = -0.5 - j1.2$ A, $\omega = 10^3$

Example – 7

Simplify the following:

(a) $f(t) = 5\cos(2t + 15^{o}) - 4\sin(2t - 30^{o})$

(b) $g(t) = 8\sin t + 4\cos(t + 50^{o})$

(c) $h(t) = \int_0^t (10\cos 40t + 50\sin 40t)dt$

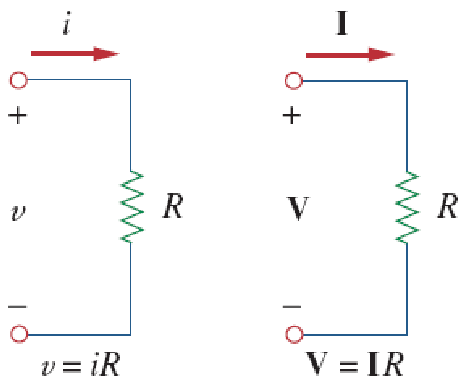
Example – 8

Using phasors, determine $i(t)$ in the following equations:

(a) $2\frac{di}{dt} + 3i(t) = 4\cos(2t - 45^{o})$

(b) $10\int i\,dt + \frac{di}{dt} + 6i(t) = 5\cos(5t + 22^{o})$

Phasor Relationships for Circuit Elements



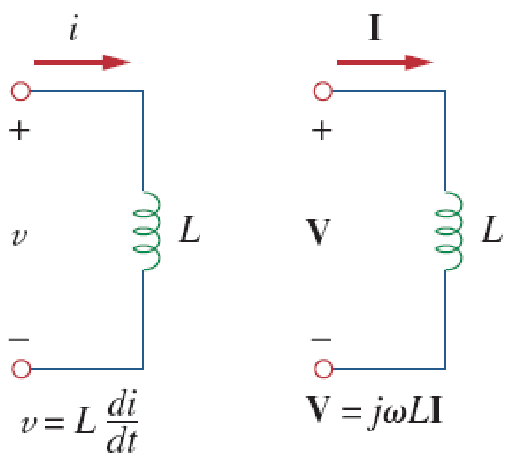
If the current through a resistor R is $i = I_m(\cos\omega t + \varphi)$, then the voltage across it is given by Ohm's law as:

$$v = iR = RI_m \cos(\omega t + \phi)$$

$$\mathbf{V} = RI_m \angle \phi$$

$$\mathbf{V} = R\mathbf{I}$$

∴ voltage-current relation for the resistor in the phasor domain continues to be Ohm's law



For the inductor L, assume current $i = I_m(\cos\omega t + \varphi)$, then the voltage across it is:

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

$$v = \omega L I_m \cos(\omega t + \phi + 90°)$$

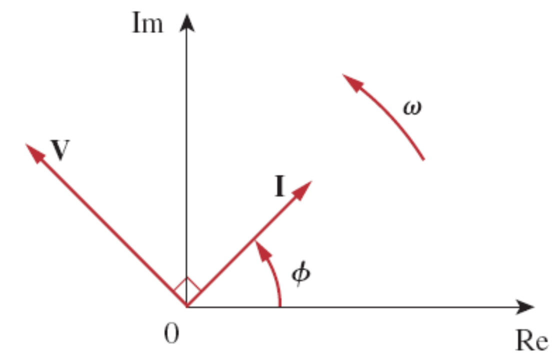
Phasor Relationships for Circuit Elements

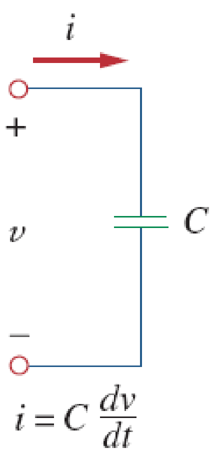
$$v = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

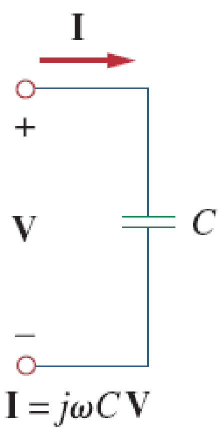
$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \underline{/\phi + 90^\circ}$$

$$\mathbf{V} = j\omega L \mathbf{I}$$

the voltage has a magnitude of $\omega L I_m$ and a phase of φ. The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°.

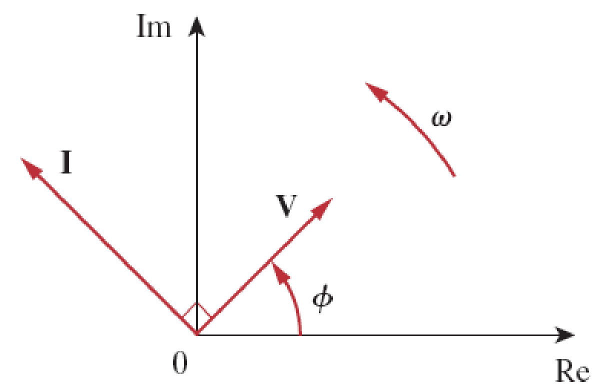






$$\mathbf{I} = j\omega C \mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

the current leads the voltage by 90°.



Example – 9

What is the instantaneous voltage across a 2μF capacitor when the current through it is $i = 4\sin(10^6 t + 25^o)\ A$?

Example – 10

A voltage $v(t) = 100\cos(60t + 20^o)$V is applied to a parallel combination of a 40kΩ resistor and a 50μF capacitor. Find the steady-state currents through the resistor and the capacitor.

Example – 11

A series *RLC* circuit has *R* = 80 Ω, *L* = 240mH, and *C* = 5mF. If the input voltage is $v(t) = 100\cos(2t)$, find the current flowing through the circuit.