Lecture – 19 Date: 31.10.2016

- Inverse Laplace Transform
- Convolution Integral
- Circuit Models and Analysis

Inverse Laplace Transform

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Inverse Laplace Transform (contd.)

Q: Given $F(s)$, how do we transform it back to the time domain $f(t)$?

Suppose $F(s)$ has the general form: $F(s) = \dfrac{N(s)}{D(s)}$

Steps to Find the Inverse Laplace Transform:

1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Table.

Solution Methods

1. Simple Poles:

$$F(s) = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

- the poles are distinct.
- Degree of N(s) is smaller than degree of D(s).

Inverse Laplace Transform (contd.)

$$F(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n}$$

The expansion coefficients k_1, k_2, ….., k_n are known as the *residues* of *F*(*s*).

- There are many ways of finding the expansion coefficients. One way is using the *residue method*.

$$(s + p_1)F(s) = k_1 + \frac{(s + p_1)k_2}{s + p_2} + \cdots + \frac{(s + p_1)k_n}{s + p_n}$$

Since $p_i \neq p_j,$ setting $s = -p_1$ in the above leaves only k_1:

$$(s + p_1)F(s)\,|_{s=-p_1} = k_1$$

- **In general** $k_i = (s + p_i)F(s)\,|_{s=-p_i}$ ***Heaviside's theorem***

Inverse Laplace Transform (contd.)

- Once the values of k_i are known, we proceed to find the inverse of $F(s)$.

$$f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t})u(t)$$

2. Repeated Poles:

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s)$$

$F_1(s)$ is the remaining part of $F(s)$ that doesn't have a pole at $s = -p$.

$$k_n = (s+p)^n F(s) \big|_{s=-p}$$

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)] \big|_{s=-p}$$

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)] \big|_{s=-p}$$

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)] \big|_{s=-p}$$

Inverse Laplace Transform (contd.)

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!}u(t) \Rightarrow f(t) = \left(k_1e^{-pt} + k_2te^{-pt} + \frac{k_3}{2!}t^2e^{-pt} + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + f_1(t)$$

3. Complex Poles: $F(s) = \dfrac{A_1s + A_2}{s^2 + as + b} + F_1(s)$

$F_1(s)$ is the remaining part of $F(s)$ that doesn't have a pole at $s = -p$.

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s+\alpha)^2 + \beta^2$$

$$A_1s + A_2 = A_1(s+\alpha) + B_1\beta$$

$$\Rightarrow F(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{B_1\beta}{(s+\alpha)^2+\beta^2} + F_1(s)$$

Inverse Laplace Transform (contd.)

$$f(t) = (A_1 e^{-\alpha t} \cos \beta t + B_1 e^{-\alpha t} \sin \beta t) u(t) + f_1(t)$$

Example – 1

Find $f(t)$ given that: $F(s) = \frac{s^2+12}{s(s+2)(s+3)}$

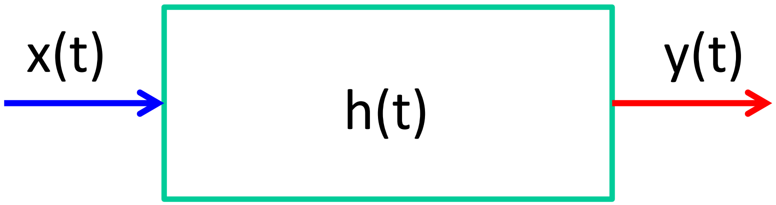
Example – 2

Find $h(t)$ given that: $H(s) = \frac{20}{(s+3)(s^2+8s+25)}$

Convolution Integral

- The term *convolution* means "folding." It is an invaluable tool to the engineer as it provides a means of viewing and characterizing physical systems.

convolution



$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda \quad \Rightarrow \quad y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\,d\lambda$$

the order in which the two functions are convolved is immaterial.

Convolution Integral (contd.)

The convolution of two signals consists of time-reversing one of the signals, shifting it, and multiplying it point by point with the second signal, and integrating the product.

- the convolution integral can be simplified if we assume that a system has two properties.
  - First, if $x(t) = 0 \;\; for\; t < 0$ then:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda = \int_{0}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda$$

  - Second, if the system's impulse response is *causal* (i.e., h(t)=0 for $t < 0$), then $h(t-\lambda) = 0$ for $t - \lambda < 0$ or $\lambda > t$.

$$y(t) = h(t) * x(t) = \int_{0}^{t} x(\lambda)h(t-\lambda)\,d\lambda$$

Convolution Integral (contd.)

- Given two functions $f_1(t)$ and $f_2(t)$ with Laplace transforms $F_1(s)$ and $F_2(s)$, respectively. Then their convolution is:

$$F(s) = \mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

convolution in the time domain is equivalent to multiplication in the *s*-domain.

- For example, if $x(t) = 4e^{-t}$ and $h(t) = 5e^{-2t}$ then:

$$h(t) * x(t) = \mathcal{L}^{-1}[H(s)X(s)] = \mathcal{L}^{-1}\left[\left(\frac{5}{s+2}\right)\left(\frac{4}{s+1}\right)\right]$$

$$= \mathcal{L}^{-1}\left[\frac{20}{s+1} + \frac{-20}{s+2}\right] \Rightarrow = 20(e^{-t} - e^{-2t}), \qquad t \geq 0$$

Convolution Integral (contd.)

- if the product $F_1(s)F_2(s)$ is very complicated, then finding the inverse may be tough.
- Also, there might be situations in which $f_1(t)$ and $f_2(t)$ could be in the form of experimental data and there are no explicit Laplace transforms. In these cases, one must do the convolution in the time domain.
- Often, graphical approach is preferred.

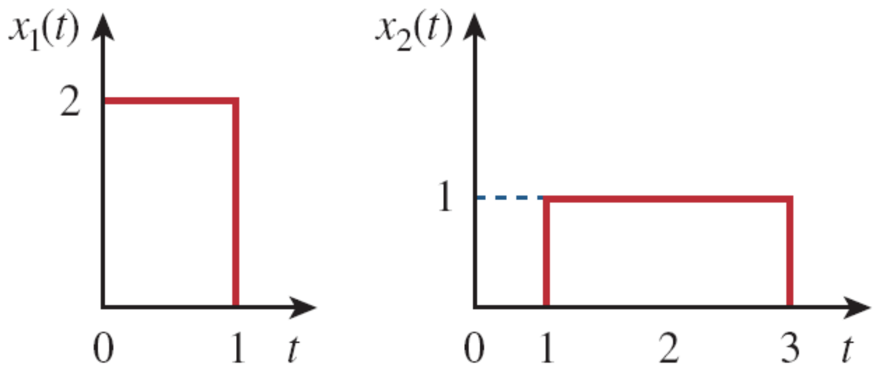
Steps to Evaluate the Convolution Integral:

1. Folding: Take the mirror image of $h(\lambda)$ about the ordinate axis to obtain $h(-\lambda)$.
2. Displacement: Shift or delay $h(-\lambda)$ by t to obtain $h(t-\lambda)$.
3. Multiplication: Find the product of $h(t-\lambda)$ and $x(\lambda)$.
4. Integration: For a given time t, calculate the area under the product $h(t-\lambda)x(\lambda)$ for $0 < \lambda < t$ to get $y(t)$ at t.

The folding operation in step 1 is the reason for the term *convolution.* The function $h(t-\lambda)$ scans or slides over $x(\lambda)$. The convolution integral is also known as the *superposition integral.*

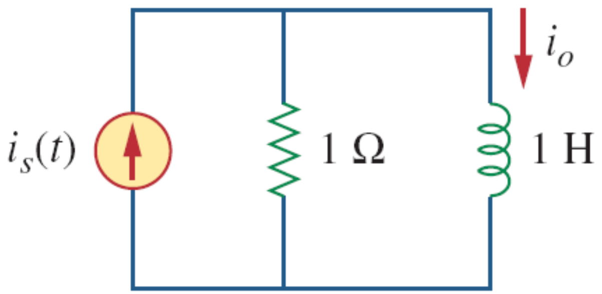
Example – 3

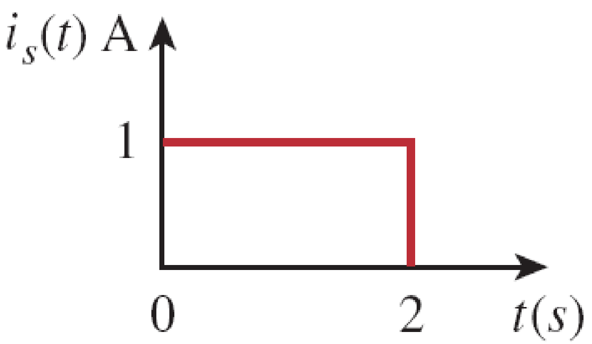
- Find the convolution of these two signals.



Example – 4

- For the following *RL* circuit, use the convolution integral to find the response $i_0(t)$ due to the given excitation.





Example – 5

Solve for the response $y(t)$ in the following integro-differential equation.

$$\frac{dy}{dt} + 5y(t) + 6\int_0^t y(\tau)\,d\tau = u(t), \qquad y(0) = 2$$

Circuit Element Models and Circuit Analysis

Steps in Applying the Laplace Transform:

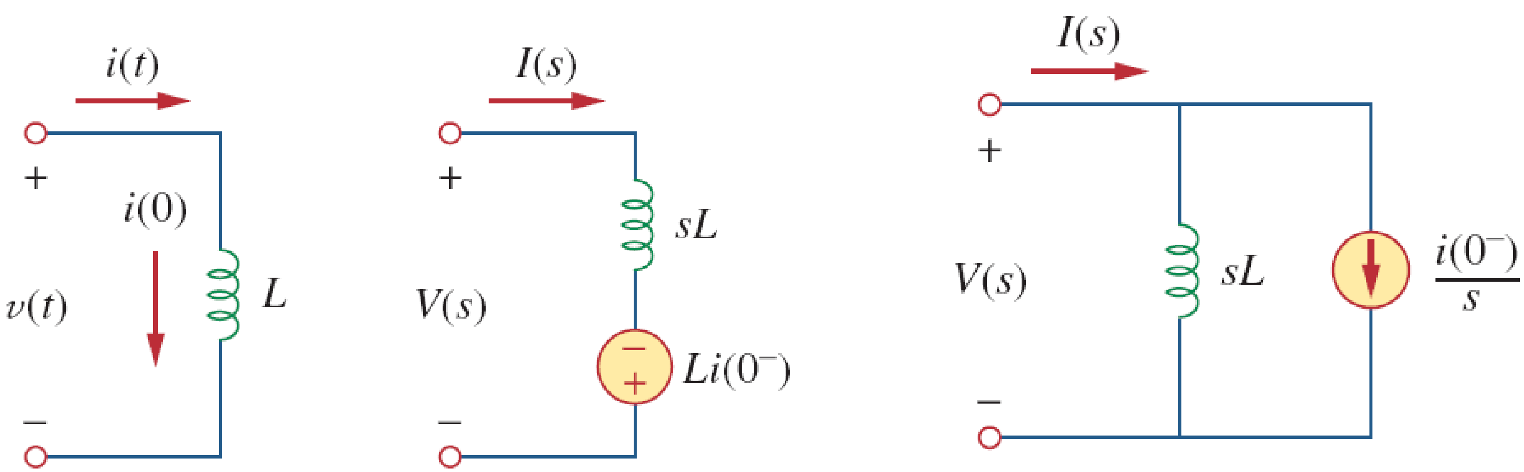
1. Transform the circuit from the time domain to the s-domain.
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Circuit Element Models and Circuit Analysis

For a resistor: $v(t) = Ri(t)$ ➡ $V(s) = RI(s)$

For an inductor: $v(t) = L\dfrac{di(t)}{dt}$ ➡ $V(s) = L[sI(s) - i(0^-)]$

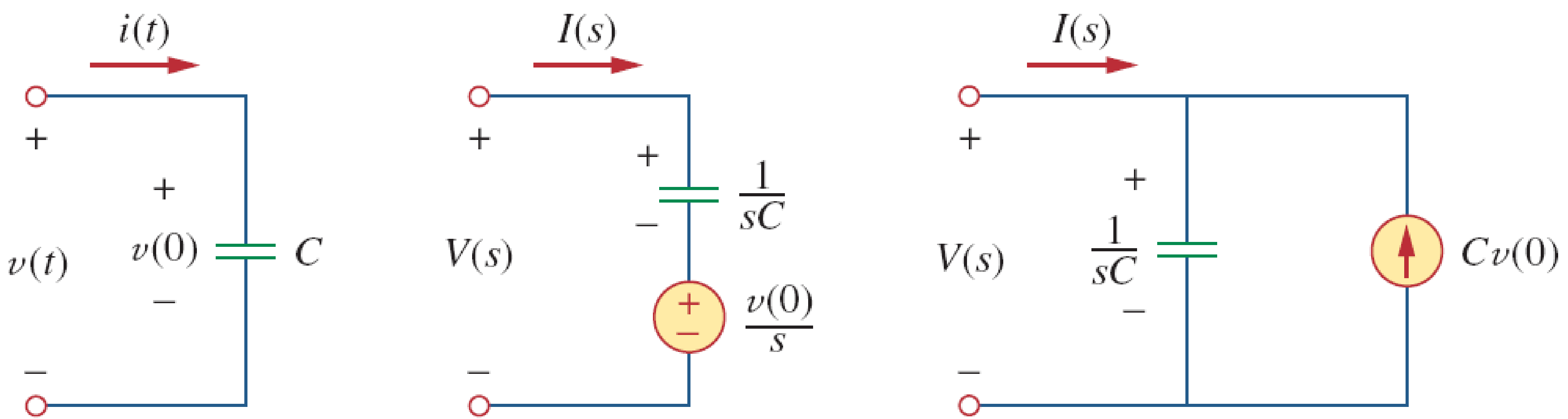
$= sLI(s) - Li(0^-)$ ➡ $I(s) = \dfrac{1}{sL}V(s) + \dfrac{i(0^-)}{s}$



Circuit Element Models and Circuit Analysis

For a capacitor: $i(t) = C\frac{dv(t)}{dt}$ → $I(s) = C[sV(s) - v(0^-)]$

$= sCV(s) - Cv(0^-)$ → $V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$



Circuit Element Models and Circuit Analysis

- For zero initial conditions for the inductor and the capacitor:

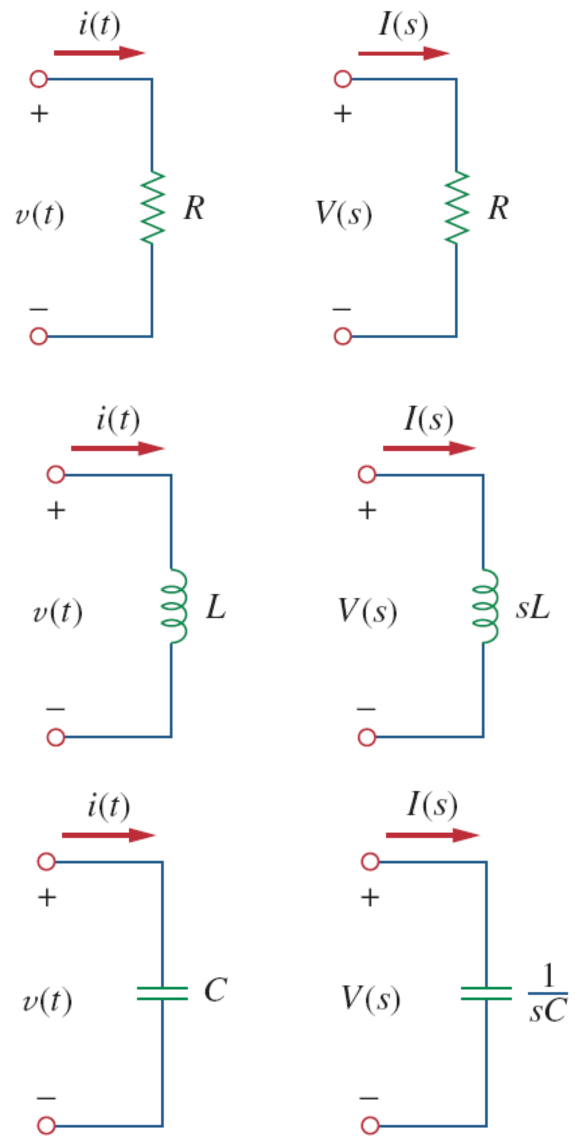
$$\text{Resistor:} \quad V(s) = RI(s)$$

$$\text{Inductor:} \quad V(s) = sLI(s)$$

$$\text{Capacitor:} \quad V(s) = \frac{1}{sC} I(s)$$

- the impedance in the *s*-domain is the ratio of the voltage transform to the current transform under zero initial conditions:

$$Z(s) = \frac{V(s)}{I(s)}$$



Example – 6

Find $v_o(t)$ by assuming zero initial conditions.