Lecture – 2 Date: 04.08.2016

- First-Order Circuit – Review

First Order Circuit

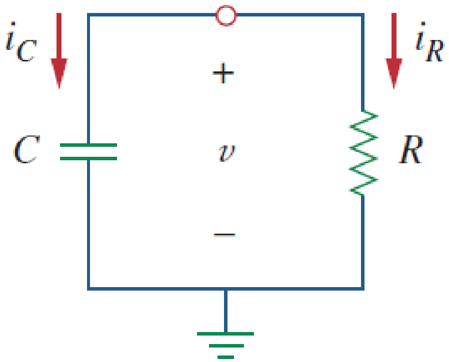
- A first order circuit is characterized by first-order differential equation

This comes in two types of configs:

a) RL Circuit

b) RC Circuit

Two possible scenarios: (a) without excitation, (b) with excitation



Need to determine $v(t)$, $i_c(t)$, and $i_R(t)$

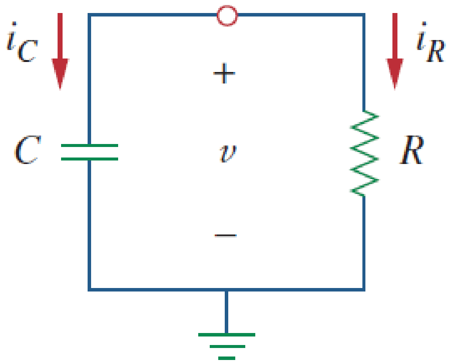
This will result when the dc source is suddenly disconnected and in this scenario the energy stored in the capacitor gets released to the resistors.

Assume: $v(0) = V_0$ ← Initial charging voltage of the capacitor

Therefore the energy stored in the capacitor:

$$w(0) = \frac{1}{2} C V_0^2$$

First Order Circuit (contd.)



KCL gives: $i_C + i_R = 0$

$$\Rightarrow C\frac{dv}{dt} + \frac{v}{R} = 0 \quad \Rightarrow \frac{dv}{v} = -\frac{1}{RC}dt$$

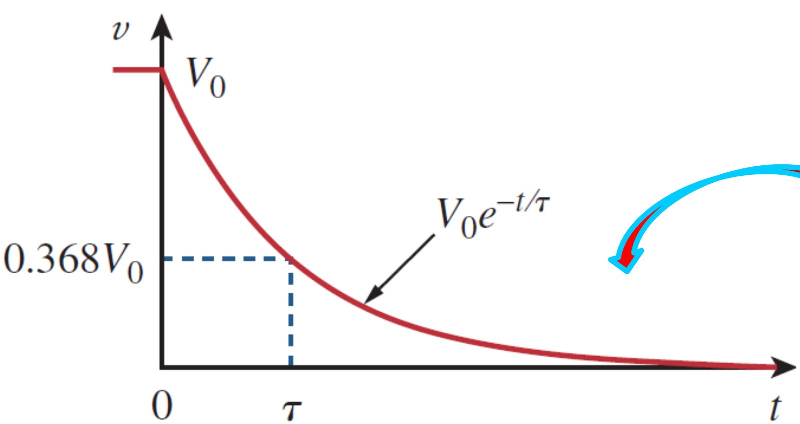
$$\Rightarrow \ln v = -\frac{t}{RC} + \ln A \quad \Rightarrow v(t) = Ae^{-t/RC}$$

From initial condition: $v(0) = A = V_0$

$$\therefore v(t) = V_0 e^{-t/RC}$$

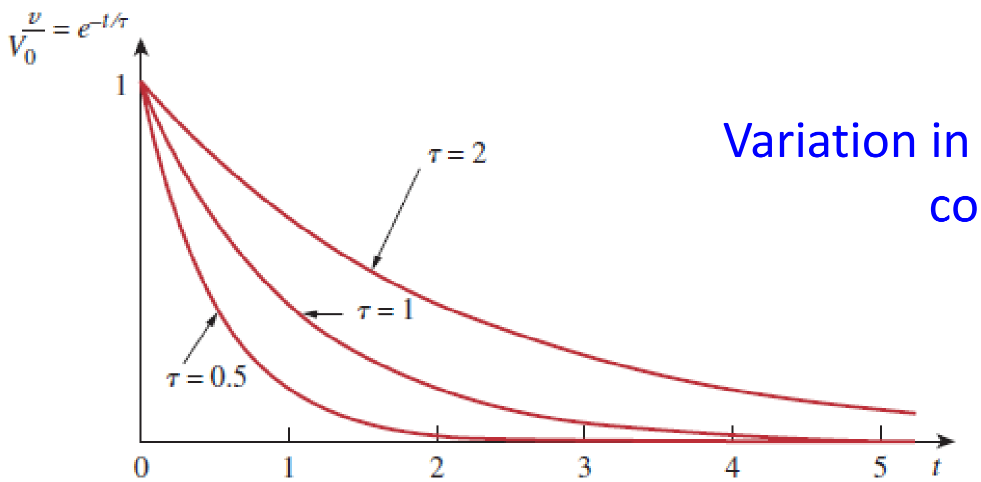
Exponential Decay

Natural Response



The time constant of a circuit is the time required for the response to decay to a factor of $1/e$

First Order Circuit (contd.)



Variation in voltage with various time constant ($\tau = RC$)

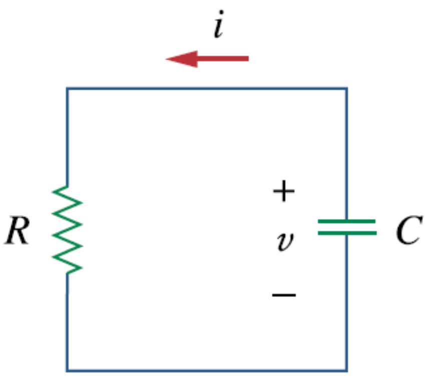
The power dissipated in the resistor

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/RC} \quad \Rightarrow \quad p(t) = v i_R(t) = \frac{V_0^2}{R} e^{-2t/RC}$$

- The energy absorbed by the resistor up to time t is: $w_R(t) = \int_0^t p(j)dj = \frac{1}{2} C V_0^2 \left(1 - e^{-2t/\tau}\right)$

For $t \to \infty$ we get $w_R(\infty) \to \frac{1}{2} C V_0^2$

Example – 1



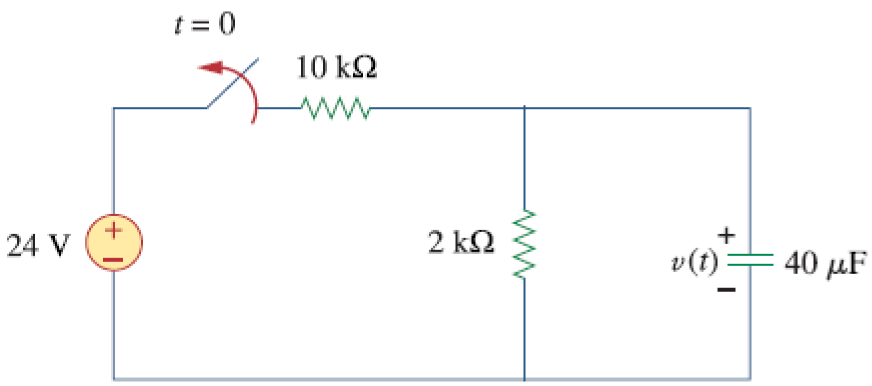
In the circuit shown:

$$v(t) = 56e^{-200t}\,\text{V}, \quad t > 0$$

$$i(t) = 8e^{-200t}\,\text{mA}, \quad t > 0$$

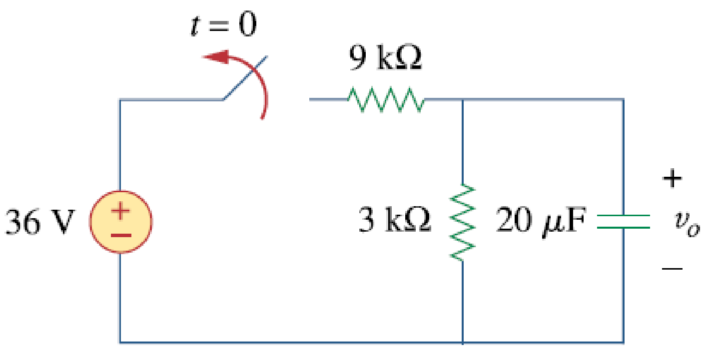
(a) Find the values of *R* and *C*.
(b) Calculate the time constant τ.
(c) Determine the time required for the voltage to decay half its initial value at *t* = 0.

Example – 2



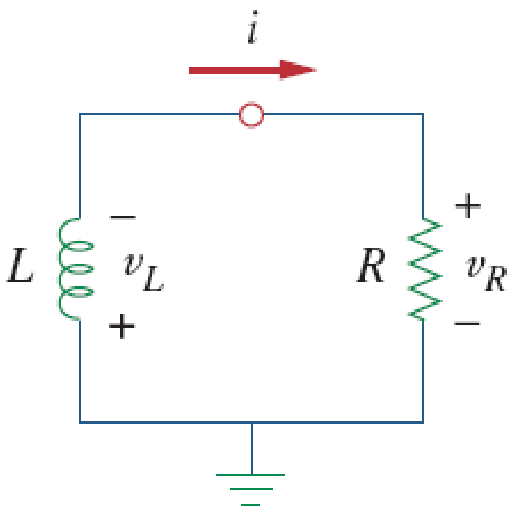
The switch in this circuit has been closed for a long time, and it opens at $t = 0$. Find $v(t)$ for $t \geq 0$.

Example – 3



Find $v_0(t)$ for $t > 0$. Determine the time necessary for the capacitor voltage to decay to one-third of its value at $t = 0$.

First Order Circuit (contd.)



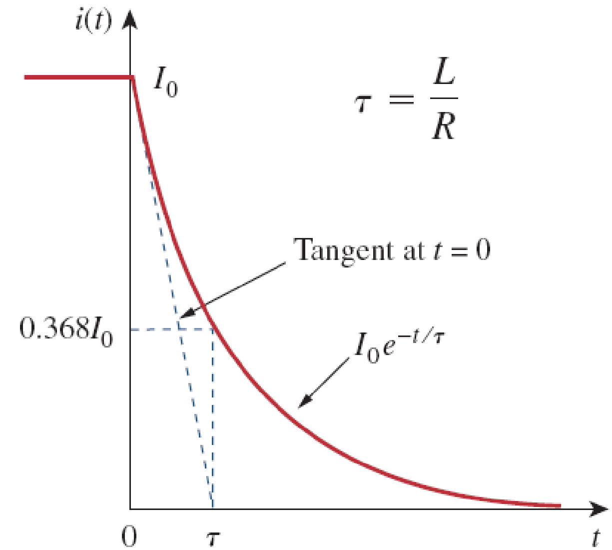
Assume, at t = 0: $i(0) = I_0$

Stored energy in the inductor: $w(0) = \frac{1}{2} L I_0^2$

$$v_L + v_R = 0 \quad \Rightarrow \quad L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0 \quad \Rightarrow \quad \int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \quad \Rightarrow \quad i(t) = I_0 e^{-Rt/L}$$



First Order Circuit (contd.)

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

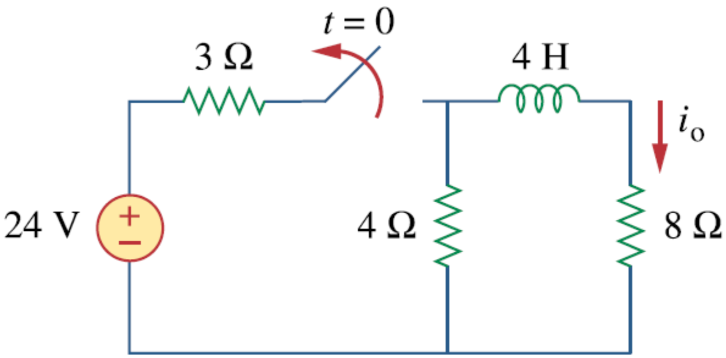
Power dissipated in the resistor →

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

- The energy absorbed by the resistor up to time t is: $w_R(t) = \int_0^t p(j)dj = \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau}\right)$

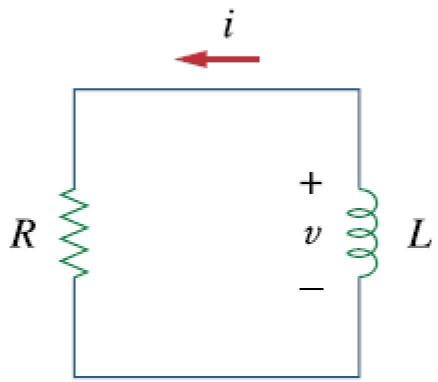
For $t \rightarrow \infty$ we get $w_R(\infty) \rightarrow \frac{1}{2} L I_0^2$

Example – 4



find i_0 for $t > 0$.

Example – 5



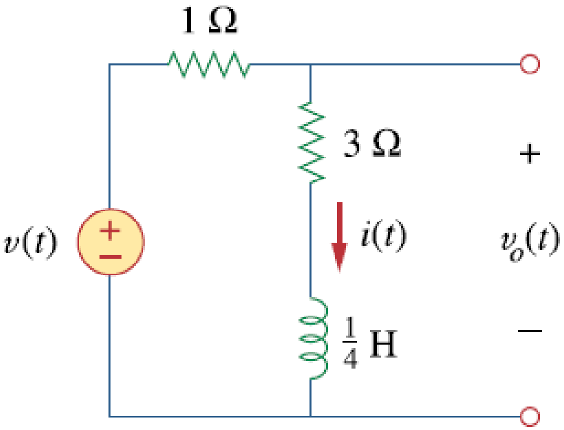
$$v(t) = 20e^{-10^3 t} \text{ V}, \qquad t > 0$$

$$i(t) = 4e^{-10^3 t} \text{ mA}, \qquad t > 0$$

(a) Find *R*, *L*, and τ .

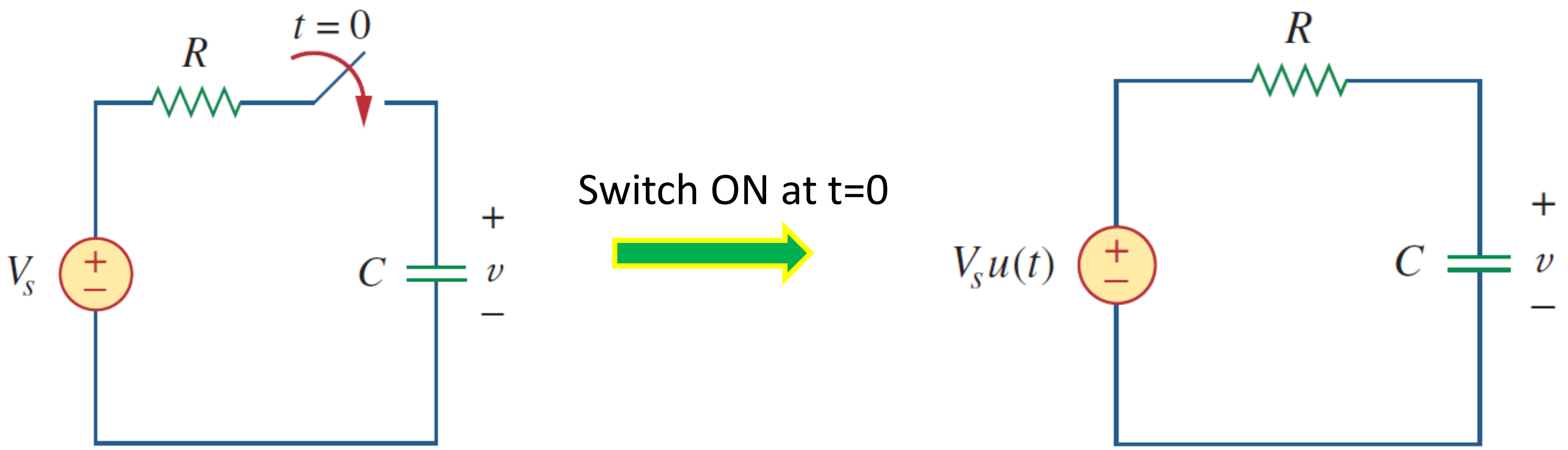
(b) Calculate the energy dissipated in the resistance for 0 < *t* < 0.5 ms.

Example – 6



Find $v_0(t)$ if $i(0)$ = 2A and $v(t)$ = 0.

Step Response of an RC Circuit



Assume an initial voltage V_0 on the capacitor

- Voltage on a capacitor cannot change instantaneously: $v(0^-) = v(0^+) = V_0$

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0 \quad \Rightarrow \quad \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

- For $t > 0$: $\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \quad \Rightarrow \quad \frac{dv}{v - V_s} = -\frac{dt}{RC}$

Step Response of an RC Circuit (contd.)

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t \Rightarrow \ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

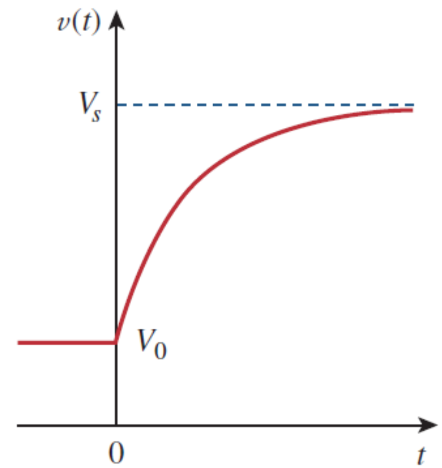
$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \Rightarrow v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}$$ For $t > 0$

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$ ⇐ Complete Response

Step Response of an RC Circuit (contd.)

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$



For $V_0 = 0$

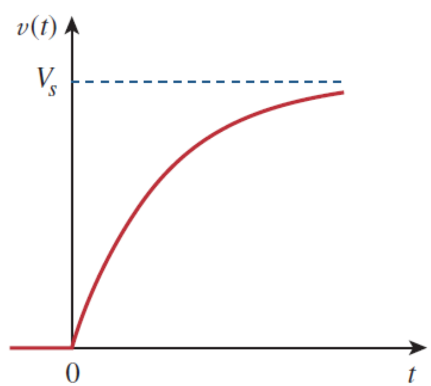
$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

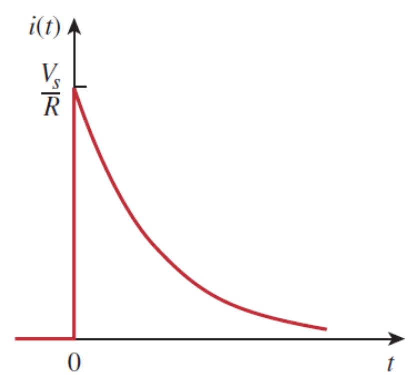
$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$

The current through the capacitor:

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}$$

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t)$$





Step Response of an RC Circuit (contd.)

$$\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$$

$$v = v_n + v_f$$

$$v_n = V_o e^{-t/\tau}$$

$$v_f = V_s(1 - e^{-t/\tau})$$

$$\text{Complete response} = \underset{\text{temporary part}}{\text{transient response}} + \underset{\text{permanent part}}{\text{steady-state response}}$$

$$v = v_t + v_{ss}$$

$$v_t = (V_o - V_s)e^{-t/\tau}$$

$$v_{ss} = V_s$$

- Essentially we can express: $v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$

Step Response of an RC Circuit (contd.)

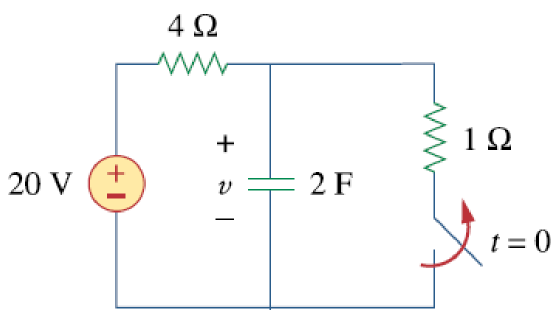
- For the determination of the step response of an *RC* circuit we require:

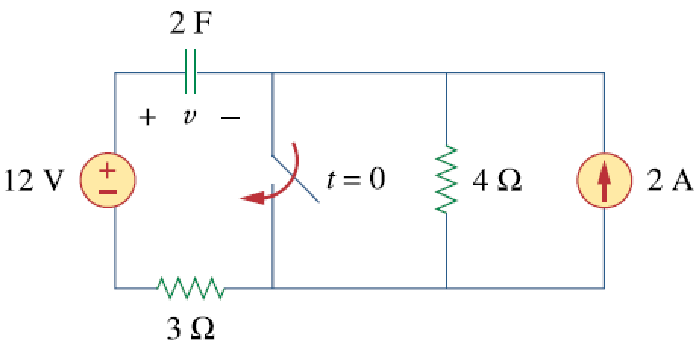
  1. The initial capacitor voltage $v(0)$
  2. The final capacitor voltage $v(\infty)$
  3. The time constant τ.

We can obtain $v(0)$ from the given circuit for t<0 and $v(\infty)$and τ from the circuit for t>0.

Example – 7

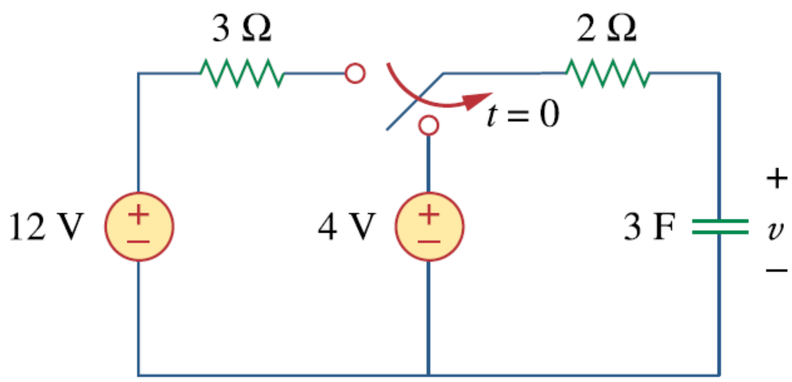
Calculate the capacitor voltage for $t < 0$ and $t > 0$ for both the circuits.

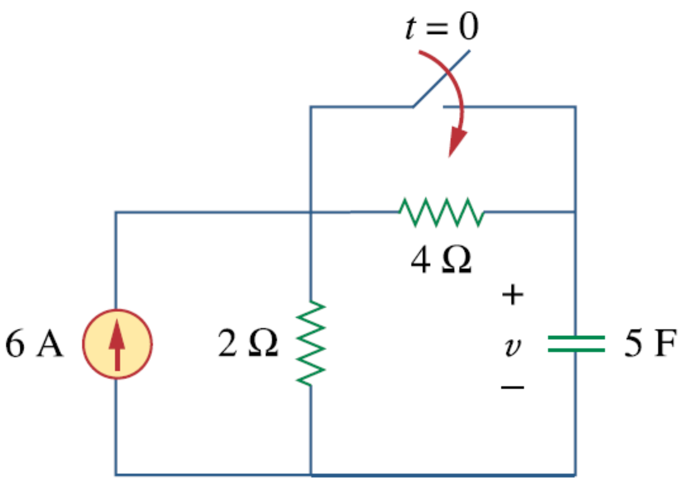




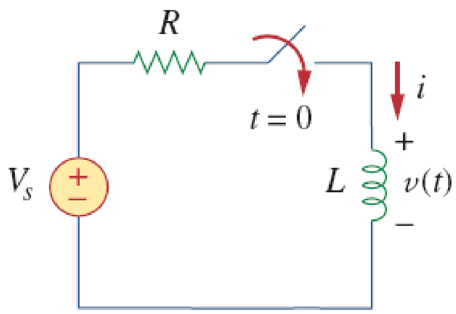
Example – 8

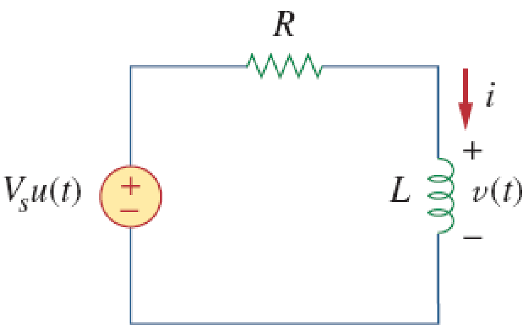
Calculate the capacitor voltage for $t < 0$ and $t > 0$ for both the circuits.





Step Response of an RL Circuit





Complete Response: $i = i_t + i_{ss}$

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \qquad\qquad i_{ss} = \frac{V_s}{R}$$

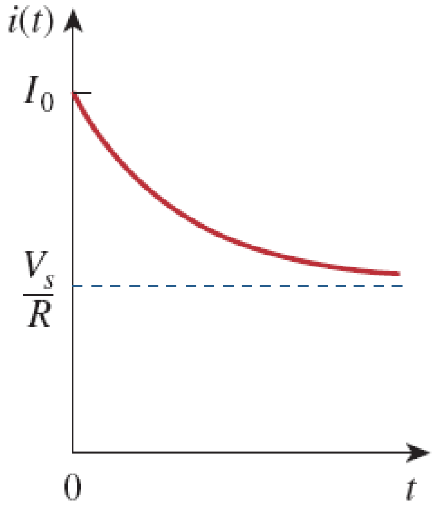
- Now the current through the inductor cannot change instantaneously:

$$i(0^+) = i(0^-) = I_0$$

Step Response of an RL Circuit (contd.)

At t=0: $I_0 = A + \frac{V_s}{R}$ → $A = I_0 - \frac{V_s}{R}$ → $i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau}$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$



- For $I_0=0$

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$$

Step Response of an RL Circuit (contd.)

- The voltage across the inductor:

$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0 \quad \Rightarrow \quad v(t) = V_s e^{-t/\tau}u(t)$$